Complete system of equations for the state vector of a relativistic composite system on a light front

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An equation for the state vector defined on an invariant light front $\omega x = \sigma(\omega^2 = 0)$ is derived by considering four-dimensional rotations of the light-front hypersurface. This equation supplements the Schrödinger equation and eliminates the ambiguities that arise in the determination of the wave functions of relativistic composite systems (in particular, in the Weinberg equation for a nonzero angular momentum).

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1. INTRODUCTION

This paper continues the development, initiated in Refs. 1–5, of a formalism of wave functions (WF) of relativistic composite systems—the Fock components of a state vector defined on a light-front hypersurface $\omega x = 0$ $(\omega^2 = 0)$. The formalism developed is needed for the investigation of high-momentum components of large nuclear WF and hadronic WF in quark models. A WF on a light front (which is non-equal-time in an arbitrary system) coincides with the customarily used WF in a system with infinite momentum (which is equal-time in this system) if the infinite momentum has the arbitrary direction ω. In the particular case $\omega = (1, 0, 0, -1)$ we obtain a WF on the "null plane" $t + z = 0$. The choice of an arbitrary invariant light-front surface $\omega x = 0$ leads to explicit covariance of the corresponding Fock components, making it convenient to parametrize the WF and simplifying considerably the construction of states with definite total angular momentum.

Parametrization of a relativistic WF differs from that of a nonrelativistic one in that, besides the relative momenta, the relativistic WF depends on an additional variable in the form of a unit vector $\mathbf{n}$ (Ref. 1). This dependence vanishes in the nonrelativistic limit. The origin of this dependence can be easily understood by considering, e.g., the WF in a system with infinite momentum. The WF depends on the direction of the infinite momentum $\mathbf{n} = \mathbf{p}/p$ as $\mathbf{p} \to \infty$, inasmuch as prior to the transition to the limit the equal-time WF had a dynamic dependence on the total momentum $\mathbf{p}$ of the system. The existence of the last dependence is obvious: if the relativistic WF were independent of the total momentum of the system, it would be meaningless to introduce WF in a system with infinite momentum, since the WF would be the same in any system (with arbitrary $\mathbf{p}$).

The indicated properties of WF on a light front are kinematic, i.e., they follow from the transformation laws of the state vector under translations and rotations of the coordinate system. The dynamics, on the other hand, is contained in the solution of the Schrödinger equation that arises when infinitely small translations of the surface of the wave front $\omega x = \sigma$ relative to the coordinate frame is considered: $\sigma \to \sigma + \delta\sigma$. However, besides translations, dynamic transformations of the state vector include also four-dimensional rotations of the surface:

$$\omega_\mu \to \omega_\mu + \delta\omega_\mu, \quad \delta\omega_\mu = \varepsilon_{\nu\mu}\omega_\nu.$$

This leads to one more dynamic equation that is satisfied by the state vector. The present paper is devoted to the study of this equation and to its role in the classification of the states and in the calculation of the WF.

The plan of the exposition is the following. In Sec. 2 we derive an equation for the dependence of the state vector on the hypersurface rotations. We note that in general a distinction must be made between three types of state-vector transformations: 1) Transformations in translations and rotations of the hypersurface relative to a given coordinate system, which is as though fixed in space. 2) Transformations in translations and rotation of the coordinate frame, with the state vector remaining specified on one and the same hypersurface whose position relative to the coordinate axes varies on going from one system to another. The hypersurface is as though fixed in space. 3) Transformations of the coordinate system and simultaneous transformations of the hypersurface, such that the position of the hypersurface relative to the coordinate axes remains unchanged in all systems. In this case the hypersurface is rigidly bound to the coordinate axes. Transformations of this types are combinations of the first two. Obviously, these are the only ones that take place in the theory on the null plane $t + z = 0$.

While the questions dealing with transformations of the state vectors are trivial, the three aforementioned types of transformation are frequently confused, leading to confusion and to difficulty in the understanding of the entire construction as a whole. We shall therefore dwell in detail on the transformation properties of the state vector in Sec. 3. It is shown in Sec. 4 that the equation obtained in Sec. 2 for the dependence of the state vector on the rotations of the hypersurface $\omega x = 0$ eliminates the ambiguity in the classification of the states of a relativistic composite system. This problem is illustrated in Sec. 5 by using as an example a system of two scalar particles interacting in the ladder approximation via exchange of a scalar massless particle. Section 6 contains the concluding remarks.

2. FORMULATION OF THE BOUND-STATE PROBLEM

Corresponding to a system of interacting fields are the 4-dimensional momentum and angular-momentum oper-

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ators $\hat{P}_\mu$ and $\hat{J}_{\mu\nu}$, which satisfy the commutation relations of the Poincaré group:

$$[\hat{P}_\mu, \hat{P}_\nu]=0, \tag{1a}$$

$$\frac{1}{i}[\hat{P}_\mu, \hat{J}_{\varkappa\rho}]=g_{\mu\rho}\hat{P}_\varkappa-g_{\mu\varkappa}\hat{P}_\rho, \tag{1b}$$

$$\frac{1}{i}[\hat{J}_{\mu\nu}, \hat{J}_{\rho\tau}]=g_{\mu\rho}\hat{J}_{\nu\tau}-g_{\nu\rho}\hat{J}_{\mu\tau}+g_{\nu\tau}\hat{J}_{\mu\rho}-g_{\mu\tau}\hat{J}_{\nu\rho}. \tag{1c}$$

The operators $\hat{P}_\mu$ and $\hat{J}_{\mu\nu}$ are represented in the form of the sums

$$\hat{P}_\mu=\hat{P}_\mu^0+\hat{P}_\mu^{int}, \tag{2}$$

$$\hat{J}_{\mu\nu}=\hat{J}_{\mu\nu}^0+\hat{J}_{\mu\nu}^{int}, \tag{3}$$

where $\hat{P}_\mu^0$ and $\hat{J}_{\mu\nu}^0$ are the operators corresponding to the free fields, while $\hat{P}_\mu^{\rm int}$ and $\hat{J}_{\mu\nu}^{\rm int}$ contain the interaction, and are expressed in the interaction representation in terms of the free fields. On the light front $\omega x=\sigma$, the operators $\hat{P}_\mu^{\rm int}$ and $\hat{J}_{\mu\nu}^{\rm int}$ take the form

$$\hat{P}_\mu^{int}=\omega_\mu\int H^{int}(x)\delta(\omega x-\sigma)d^4x, \tag{4}$$

$$\hat{J}_{\mu\nu}^{int}=\int H^{int}(x)(x_\mu\omega_\nu-x_\nu\omega_\mu)\delta(\omega x-\sigma)d^4x, \tag{5}$$

where $H^{\rm int}(x)$ is the density of the interaction Hamiltonian, expressed in terms of the free fields. Thus, in the Wick-Cutkosky model

$$H^{int}(x)=-g\varphi^2(x)\chi(x), \tag{6}$$

where $\varphi(x)$ is a scalar field with mass m and $\chi(x)$ is a massless scalar field; for $\varphi(x)$, e.g., we have

$$\varphi(x)=(2\pi)^{-3/2}\int[a(\mathbf{k})e^{-ikx}+a^+(\mathbf{k})e^{ikx}]\frac{d^3k}{(2\varepsilon_k)^{1/2}}=\varphi^-(x)+\varphi^+(x). \tag{7}$$

From the operators $\hat{P}_\mu$ and $\hat{J}_{\mu\nu}$ we construct the Pauli-Lubanski vector

$$\hat{S}_\mu={}^1/_2\varepsilon_{\mu\nu\rho\tau}\hat{P}_\nu\hat{J}_{\rho\tau}. \tag{8}$$

The state vector Φ_s^J defined on the hypersurface $\omega x=\sigma$ and describing a state with a definite 4-momentum p, mass $M^2=p^2$, total angular momentum J, and projection s of the angular momentum on the z axis in the rest system $\mathbf{p}=0$, satisfies the equations

$$\hat{P}_\mu\Phi_s^J(p)=p_\mu\Phi_s^J(p), \tag{9}$$

$$\hat{P}_\mu^2\Phi_s^J(p)=M^2\Phi_s^J(p), \tag{10}$$

$$\hat{S}_\mu^2\Phi_s^J(p)=-M^2J(J+1)\Phi_s^J(p), \tag{11}$$

$$\hat{S}_3\Phi_s^J(p)=Ms\Phi_s^J(p). \tag{12}$$

The state vector, being in the general case a functional of the surface on which it is defined, is in the case of the plane surface $\omega x=\sigma$ a function of the parameters σ and ω. The equations that determine the dependence of Φ on σ and ω can be easily obtained by starting from the Tomonaga-Schwinger equation

$$i\delta\Phi/\delta\sigma(x)=H^{int}(x)\Phi. \tag{13}$$

From the definition of the variational derivative in (13) it follows that

$$i\delta\Phi=H^{int}(x)\Phi\delta V(x), \tag{14}$$

where $\delta V(x)$ is the volume contained between the initial surface and the surface obtained from the initial by variation $\delta\sigma(x)$ about the point x.

Following a translation $\sigma\to\sigma+\delta\sigma$ of the surface, the total increment of the state vector is the results of the increments at each point of the surface:

$$i\delta\Phi=\int H^{int}(x)\delta(\omega x-\sigma)d^4x\Phi\delta\sigma,$$

or

$$id\Phi(\sigma)/d\sigma=H(\sigma)\Phi(\sigma),\qquad H(\sigma)=\int H^{int}(x)\delta(\omega x-\sigma)d^4x. \tag{15}$$

The four-dimensional rotations

$$\omega_\mu\to\omega_\mu'=\omega_\mu+\delta\omega_\mu,\qquad \delta\omega_\mu=\varepsilon_{\nu\mu}\omega_\nu$$

lead to

The increment of the volume over the point x is

$$\delta V=\varepsilon_{\mu\nu}x_\mu\omega_\nu\delta(\omega x-\sigma)d^4x,$$

and it follows from (14) that

$$\hat{L}_{\mu\nu}(\omega)\Phi(\omega)=\hat{J}_{\mu\nu}^{int}\Phi(\omega), \tag{16}$$

where

$$\hat{L}_{\mu\nu}(\omega)=i\left(\omega_\mu\frac{\partial}{\partial\omega_\nu}-\omega_\nu\frac{\partial}{\partial\omega_\mu}\right), \tag{17}$$

and $\hat{J}_{\mu\nu}^{\rm int}$ is given by Eq. (5).

Equation (16) is the sought equation that supplements the Schrödinger Eq. (15).

If (16) is satisfied, the operator $\hat{J}_{\mu\nu}^{\rm int}$ contained in the Pauli-Lubanski vector $\hat{S}_\mu$ in (11) and (12) can be replaced by $\hat{L}_{\mu\nu}(\omega)$. Next, introducing the notation

$$\hat{M}_{\mu\nu}=\hat{J}_{\mu\nu}^0+\hat{L}_{\mu\nu}(\omega), \tag{18}$$

where $\hat{J}_{\mu\nu}^0$ is a free operator, $\hat{L}_{\mu\nu}(\omega)$ is given by Eq. (17), and

$$\hat{W}_\mu={}^1/_2\varepsilon_{\mu\nu\rho\tau}\hat{P}_\nu\hat{M}_{\rho\tau}, \tag{19}$$

we obtain in place of (11) and (12)

$$\hat{W}_\mu^2\Phi_s^J(p)=-M^2J(J+1)\Phi_s^J(p), \tag{20}$$

$$\hat{W}_3\Phi_s^J(p)=Ms\Phi_s^J(p). \tag{21}$$

The construction of states with definite total angular momentum, which reduces to solution of Eqs. (20) and (21), is a purely kinematic problem, i.e., it does not depend on the form of the interaction Hamiltonian [provided that the state vector $\Phi_s^J(p)$ corresponds to a definite 4-momentum, i.e., satisfies Eqs. (9) and (10), as well as Eq. (16)]. Writing down in accord with Ref. 3 the general expression for the WF on the light front with total angular momentum J, we can determine next from the dynamic Eqs. (9) and (19) the invariant functions it contains. The role of Eq. (16) in this procedure is traced in Sec. 4 and is illustrated by an example in Sec. 5. The kinematic part is thus maximally separated from the dynamic in this approach.

To conclude this section we present the commutation relations of the operators $\hat{M}_{\mu\nu}$ with $\hat{P}_\nu$ and $\hat{J}_{\mu\nu}$:

$$\frac{1}{i}[\hat{P}_\mu, \hat{M}_{\varkappa\rho}]=g_{\mu\rho}\hat{P}_\varkappa-g_{\mu\varkappa}\hat{P}_\rho, \tag{22a}$$

$$\frac{1}{i}[\hat{J}_{\mu\nu}, \hat{M}_{\rho\gamma}]=g_{\mu\rho}\hat{J}_{\nu\gamma}-g_{\nu\rho}\hat{J}_{\mu\gamma}+g_{\nu\gamma}\hat{J}_{\mu\rho}-g_{\mu\gamma}\hat{J}_{\nu\rho}. \tag{22b}$$

They can be easily obtained if its recognized that the operators $\hat{J}^0_{\mu\nu}$ and $\hat{L}_{\mu\nu}(\omega)$, which enter in $\hat{M}_{\mu\nu}$, are respectively generators of infinitely small transformations of the free field and of the 4-vector ω, from which the operators $\hat{P}_\mu$ and $\hat{J}_{\mu\nu}$ are constructed. The commutator $[\hat{M}_{\mu\nu}, \hat{M}_{\rho\gamma}]$ is similar to the commutator (1c).

3. TRANSFORMATION PROPERTIES OF THE STATE VECTOR

Equations (15) and (16) determine the transformation properties of the state vectors in translations and rotations of the hypersurface relative to a fixed coordinate system (type 1 in the classification given in the Introduction). We obtain now the law of transformation of a state vector defined on a fixed surface under transformations of the coordinate system (type 2).

We consider first the translations of the coordinate system, $x \to x'=x+a$. The equation of the hypersurface $\omega x=\sigma$ takes in the new system the form $\omega x'=\sigma'$, where $\sigma'=\sigma+\omega a$. The state vector transforms as

$$\Phi(\sigma)\to\Phi'(\sigma+\omega a)=U(a)\Phi(\sigma). \tag{23}$$

Our task is to find the operator $U(a)$.

By virtue of the translational invariance, the Tomonaga-Schwinger equation retains its form in the new system:

$$i\frac{\delta\Phi'(\sigma')}{\delta\sigma(x')}=H^{int}(x')\Phi'(\sigma'). \tag{24}$$

Substituting (23) in (24), we obtain

$$i\frac{\delta\Phi(\sigma)}{\delta\sigma(x)}=U^{-1}(a)H^{int}(x+a)U(a)\Phi(\sigma).$$

Since $H^{int}(x)$ is expressed in terms of free-field operators, we have

$$H^{int}(x+a)=\exp(i\hat{P}_a{}^0)H^{int}(x)\exp(-i\hat{P}_a{}^0),$$

therefore

$$U(a)=\exp(iP^0a), \tag{25}$$

where $\hat{P}^0$ is the free-field 4-momentum operator.

Repeating this reasoning for the case of infinitely small rotations $x_\mu \to x'_\mu=gx_\mu=x_\mu+\varepsilon_{\nu\mu}x_\nu$, we obtain

$$\Phi(\omega)\to\Phi'(\omega')=U(g)\Phi(\omega), \tag{26}$$

where

$$\omega_\mu'=\omega_\mu+\varepsilon_{\nu\mu}\omega_\nu,\quad U(g)=1+{}^1/{}_2 i\hat{J}_{\mu\nu}{}^0\varepsilon_{\mu\nu}. \tag{27}$$

Here $\hat{J}^0_{\mu\nu}$ is a free operator.

We see thus that the generators of the transformation of a state vector specified on an invariant surface do not contain the interaction when the coordinate system is transformed.

Corresponding to a transformation of the coordinate system and to a simultaneous transformation of a hypersurface rigidly tied to the coordinate axes (type 3) is a successive application of the transformations of the two types considered above. The corresponding generators are given by Eqs. (2) and (3). Thus, at infinitely small translations from the coordinate system A into a system $A': x\to x'=x+a$ we have

$$\Phi(\sigma)\to\Phi'(\sigma)=(1+i\hat{P}a)\Phi(\sigma), \tag{28}$$

where $\Phi(\sigma)$ is defined in the system A on the surface $\omega x=\sigma$, while $\Phi'(\sigma)$ is defined in the system A' on the surface $\omega x'=\sigma$ and $\hat{P}$ is the 4-momentum operator (2).

We change now to the Fock representation

$$\Phi_s{}^J(p,\omega)=\int C_s{}^J(x_1,x_2,p,\omega,\sigma)\delta(\omega x_1-\sigma)\delta(\omega x_2-\sigma)\varphi^+(x_1)\times\varphi^+(x_2)|0\rangle d^4x_1\,d^4x_2+\dots. \tag{29}$$

We have written out explicitly only the two-particle component. We assume for simplicity that all the particles that make up the coupled system are spinless, and the total angular momentum of the system is equal to J.

From (23) and (26) we easily obtain the transformation properties of the wave functions C^J_s. Thus, the transformation $x\to x'=gx$ leads to

$$\Phi_s{}^J(p,\omega)\to\Phi_s{}'^J(gp,g\omega)=U(g)\Phi_s{}^J(p,\omega), \tag{30}$$

where $U(g)$ is defined by (27). Expanding Φ' in terms of the states Φ:

$$\Phi_s{}'^J(gp,g\omega)=\sum_{s'}D^{(J)}_{s's}\{R(g,p)\}\Phi_{s'}{}^J(gp,g\omega)$$

and using the fact that

$$U(g)\varphi^+(x)U^{-1}(g)=\varphi^+(gx),$$

we obtain

$$C_s{}^J(gx_1,gx_2,gp,g\omega)=\sum_{s'}D^{*(J)}_{ss'}\{R(g,p)\}C_{s'}{}^J(x_1,x_2,p,\omega). \tag{31}$$

It is now easy to verify that the state vector (29) with Fock components that transform in accord with (31) is actually a solution of Eqs. (20) and (21). Wave functions of particles with arbitrary spins were constructed in Ref. 3.

Expansion (39) takes in momentum space the form

$$\Phi_s^{(J)}(p)=\int\psi_s{}^J(k_1,k_2,p,\omega\tau)\delta^{(4)}(k_1+k_2-p-\omega\tau)e^{i\tau\sigma}d\tau a^+(\mathbf{k}_1)a^+(\mathbf{k}_2)|0\rangle\times\frac{d^3k_1}{(2\varepsilon_1)^{1/2}}\frac{d^3k_2}{(2\varepsilon_2)^{1/2}}+\dots. \tag{32}$$

The equality $k_1+k_2=p+\omega\tau$ defined by the δ function in (32) is a consequence of translational invariance. Indeed, from (28), with allowance for the fact that is an eigenvector of the operator P, it follows that

$$\Phi'(\sigma+\omega a)=e^{ipa}\Phi(\sigma+\omega a),$$

and from (23) we obtain

$$\exp(i\hat{P}^0a)\Phi(\sigma)=\exp(ipa)\Phi(\sigma+\omega a). \tag{33}$$

It is easily seen that the state vector (32) satisfies the condition (33). The action of the operator $\exp(i\hat{P}^0a)$

leads to the appearance of $\exp(i(k_1+k_2)a)$ under the integral sign in (32), while the factor $\exp(i(pa+\tau\omega a))$ appears in the right-hand side of (33) under the integral sign. The δ-function in (32) ensures equality of these factors and satisfaction of the condition (33). Thus, by starting with the transformation formulas (23), (28) and and (26), we reproduce the Fock-component properties obtained in Refs. 1 and 3.

4. CLASSIFICATION OF RELATIVISTIC BOUND STATES

With the aid of the Pauli-Lubanski vector (19) we can construct the operator

$$\hat{A}=(\omega_\mu \hat{W}_\mu)^2. \tag{34}$$

It is easy to verify that this operator commutes with $\hat{P}_\mu$, $\hat{M}_{\mu\nu}$, and with $\hat{W}_\mu$. It appears therefore at first glance that the state vector should be characterized not only by the quantum numbers p_μ, M, J, and s but also by an eigenvalue of the operator $\hat{A}$:

$$\hat{A}\Phi_\alpha=\alpha\Phi_\alpha, \tag{35}$$

and, as will be shown below, states with different α correspond to the same energy, i.e., degeneracy is present.

The reason for this degeneracy is that no account has been taken so far of Eq. (16). We shall show that a state vector that satisfies (16) cannot be an eigenvector of the operator $\hat{A}$, but is a superposition of states with different α (except for the trivial case $J=0$).

We rewrite (16) in the form

$$\Delta\hat{J}_{\mu\nu}\Phi=0, \tag{36}$$

where

$$\Delta\hat{J}_{\mu\nu}=\hat{M}_{\mu\nu}-\hat{J}_{\mu\nu}=\hat{L}_{\mu\nu}(\omega)-\hat{J}_{\mu\nu}^{int}, \tag{37}$$

$\hat{M}_{\mu\nu}$ and $\hat{J}_{\mu\nu}$ are given by (18) and (3). Let B be some operator (e.g., $\hat{B}=\hat{P}_\mu, \hat{W}_\mu^2, \hat{W}_3, \hat{A}$) whose eigenvector is Φ:

$$\hat{B}\Phi=b\Phi. \tag{38}$$

We shall show that Eqs. (36) and (38) are compatible only in the case when the action of the commutator $[\Delta\hat{J}_{\mu\nu},\hat{B}]$ on Φ produces zero:

$$[\Delta\hat{J}_{\mu\nu},\ \hat{B}]\,\Phi=0. \tag{39}$$

Assume that Eqs. (36) and (38) are simultaneously satisfied. Then

$$\Delta\hat{J}_{\mu\nu}\hat{B}\Phi=\Delta\hat{J}_{\mu\nu}b\Phi=b\Delta\hat{J}_{\mu\nu}\Phi=0.$$

On the other hand

$$0=\Delta\hat{J}_{\mu\nu}\hat{B}\Phi=\hat{B}\Delta\hat{J}_{\mu\nu}\Phi+[\Delta\hat{J}_{\mu\nu},\ \hat{B}]\,\Phi=[\Delta\hat{J}_{\mu\nu},\ \hat{B}]\,\Phi.$$

If the condition (39) is not satisfied, we arrive at a contradiction. This result is a particular case of a theorem from the theory of group representations (see Ref. 6).

Putting $\hat{B}=\hat{A}$ and calculating with the aid of relations (1) and (22) the commutator

$$[\Delta\hat{J}_{\mu\nu},\ \omega_\tau\hat{W}_\tau]=i\omega_\tau\hat{P}_\beta(\varepsilon_{\tau\beta\mu\lambda}\Delta\hat{J}_{\nu\lambda}-\varepsilon_{\tau\beta\nu\lambda}\Delta\hat{J}_{\mu\lambda})+i(\hat{W}_\mu\omega_\nu-\hat{W}_\nu\omega_\mu), \tag{40}$$

we see that for a nonzero spin the condition (39) is not satisfied, and a state vector that satisfies (36) is not an eigenvector of the operator $\hat{A}=(\omega_\mu\hat{W}_\mu)^2$.

As already noted, the states Φ_α are degenerate. In fact, the state $\Phi'=\Delta\hat{J}_{\mu\nu}\Phi_\alpha$ is by virtue of (40) not an eigenvector of the operator A, i.e., it is represented by the superposition

$$\Phi'=\sum_\alpha \beta_\alpha\Phi_\alpha,$$

But since $[\Delta\hat{J}_{\mu\nu},\hat{P}_\gamma]=0$, it follows that Φ' corresponds to the same mass as Φ_α. The condition (16) or (36) separates in fact a definite superposition of the states Φ_α:

$$\Phi=\sum_\alpha c_\alpha\Phi_\alpha, \tag{41}$$

such that $\Delta\hat{J}_{\mu\nu}\Phi=0$, and enables us to find the coefficients in Eq. (41). Thus, allowance for (16) eliminates the problem of "redundant" states of a relativistic composite system.

We note that since the commutators

$$[\Delta\hat{J}_{\mu\nu},\ \hat{M}_{\gamma\delta}]\sim\Delta\hat{J},\qquad [\Delta\hat{J}_{\mu\nu},\ \hat{W}_\tau]\sim\Delta\hat{J}$$

vanish as the state vector, Eq. (16) does not prevent the state vector from being an eigenvector of the operators $\hat{W}_\mu^2$ and $\hat{W}_3$, i.e., from having a definite spin. This, incidentally, follows even from the fact that Eq. (16) is clearly covariant.

5. LADDER APPROXIMATION

Equation (16) leads to a condition on the Fock components. The concrete form of the condition depends on the Hamiltonian contained in (5) and on the approximation with which the problem with the given initial Hamiltonian is solved. By way of example we solve the problem with the Hamiltonian (6) in the ladder approximation. Within the framework of this model the WF on a light front were obtained in Ref. 4. We shall use this example to resolve problem discussed in Sec. 4, obtain next the condition that follows from (16) on the two-particle WF, and show that the WF obtained in Refs. 4 and 5 indeed satisfy this condition.

The equation for the two-particle WF is of the form[1,4]

$$[4(\mathbf{q}^2+m^2)-M^2]\psi(\mathbf{q},\mathbf{n})=-\frac{m^2}{2\pi^3}\int\psi(\mathbf{q}',\mathbf{n})V(\mathbf{q}',\mathbf{q},\mathbf{n},M^2)\frac{d^3q'}{\varepsilon(\mathbf{q}')}. \tag{42}$$

In place of the variables k_1, k_2, p, and ωt we have introduced in (42) the variables $\mathbf{q}$ and $\mathbf{n}$ (see Refs. 1 and 4), where $\mathbf{q}$ has the meaning of the particle 1 in the rest system of the pair 1 and 2 (i.e., at $k_1+k_2=0$), and $\mathbf{n}$ is a unit vector in the $\boldsymbol{\omega}$ direction in this system. Equation (42) rewritten in terms of the variables $\mathbf{k}_\perp$ and x of the infinite-momentum system (see Ref. 4) coincides with Weinberg's equation.[7]

The angular momentum operator $\hat{\mathbf{J}}$ as a function of the variables $\mathbf{q}$ and $\mathbf{n}$ is of the form[3]

$$\hat{\mathbf{J}}=-i\left[\mathbf{q}\times\frac{\partial}{\partial\mathbf{q}}\right]-i\left[\mathbf{n}\times\frac{\partial}{\partial\mathbf{n}}\right]. \tag{43}$$

In the nonrelativistic limit, the WF ceases to depend on n, therefore the term $-i[\mathbf{n}\times\partial/\partial\mathbf{n}]$ can be left out of (43) at $q\ll m$.

We consider now the operator[1)]

$$\hat{A}'=(\mathbf{n}\hat{\mathbf{J}})^2.$$

It commutes with the angular momentum and with the kernel $V(\mathbf{q}',\mathbf{q},\mathbf{n},M^2)$ (the kernel V is a scalar). Therefore the solution of Eq. (42) is characterized by the mass M and by the angular momentum J and its projections, as well as by the eigenvalue α of the operator $\hat{A}'$. The state $J\neq 0$ always corresponds to several eigenfunctions with different values of α. For example, for $J=1$ we have $\alpha=0$ and 1. Since this property of the spectrum does not depend on the kernel V, it is clear that some of the states are not physical and are redundant. The appearance of redundant states is due to the failure to use Eq. (16).

Before we obtain the corollary of (16), we illustrate how Eq. (9) for the state vector leads to Eq. (42), since the condition for the two-particle WF from (16) is obtained in similar fashion. We rewrite (9), taking (2) and (4) into account, in the form

$$(\hat{P}_\mu^0-p_\mu)\Phi=-\omega_\mu\int H^{int}(x)\delta(\omega x)d^4x\Phi. \tag{44}$$

Here and below we assume $\sigma=0$. The action of the operator $\hat{P}_\mu^0-p_\mu$ on the state vector causes each of the Fock components in (32) to be multiplied under the integral sign by $\omega_\mu\tau$. We put

$$\hat{P}_\mu^0-p_\mu=\omega_\mu\hat{\tau}.$$

We have introduced an operator $\hat{\tau}$ such that $\hat{\tau}\Phi=\Phi'$, where the Fock components Φ' are obtained from Φ by multiplication by τ. Introducing the Fourier transform of the interaction Hamiltonian

$$\tilde{H}(p)=\int e^{-ipx}H^{int}(x)d^4x,$$

we obtain from (44)

$$\hat{\tau}\Phi=-\int\tilde{H}(-\omega\tau)\frac{d\tau}{2\pi}\Phi. \tag{45}$$

Corresponding to the Hamiltonian (6) is the operator

$$\tilde{H}(-\omega\tau)=-\frac{g}{(2\pi)^{1/2}}\int\delta^{(4)}(\omega\tau+k_1+k_2+k_3)\tilde{\varphi}(k_1)\tilde{\varphi}(k_2)\tilde{\chi}(k_3)d^4k_1d^4k_2d^4k_3, \tag{46}$$

where

$$\tilde{\varphi}(k)=(2\pi)^{-5/2}\int e^{-ikx}\varphi(x)d^4x.$$

We confine ourselves in the expansion (32) to the contribution of only the two- and three-particle components ψ_2 and ψ_3. The component ψ_2 corresponds to two particles described by the field φ, while ψ_3 contains in addition a particle described by the field χ. From Eq. (45) with the Hamiltonian (46) follows a system of two equations for these components. Eliminating from them the three-particle component, we arrive at the equation

$$\int_{-\infty}^{+\infty}\tau\psi(k_1,k_2,p,\omega\tau)\delta^{(4)}(k_1+k_2-p-\omega\tau)d\tau$$
$$=\int_{-\infty}^{+\infty}R(k_1,k_2,p,\omega\tau)\delta^{(4)}(k_1+k_2-p-\omega\tau)d\tau, \tag{47}$$

where

$$R=\int\psi(k_1',k_2',p,\omega\tau')K(k_1',k_2',\omega\tau',k_1,k_2)$$
$$\times\delta^{(4)}(k_1'+k_2'-p-\omega\tau')\frac{d^3k_1'}{2\varepsilon_1'}\frac{d^3k_2'}{2\varepsilon_2'}\frac{d\tau'}{2\pi}, \tag{48}$$

and the kernel K is given by the formula[4]

$$K=\frac{g^2}{(2\pi)^2}\frac{\theta(\omega(k_1'-k_1))}{2\tau'\omega(k_1'-k_1)-(k_1'-k_1)^2-i0}$$
$$+\frac{g^2}{(2\pi)^2}\frac{\theta(\omega(k_2'-k_2))}{2\tau'\omega(k_2'-k_2)-(k_2'-k_2)^2-i0}. \tag{49}$$

The kernel V in (42) is proportional to K: $V=-\pi^2K/m^2$, and an expression corresponding to (49), for V in terms of the variables $\mathbf{q}'$, $\mathbf{q}$, and $\mathbf{n}$, is given in Ref. 4. Equation (42) follows from (47) when the change is made to the variables $\mathbf{q}$ and $\mathbf{n}$.

We obtain now the condition arrived at from (16) in the ladder approximation. We transform expression (5) for $\hat{J}_{\mu\nu}^{int}$

$$J_{\mu\nu}^{int}=i\left(\omega_\mu\frac{\partial}{\partial\omega_\nu}-\omega_\nu\frac{\partial}{\partial\omega_\mu}\right)\int_{-\infty}^{+\infty}\tilde{H}(-\omega\tau)\frac{d\tau}{2\pi\tau}.$$

Equation (16) takes the form

$$\hat{L}_{\mu\nu}(\omega)\Phi(\omega)=\left(\hat{L}_{\mu\nu}(\omega)\int_{-\infty}^{+\infty}\tilde{H}(-\omega\tau)\frac{d\tau}{2\pi\tau}\right)\Phi(\omega), \tag{50}$$

where the operator $\hat{L}_{\mu\nu}(\omega)$ is defined in (17), and acts in the right-hand side of (5) only on $\hat{H}(-\omega t)$. An equation for a two-particle WF can be obtained from (50) in exactly the same manner as Eq. (47) from (45). It is possible, however, even without two cumbersome manipulations, to obtain this equation by comparing (45), (50), and (47):

$$\hat{L}_{\mu\nu}(\omega)\int\psi(k_1,k_2,p,\omega\tau)\delta^{(4)}(k_1+k_2-p-\omega\tau)d\tau$$
$$=-\int R(k_1,k_2,p,\omega\tau)\delta_{\mu\nu}^{(4)}(k_1+k_2-p-\omega\tau)d\tau, \tag{51}$$

where

$$\delta_{\mu\nu}^{(4)}(k)=\omega_\mu\frac{\partial}{\partial k_\nu}\delta^{(4)}(k)-\omega_\nu\frac{\partial}{\partial k_\mu}\delta^{(4)}(k),$$

and R is defined in Eq. (48).

The left-hand sides of (45) and (50) differ in fact in that the operator $\hat{\tau}$ in (45) is replaced by $\hat{L}_{\mu\nu}(\omega)$ in (50). This is also the difference between the left-hand sides of (47) and (51). The right-hand side of (50), in contrast to (45), contains the operator $L_{\mu\nu}(\omega)$ that acts on $\tilde{H}(-\omega\tau)$. The operator $\hat{L}_{\mu\nu}$ differentiates the δ function contained in (46), and this yields $\delta_{\mu\nu}^{(4)}$ in (51) in lieu of the δ function in (47). The factor $1/\tau$ in (50) is cancelled when the δ function is differentiated.

We multiply both halves of (51) by $\delta(\omega^2-\eta^2)\theta(\omega_0)d^4\omega$, integrate with respect $d^4\omega$, and let η^2 go to zero. The integral of the left-hand side of the equation gives zero, and the integral of the right-hand side leads to the condition

$$(\hat{L}_{\mu\nu}(\omega)R(k_1,k_2,p,\omega\tau))|_{k_1+k_2=p+\omega\tau}=0. \tag{52}$$

We emphasize that in (52) it is necessary first to carry out the differentiation and then put $k_1+k_2=p+\omega\tau$.

Starting from Eq. (42) and using the convergence of the integral in (42) at $q' \ll m$ we have obtained in Refs. 4 and 5 an approximate expression for the WF by substituting in the right-hand side of (42) the nonrelativistic WF, and taking the kernel V outside the integral sign at the point $\mathbf{q}'=0$. In the same approximation we shall calculate (48) for R and verify that the condition (52) is indeed satisfied. In the system where $\mathbf{p}=0$, the kernel K can be taken outside the integral sign at the point $\mathbf{k}_1' = \mathbf{k}_2' = \tau' = 0$, and this yields in an arbitrary system

$$R \sim \frac{\theta(^1/_2-\omega k_1/\omega p)}{2m^2-(k_1p)} + \frac{\theta(^1/_2-\omega k_2/\omega p)}{2m^2-(k_2p)}. \tag{53}$$

It is seen from (53) that R does not depend on ω, and it is this which leads to satisfaction of the condition (52). Exceptions are the points $\omega k_{1,2}/\omega p = \frac{1}{2}$, in the vicinity of which, however, the approximation that leads to expression (53) for R no longer holds.

It follows from (47) that

$$\psi \sim R(k_1, k_2, p, \omega\tau)/\mathbf{q}^2, \tag{54}$$

in which the arguments R are connected by the relation $k_1 + k_2 = p + \omega\tau$. It is convenient to change to the variables $\mathbf{q}$ and $\mathbf{n}$ with the aid of the following equations (at $M \approx 2m$):

$$t=(p-k_1)^2=M^2+m^2-2(pk_1)=m^2-2\tau(\omega k_2),$$
$$2m^2-(pk_1)\approx-\tau(\omega k_2)\approx-2\mathbf{q}^2\omega k_2/\omega p=-\mathbf{q}^2(1+\mathbf{nq}/\varepsilon(\mathbf{q})),$$

where $\varepsilon(\mathbf{q}) = (\mathbf{q}^2 + m^2)^{1/2}$. From this, taking (53) and (54) into account, we obtain an expression for the WF:

$$\psi(\mathbf{q}, \mathbf{n}) \sim [\mathbf{q}^4(1+|\mathbf{nq}|/\varepsilon(\mathbf{q}))]^{-1},$$

which agrees with that obtained in Ref. 4. We emphasize that although satisfaction of condition (52) is ensured by the independence of R of ω, by virtue of the relation $k_1 + k_2 = p + \omega\tau$ between the arguments in (54) R becomes a function of ω. This leads to a dependence of the WF on the argument $\mathbf{n}$. It can be similarly verified that the condition (52) is satisfied also for the solutions with nonzero angular momentum and spin, obtained in Refs. 4 and 5.

We have thus shown how to obtain from (16) the condition on the Fock component, and have verified that the WF obtained in Refs. 4 and 5 with kernel (49) in the ladder approximation satisfy this condition. In a higher order, the kernel of Eq. (42), its solution, and the WF condition that follows from (16) should be obtained in approximations that make them mutually consistent. If some phenomenological kernel is used in (42), we must see to it that the corresponding 4-momentum operator $\hat{J}_{\mu\nu}$ ensures the correct commutation relations of the Poincaré group. In this way we arrive at the quantum-mechanical approach developed in Refs. 8–11, in which the number of particles was fixed from the very outset. If it is assumed within the framework of this approach that the kernel of Eq. (42) does not depend on $\mathbf{n}$, the operator $\hat{J}_{\mu\nu}$ in the WF is likewise independent of n, and we obtain the relativistic quantum-mechanical-approach variant developed in Refs. 10 and 11. This assumption, however, seems unrealistic to us since it is not confirmed even by the simplest models.

6. CONCLUSION

We have shown that allowance for Eq. (16) rids the determined the state vector of the ambiguity that arises when the angular momentum is not zero. Thus, the developed formalism of relativistic WF on a light front is closed and self-consistent. This circumstance, in conjunction with knowledge of the general properties of the WF (such as the character of the dependence of the WF on the variable $\mathbf{n}$ or the spin structure of the WF), investigated in Refs. 1–5, provides a solid base both for the calculation of the WF in dynamic models and for their phenomenological analysis, i.e., for finding them directly from the experimental data. Since the information on the interaction of the components in a bound system is most frequently scanty, the determination of the WF from an analysis of the experimental data would apparently produce more results so far. Of particular interest to researach in this direction are the prediction and observation of the qualitative consequences of the characteristic dependences of the relativistic WF (change of the parametrization and of the spin structure) due to the relativistic character of the motion of the components, and therefore weakly dependent on the details of the dynamics.

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[1]V. A. Karmanov, Zh. Eksp. Teor. Fiz. **71**, 399 (1976) [Sov. Phys. JETP **44**, 210 (1976)].
[2]V. A. Karmanov, *ibid.* **75**, 1187 (1978) [**48**, 598 (1978)].
[3]V. A. Karmanov, *ibid.* **76**, (1979) [**49**, 954 (1979)].
[4]V. A. Karmanov, Nucl. Phys. **B166**, 378 (1980).
[5]V. A. Karmanov, Nucl. Phys. **A352** (1981).
[6]A. O. Barut and R. Raczka, Theory of Group Representations and Applications, PWN, Warsaw, 1977 [Russ. transl. Nauka, Vol. 2, p. 9].
[7]S. Weinberg, Phys. Rev. **150**, 1313 (1966).
[8]S. N. Sokolov, Teor. Mat. Fiz. **18**, 56 (1974).
[9]H. Leutwyler and S. Stern, Ann. Phys. (N.Y.) **112**, 94 (1978).
[10]M. V. Terent'ev, Yad. Fiz. **24**, 207 (1976) [Sov. J. Nucl. Phys. **24**, 106 (1976)].
[11]B. L. G. Bakker, L. A. Kondratyuk, and M. V. Terent'ev, Nucl. Phys. **B158**, 497 (1979).

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