Hexagonal restructuring of the charged surface of liquid helium

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It is shown that the structure of the charged surface of helium is determined by the joint action of the flexural nonlinearity due to the finite deviation of the surface from horizontal, and the charge instability due to the onset of surface sections free of electric charge. In the limit of low average charge density, $\bar{n} \ll E_c$, and low supercriticality of the clamping field $|E - E_c| \ll E_c$ (E_c is the threshold field at which a plane surface becomes unstable), the sizes of the uncharged regions, the energy of the restructured surface, and the period of the structure are obtained as functions of n and $E - E_c$. It is shown that a hexagonal restructuring is favored over other types of stucture. The field interval in which a connective charge distribution exists is estimated.

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The investigation of the properties of a charged helium surface has drawn particular interest following the experiments of Leiderer and Wanner,[1] who have shown that an electric field pushes the electrons to the surface produces on the latter a structure with hexagonal symmetry. This phenomenon was theoretically predicted earlier by Gor'kov and Chernikova.[2]

It was noted in our earlier paper[3] that in the limit of low charge density the main nonlinearity of the charged surface is due to formation of sections free of charge. The system is then described by linear equations for the deformation of the surface $\xi(\mathbf{r})$ and the charge density $n(\mathbf{r})$ ($\mathbf{r}$ is the coordinate on the surface), and the nonlinearity of the problem stems from the condition that $n(\mathbf{r}) \geqslant 0$ be non-negative. The correct solution is determined by the minimum of the energy under the additional condition $n(\mathbf{r}) \geqslant 0$. In addition to a hexagonal structure, other equilibrium configurations are possible. In Ref. 4 we have calculated numerically a surface state such that the electrons are gathered in a single dimple. Dimples are investigated in experiment by optical methods,[1] and we therefore calculated in Ref. 4 also the intensity of the light refracted by the dimple.

In the present article we study the restructuring of a weakly charged helium surface with formation of a two-dimensional periodic structure (such a structure is analogous to some degree to a crystal made up of dimples). The direction of the evolution of the instability and the energy of the reconstructed surface, if the homogeneous distribution is not excessively distorted, are determined by the nonlinearity due to the finite amplitude of the flexure of the surface.[2] (We shall call this "flexural nonlinearity.") With increasing amplitude of the produced charge-density wave (CDW), charge-free sections are produced,[3] and what will be referred to below as charge nonlinearity comes into play. It is under the influence of these mechanism that the final restructured surface takes shape.

1. ONSET OF INSTABILITY OF THE HOMOGENEOUS STATE

The instability of the homogeneous state of a charged surface was investigated by Gor'kov and Chernikova[2] by analyzing the ripplon spectrum. Here we consider the instability onset by starting from energy considerations.

The field E and the charge density $n(\mathbf{r})$ are given throughout in $(\alpha\rho g)^{1/4}$ units, the coordinate $\mathbf{r}$ on the unperturbed surface and the vertical deflection of the surface $\xi(\mathbf{r})$ in $(\alpha/\rho g)^{1/2}$ units, and the energy in $\alpha^2/\rho g$ units, where α and ρ are the surface-tension coefficient and the density of the helium, and g is the free-fall acceleration. Detailed numerical estimates of the corresponding quantities were given earlier.[4]

The surface energy is given in the general case by

$$\mathscr{E} = \int [1+(\nabla\xi)^2]^{1/2} d^2r + \frac{1}{2}\int \xi^2 d^2r + E\int n\xi\, d^2r + \frac{1}{2}\int \frac{n(\mathbf{r})n(\mathbf{r}')}{|\mathbf{l}-\mathbf{l}'|} d^2r d^2r', \tag{1}$$

where $\mathbf{l} \equiv (\mathbf{r}, \xi)$ is the three-dimensional coordinate of a point on the surface.

At small deformations of a weakly charged surface, when $|\nabla\xi| \ll 1$ and $n(\mathbf{r}) \ll 1$, it suffices to retain in (1) only the terms quadratic in $\xi(\mathbf{r})$ and $n(\mathbf{r})$:

$$\mathscr{E} = \frac{1}{2}\int [(\nabla\xi)^2+\xi^2] d^2r + E\int n\xi d^2r + \frac{1}{2}\int \frac{n(\mathbf{r})n(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^2r d^2r'. \tag{2}$$

The homogeneous state is stable at field values such that the contribution to the energy (2) from the inhomogeneity of $\xi(r)$ and $n(r)$ is positive-definite. In the Fourier representation

$$n(\mathbf{r}) = \int n_{\mathbf{k}} e^{i\mathbf{k}\mathbf{r}} \frac{d^2k}{(2\pi)^2}, \quad \xi(\mathbf{r}) = \int \xi_{\mathbf{k}} e^{i\mathbf{k}\mathbf{r}} \frac{d^2k}{(2\pi)^2}$$

the energy (2) per unit area takes the form

$$\mathscr{E} = \frac{1}{4}\int \left[(k^2+1)|\xi_{\mathbf{k}}|^2 + 2E\,\mathrm{Re}(\xi_{\mathbf{k}}^* n_{\mathbf{k}}) + \frac{2\pi}{k}|n_{\mathbf{k}}|^2 \right] \frac{d^2k}{(2\pi)^2}, \tag{3}$$

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where we have left out the infinite term corresponding to the energy of a uniformly charge plane surface.

The functional (3) is positive-definite if the condition that the integrand quadratic form be positive, namely

$$E^2<2\pi(k+k^{-1}) \tag{4}$$

is satisfied at all k.

The minimum value of the right-hand side of the inequality is equal to 4π and is reached at $k=1$. The critical field value is therefore[2] $E_c=(4\pi)^{1/2}$. At $E<E_c$ the energy functional is positive-definite and the system is stable. At $E>E_c$ the stability is lost, since energywise favored perturbations of the homogeneous states appear. In the case $E=E_c$ the system is in an indifferent equilibrium with respect to perturbations with a unity wave vector.

To analyze the inhomogeneous state that results from the development of instability of such waves, account must be taken of the nonlinear interactions. In the next section we consider in detail the flexural nonlinearity.[2] In Secs. 3–7 we study the effect of the charge nonlinearity and show, in particular, that the result proved in Ref. 2 with only flexural nonlinearity taken into account, namely that a hexagonal structure is energywise favored, remains in force also under conditions when the charge nonlinearity is substantial. In Sec. 8 we investigate a hexagonal structure made up of widely spaced individual dimples.

2. FLEXURAL NONLINEARITY

As shown in Ref. 2, at the instability threshold the surface energy decreases upon appearance, against the background of a homogeneous charge distribution $n(\mathbf{r})=\bar{n}$, of a CDW with hexagonal symmetry of the form

$$\tilde{n}(\mathbf{r})=\tilde{n}[\cos x+\cos(x/2+3^{1/2}y/2)+\cos(x/2-3^{1/2}y/2)],\ \xi(\mathbf{r})={}^1/_2E\tilde{n}(\mathbf{r}). \tag{5}$$

Calculation of the energy (1) with terms of fourth order in $\bar{n}$ and $\tilde{n}$ inclusive leads to the following for the energy of the CDW (7) per unit surface

$$\mathscr{E}={}^3/_8(E_c^2-E^2)\tilde{n}^2-{}^3/_2\pi^2\bar{n}\tilde{n}^3+{}^3/_8\pi^2\gamma\tilde{n}^4;\ \gamma={}^{63}/_8-4\sqrt{3}\approx 0.947, \tag{6}$$

where $E_c=[4\pi-(2\pi\bar{n})^2]^{1/2}$ is the critical field for the onset of instability at a finite charge density n.

The relatively simple expression (6) permits a complete investigation of the effect of the flexural instability on the stability and the structure of the surface near the threshold, when $(E-E_c)\sim\bar{n}^2$.

It is seen from (6) that $\tilde{n}=0$ corresponds to a local minimum of the energy so long as $E<E_c$. At $E=E_c$, a CDW of finite amplitude appears jumpwise:

$$\tilde{n}=3\bar{n}/\gamma\approx 3.16\ \bar{n}. \tag{7}$$

The cubic term in the energy (6) leads not only to the amplitude jump (7), but also to hysteresis effects.[5] Thus, the condition for the extremum of the energy (6) is of the form

$$\gamma\tilde{n}^2-3\bar{n}\tilde{n}=\frac{2\Delta}{\pi},\quad \Delta\equiv E^2/E_c^2-1. \tag{8}$$

It follows from this that

$$\tilde{n}=[3\bar{n}+(9\bar{n}^2+8\gamma\Delta/\pi)^{1/2}]/2\gamma. \tag{9}$$

Substitution of (9) in (6) shows that the formation of a CDW is energywise favored in fields $E>E_c-5.58\bar{n}^2$. In the field range $E_c-6.62\bar{n}^2<E<C_c-5.58\bar{n}^2$ a CDW of finite amplitude can exist as a metastable surface state. At the metastability boundary we have $\tilde{n}=1.58\bar{n}$. These results are illustrated by the $\mathscr{E}(\tilde{n})$ curves for different E in Fig. 1.

We note that our results agree with those of Gor'kov and Chernikova[2,5] but differ from the results of Ikezi.[6]

3. CHARGE NONLINEARITY

We turn now to the charge nonlinearity due to the presence of regions with zero charge density (ZCR). The alternating part of the charge density (5) lies in the range $(-3\tilde{n}/2, 3\tilde{n})$. If the amplitude of the CDW is in the range $(-\tilde{n}/3, 2\tilde{n}/3)$, the charge density $n(\mathbf{r})=\bar{n}+\tilde{n}(\mathbf{r})$ is everywhere positive and there are no ZCR. They appear when $\tilde{n}$ reaches the indicated limits. Thus, at $E=E_c$ the flexural instability strives to increase $\tilde{n}$ to a value $3\bar{n}/\gamma$, but at the value $\tilde{n}=2n/3$, which is approximately five times smaller, the charge-nonlinearity mechanism goes into operation. In Fig. 1, the region of action of the charge nonlinearity is located to the right of the dashed straight line. It is seen that all the hysteresis effects due to the flexural nonlinearity lie in this region. Consideration of these effects calls therefore for simultaneous allowance for both nonlinearity mechanisms.

It is important in what follows that the flexural nonlinearity is small in terms of the parameter $\bar{n}^2$. At the same time, so long as the size of the ZCR is small, the charge nonlinearity is small but contains no other small parameters. It must therefore be concluded that at the instability threshold ($\Delta=0$) the development of the CDW should terminate at $\tilde{n}\approx 2\bar{n}/3$ as a result of formation of ZCR with dimensions that are small to the extent that $\bar{n}^2$ is small. In the limit as $\bar{n}\to 0$ the value of $\tilde{n}$ at the threshold is exactly $2\bar{n}/3$ and the charge density takes the form

$$n(\mathbf{r})=\bar{n}\left\{1+\frac{2}{3}[\cos x+\cos(x/2+3^{1/2}y/2)+\cos(x/2-3^{1/2}y/2)]\right\}. \tag{10}$$

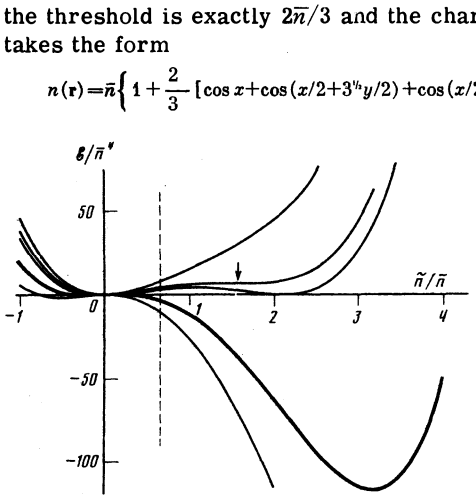

FIG. 1. Relative energy $\mathscr{E}/\bar{n}^4$ [see (6)] as a function of the relative CDW amplitude $\tilde{n}/\bar{n}$ for different $\varepsilon=(E-E_c)/\bar{n}^2$ without allowance for the restriction $n(\mathbf{r})\geq 0$. Reading upward: $\varepsilon=5$, $\varepsilon=0$ (thick line), $\varepsilon=-\pi^{3/2}\gamma^{-1}\approx -5.58$ (at the energy minimum $\mathscr{E}=0$); $\varepsilon=-9\pi^{3/2}/8\gamma\approx -6.62$ (the minimum vanishes, the arrow indicates the inflection point at $\tilde{n}/\bar{n}=3/2\gamma\approx 1.58$); $\varepsilon=-10$. To the right of the dashed line $\tilde{n}/\bar{n}=2/3$ the condition $n(\mathbf{r})\geq 0$ is violated.

The ZCR is reduced then to a honeycomb-like array of points with coordinates

$$x=4\pi(n+m/2\pm{}^1/_3),\ y=2\pi m/\sqrt{3}, \tag{11}$$

where n and m are integers.

We write down now the equations that determine the equilibrium structure of the surface at $\Delta > 0$ in the limit as $\bar{n} \to 0$, when the flexural instability can be neglected compared with the charge instability. This limit corresponds to the quadratic energy functional (2), which must be minimized with account taken of the additional condition that the charge is constant on the surface:

$$\int n(\mathbf{r})\,d^2r = S\bar{n}, \tag{12}$$

where S is the surface energy and $\bar{n}$ is the average charge density. Multiplying the left-hand side of (12) by the Lagrange multiplier λ, adding the result to (2), and varying the resultant expression with respect to ξ and n, we obtain the equations

$$\Delta\xi - \xi = E(n-\bar{n}), \tag{13a}$$

$$\int \frac{n(\mathbf{r}')-\bar{n}}{|\mathbf{r}-\mathbf{r}'|}\,d^2r' + E\xi(\mathbf{r}) = \lambda, \quad n(\mathbf{r})>0. \tag{13b}$$

Obviously, λ is the electrostatic potential in the region occupied by the electrons, reckoned from the potential of the unperturbed surface, while Eq. (13b) holds only for r such that $n(\mathbf{r}) > 0$.[3] The system (13) is so written that the volume of the liquid be constant ($\xi = 0$), and the average electrostatic potential of the surface is zero.

For small deviations of $n(\mathbf{r})$ from $\bar{n}$, so long as $n(\mathbf{r}) > 0$ at all $\mathbf{r}$, integration of the left-hand side of (13b) with respect to $\mathbf{r}$ yields zero, from which it follows that $\lambda = 0$. Under this condition, the system (13) is valid on the entire surface and is linear. Nonlinearity of (13b) on account of $\lambda \neq 0$ sets in only in the presence of ZCR. The ZCR configuration is determined in this case from two nonlinear conditions: positiveness of $n(\mathbf{r})$ in the charged region, and absence of divergence of $n(\mathbf{r})$ at the ZCR boundary.[4] The potential λ is determined from condition (12).

4. SIZES OF UNCHARGED REGIONS

The generalized forces acting on a CDW of the type (5) is given by the derivative

$$\frac{\partial \mathscr{E}(\tilde{n})}{\partial \tilde{n}} = -3\pi\left[\Delta + \frac{3}{2}\pi\tilde{n}\tilde{n} - \frac{\pi}{2}\gamma\tilde{n}^2\right]\tilde{n}. \tag{14}$$

The effective transcriticality that determines the size of the ZCR is equal to the expression in the square brackets of (14), taken at $\tilde{n} = 2\bar{n}/3$, namely,

$$\delta = \Delta + \pi(1-2\gamma/9)\tilde{n}^2 \approx \Delta + 2.48\tilde{n}^2. \tag{15}$$

At a finite charge density $\bar{n}$, a distribution of the form (10) with the ZCR degenerating into points takes place at $\delta = 0$. Near each of the points (11) at which $n(\mathbf{r})$ vanishes we have

$$n(\mathbf{r}) = \tilde{n}\rho^2/4, \quad \rho \ll 1, \tag{16}$$

where ρ is the distance from the indicated point.

At finite but small dimensions of the ZCR, expression (16) remains valid at ρ that are large compared with the size of the ZCR, but small compared with unity. It follows therefore that in the principal approximation in $\delta \ll 1$ it can be assumed that the ZCR are circles of radius $R \ll 1$ with centers at the points (11), and $n(\mathbf{r})$ at $R \ll \rho \ll 1$ is given as before by expression (16), which must in this case be used as the boundary condition. A schematic arrangement of the ZCR is shown in Fig. 2.

We describe now an approach that makes calculation of $R(\delta)$ possible. We use for this purpose the system (13), extending Eq. (13b) over the entire surface by replacing λ with $\lambda + \varphi(\mathbf{r})$, where the electrostatic potential $\varphi(\mathbf{r})$ differs from zero only in the ZCR. Of course, for points of the ZCR the relation (5) is no longer an equation but a definition of the function $\varphi(\mathbf{r})$. We shall nevertheless regard $\varphi(\mathbf{r})$ as an independent function, and find its connection with $n(\mathbf{r})$ later.

We change to a Fourier representation in accord with the relation

$$F(\mathbf{r}) = \sum_{\mathbf{k}\neq 0} F_{\mathbf{k}} \cos(\mathbf{kr}),$$

where the summation is over a triangular reciprocal lattice with wave vectors $\mathbf{k} \equiv (k_x, k_y) = (m + \frac{1}{2}n, 3^{1/2}n/2)$ (m and n are integers). Out of each pair of vectors, which are equivalent with respect to the transformation $\mathbf{k} \to -\mathbf{k}$, we choose only one.

The unit-cell area (see Fig. 2) is equal to $S_e = 8\pi^2/\sqrt{3}$ (this area subtends one charge-density maximum and two ZCR). Calculation of the Fourier components of the function $\varphi(\mathbf{r})$ in the limit $R \ll 1$ yields

$$\varphi_{\mathbf{k}} = \frac{8\pi}{S_e}\cos(\mathbf{kr}_0)\int_0^R \varphi(\rho) J_0(k\rho)\rho\,d\rho, \tag{17}$$

where J_0 is a Bessel function and $\mathbf{r}_0$ is the coordinate of any of the points of the lattice (11).

The system (13) reduces in the Fourier representation, after elimination of ξ, to a single equation

$$\left(\frac{2\pi}{k} - \frac{E^2}{k^2+1}\right) n_{\mathbf{k}} = \varphi_{\mathbf{k}}. \tag{18}$$

For our purposes it suffices to use (18) at $|\mathbf{k}| = 1$, when the coefficient of $n_k = \tilde{n}$ in the left-hand side of (18) is $-2\pi\Delta$, i.e., small at $|\Delta| \ll 1$. As shown earlier, in the principal approximation in Δ we have $\tilde{n} = 2\bar{n}/3$. For the chosen form of the CDW, $\cos(\mathbf{k}\cdot\mathbf{r}_0)$ is equal to $-1/2$, and $J_0(k\rho)$ under the integral sign in (18) must be replaced by unity, since $R \ll 1$. We take immediate ac-

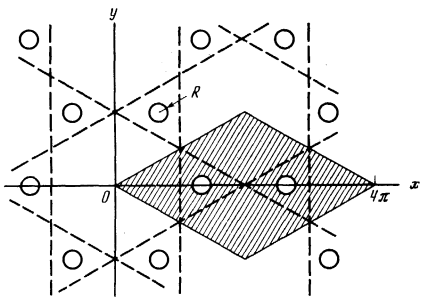

FIG. 2. Schematic representation of the ZCR at low transcriticality $\delta \ll 1$ (circles) and at $\delta = \delta_f$ (triangles). The increase of the period at $\delta > 0$ is not shown. The dashed lines delineate the unit cell.

count of the flexural nonlinearity by replacing Δ with the effective transcriticality δ, defined in accord with (15). In the upshot we get

$$\frac{4}{3}\pi\bar{n}\delta=\frac{\sqrt{3}}{2\pi}\int_0^R\varphi(\rho)\rho d\rho. \tag{19}$$

Substituting here the value of $\varphi(r)$ calculated for the given R, we obtain upon integration the sought connection between δ and R.

We propose now, and shall verify later, that $\varphi(r)$ is determined by the solution of the electrostatic problem for a conducting plane with a hold of radius R, and with the boundary condition that the charge density far from the hole is given by (16). In other words, the integral equation (13b) will be solved neglecting the term $E\xi$, after which it will be shown that the contribution of $E\xi$ to $\varphi(r)$ is small in terms of the parameter $R\ll 1$.

In practice it is convenient to solve not the integral equation (5) at $E\xi=0$, but an equivalent Laplace equation with corresponding boundary conditions.

To solve the electrostatic problem, we make the substitution $(\rho,z)\to(R\rho,Rz)$, so that the boundary of the ZCR corresponds to $\rho=1$ and $z=0$, and change over to oblate ellipsoidal coordinates (σ,τ) in accord with the relations

$$\sigma^2=[(r^2-1)^2/4+z^2]^{1/2}+(r^2-1)/2,$$
$$\tau^2=[(r^2-1)^2/4+z^2]^{1/2}-(r^2-1)/2,\qquad r^2=\rho^2+z^2.$$

The Laplace equation takes then the form

$$\left[\frac{\partial}{\partial\sigma}(1+\sigma^2)\frac{\partial}{\partial\sigma}+\frac{\partial}{\partial\tau}(1-\tau^2)\frac{\partial}{\partial\tau}\right]\Phi=0. \tag{20}$$

Near the plane $z=0$, the coordinates (σ,τ) are given by the expressions

$$\sigma\approx z(1-\rho^2)^{-1/2},\quad \tau\approx(1-\rho^2)^{1/2};\quad \rho<1,\ |z|\ll 1,$$
$$\sigma\approx(\rho^2-1)^{1/2},\quad \tau\approx z(\rho^2-1)^{-1/2};\quad \rho>1,\ |z|\ll 1.$$

From the equipotentiality condition $\Phi=$ const at $z=0$ and $\rho>1$, and from the zero-charge condition $\partial\Phi/\partial z=0$ at $z=0$ and $\rho<1$, we must choose solutions of (20) in the form

$$\Phi_n=P_{2n+1}(\tau)Q_{2n+1}(i\sigma),$$

where P_{2n+1} is a Legendre polynomial and Q_{2n+1} is a Legendre function of the second kind. The solution we need is a linear combination of the following functions:

$$\Phi_0=\tau[\sigma\operatorname{arctg}\sigma+1],$$
$$\Phi_1=(\tau^3-3\tau/5)[(\sigma^3+3\sigma/5)\operatorname{arctg}\sigma-\sigma^2+{}^4/_{15}].$$

For the potential and for the charge density at $z=0$ this yields

$$\Phi_0(\rho)=(1-\rho^2)^{1/2},\quad \Phi_1(\rho)=\frac{4}{15}\left[(1-\rho^2)^{1/2}-\frac{3}{5}(1-\rho^2)^{1/2}\right];\quad \rho<1,$$
$$n_0(\rho)=\frac{1}{2\pi}[(\rho^2-1)^{-1/2}+\arccos\rho^{-1}], \tag{21}$$
$$n_1(\rho)=\frac{3}{10\pi}\left[\left(\rho^2-\frac{2}{5}\right)\arccos\rho^{-1}+(\rho^2-1)^{1/2}+\frac{4}{15}(\rho^2-1)^{-1/2}\right];\quad \rho>1.$$

In the equilibrium situation, there should be no singularity of $n(\rho)$ on the ZCR boundary, i.e., as $\rho\to 1$ (Ref. 4). In conjunction with condition (16) this allows us to determine the correct linear combination of $n_0(\rho)$ and $n_1(\rho)$. For the potential we obtain ultimately

$$\varphi(\rho)=\frac{4\bar{n}}{9}(R^2-\rho^2)^{1/2}, \tag{22}$$

where we have returned to the previous unit of length. Substitution of (22) in (19) yields

$$R^5=10\pi^2\cdot 3^{1/2}\delta\approx 171\delta,\qquad R\approx 2.80\delta^{1/5}. \tag{23}$$

The appearance of a large numerical factor seems to indicate that the region in which the expansion of R in powers of δ is applicable is narrow.

We estimate now the correction that must be introduced in $\varphi(r)$, to account for the term $E\xi$ in (13b). Immediately at the threshold, the entire surface has a constant zero potential, i.e., $E\xi$ is exactly cancelled out by the potential produced by the surface charge. A contribution to $\varphi(r)$ is made therefore only by the change of ξ on account of the onset of ZCR of finite dimensions. The change of the charge density upon formation of the ZCR is of the order of $\bar{n}R^2$ and is concentrated in a region $\sim R$. Equation (13a) relates $\partial^2\xi/\partial\rho^2$ with $n(r)$, so that the change of ξ over distances of the order R under the influence of the change of the charge density is of the order of $\bar{n}R^4$. On the other hand, according to (21) we have $\varphi\sim\bar{n}R^3$, therefore the contribution of $E\xi$ to the ZCR electrostatic potential has a relative smallness $\sim R\ll 1$, thereby justifying the neglect of the term $E\xi$ in the solution of Eq. (13b).

To conclude this section, we present the δ-dependence of the quantity $\varphi(0)$ that characterizes the inhomogeneity of the electrostatic potential over the surface, and of the potential λ in the charged region. These dependences follow directly from expression (23) for $R(\delta)$ and from the condition that the average potential on the surface be zero:

$$\varphi(0)=\frac{4}{9}(10\pi^2\sqrt{3})^{3/5}\bar{n}\delta^{3/5}\approx 9.7\bar{n}\delta^{3/5},$$

$$\lambda=\frac{4\pi}{S_e}\int_0^R\varphi(\rho)\rho d\rho=\frac{4}{3}\pi\bar{n}\delta.$$

5. ENERGY OF RESTRUCTURED SURFACE

We begin with the calculation of the amplitude of the principal harmonic of $\tilde{n}$, corresponding to $|\mathbf{k}|=1$, on the effective transcriticality δ. To this end we express the condition $n(\mathbf{r}_0)=0$ in terms of the Fourier components $n_{\mathbf{k}}$. It is important that the contributions from $\tilde{n}$ and from the harmonics with $k\sim R^{-1}\gg 1$ must be taken into account separately. For the latter we obtain from Eq. (18), neglecting the terms $\sim k^{-1}\sim R\ll 1$, that $n_{\mathbf{k}}=k\varphi_{\mathbf{k}}/2\pi$. Calculations of $\varphi_{\mathbf{k}}$ by substituting (22) in (17) yields

$$\varphi_{\mathbf{k}}=(8/3\pi)^{1/2}\cos(\mathbf{kr}_0)R^3J_{3/2}(kR)(kR)^{-3/2}.$$

The contribution of $\varphi_{\mathbf{k}}$ to the charge density $n(r_0)$ is given by

$$\sum_{\mathbf{k}}\cos(\mathbf{kr}_0)k\varphi_{\mathbf{k}}/2\pi\approx\bar{n}R^2/6, \tag{24}$$

where account is taken in the summation of the fact that the mean value of $\cos^2(\mathbf{k}\cdot\mathbf{r}_0)$ at the lattice points $\mathbf{k}$ is equal to $1/2$, and on going from summation to integration we introduced the factor $1/\sqrt{3}$ because the summation covers half the $\mathbf{k}$ plane, while the area per reciprocal-lattice point is $\sqrt{3}/2$.

For the fundamental harmonic we have $\cos(\mathbf{k}\cdot\mathbf{r}_0) = -1/2$, and the number of equivalent waves is three. With allowance for (24) we obtain then

$$n(\mathbf{r}_0)=\bar{n}-3\tilde{n}/2+\bar{n}R^2/6=0,$$

whence

$$\tilde{n}={}^2/_3\bar{n}(1+R^2/6). \tag{25}$$

This result defines the $\tilde{n}(\delta)$ dependence in terms of the previously obtained $R(\delta)$ dependence (23).

These results are sufficient for the calculation of two terms in the dependence of the energy on δ. It is necessary to find separately the contributions of the fundamental harmonic and of the harmonics with $k \sim R^{-1} \gg 1$. The contribution of the fundamental harmonic is obtained by substituting (25) in (6). The constant part $\tilde{n} = 2\bar{n}/3$ makes in this case the main contribution to the lowering of the energy:

$$-\left[\frac{2\pi}{3}\Delta+\frac{4\pi^2}{9}(1-\gamma/6)\bar{n}^2\right]\bar{n}^2. \tag{26}$$

The change of $\tilde{n}(\propto R^2(\delta))$ makes according to (25) the contribution

$$-\frac{2\pi}{9}\delta R^2(\delta)\bar{n}^2=-\frac{R^7}{45\sqrt{3}\,\pi}\bar{n}^2. \tag{27}$$

The contribution of the higher harmonics is given by

$$\frac{1}{4}\sum_{\mathbf{k}}\frac{2\pi}{k}n_{\mathbf{k}}{}^2=\frac{1}{8\pi}\sum_{\mathbf{k}}k\varphi_{\mathbf{k}}{}^2=\frac{2R^7}{315\pi\sqrt{3}}. \tag{28}$$

We now estimate the order of magnitude of these contributions to the surface energy. The principal term (26) is of the order $\delta\bar{n}^2$ [it contains Δ and $\bar{n}^2$ in a combination different than δ (15), but of the same order]. The additional lowering of the energy (27), due to the increase of $\tilde{n}$ at $\delta > 0$, is of the order of $\delta^{7/5}\bar{n}^2$ and is partially offset by the increase of the Coulomb energy (28) when the ZCR is produced. Adding the contributions (16)–(28), we obtain for the energy per unit area

$$\begin{aligned}\mathscr{E}&=-\left[\frac{2\pi}{3}\delta-\frac{\pi}{3}(1-\gamma/3)\bar{n}^2+\frac{1}{63\pi\sqrt{3}}R^7(\delta)\right]\bar{n}^2\\&\approx-(2.09\delta-1.50\bar{n}^2+3.91\delta^{7/5})\bar{n}^2.\end{aligned} \tag{29}$$

Expression (29) allows us to investigate hysteresis phenomena in the near-threshold region. When the field reaches from below a critical value $E_c(\Delta = 0)$, the system becomes unstable, the CDW appears abruptly with $\tilde{n} \approx 2\bar{n}/3$, and the energy becomes negative

$$\mathscr{E}_{\Delta=0}\approx-(4.47+13.8\bar{n}^{4/5})\bar{n}^4$$

[according to (15) it is necessary to substitute in (27) $\delta = \pi(1-2\gamma/9)\bar{n}^2$]. When the field is now decreased, the surface remains restructured down to $\Delta = -\pi(1-2\gamma/9)\bar{n}^2$. At this point $\delta = 0$ and the local minimum of the function $\mathscr{E}(\tilde{n})$ is lost. In the region $0 < \delta < (0.344 - 0.875\bar{n}^{4/5})\bar{n}^2$ the energy is positive, i.e., the surface is in a metastable state. The energy maximum is

$$\mathscr{E}_{\delta=0}\approx 0.717\bar{n}^4.$$

When the field decreases beyond the point $\delta = 0$, the surface becomes plane jumpwise, and the charge distribution becomes homogeneous. The relative value of the hysteresis is

$$(E_c-E_{\delta=0})/E_c\approx 1.24\bar{n}^2. \tag{30}$$

We shall show now that the nontrivial contribution to the energy (29), of the order of $\delta^{7/5}$, can be obtained also by another method. In analogy with the approach developed for the solution of the problem of the multielectron dimple, we consider the energy $\mathscr{E}(\delta, R)$ for the situation when the ZCR is maintained artificially at some value of R that is generally speaking not equal to $R(\delta)$. Just as in Ref. 4, the point $R = R(\delta)$ is special because it is there that the singularity of $n(\mathbf{r})$ as $\rho \to R + 0$ vanishes, as do the derivatives $\partial\mathscr{E}/\partial R$ and $\partial^2\mathscr{E}/\partial R^2$. Assume that $\mathscr{E}(\delta, R)$ is a quadratic trinomial of the form

$$\mathscr{E}(\delta, R)=\bar{n}^2R^7(A+B\delta/R^5+C\delta^2/R^{10}).$$

From the condition that the first and second derivatives of this expression vanish at $R = R(\delta)$, we can express B and C in terms A. Calculation of A is then sufficient to reconstruct the function $\mathscr{E}(\delta, R)$.

We calculate first $A = \mathscr{E}(0, R)$. In this case the left-hand side of (19) is zero, and to reduce the right-hand side to zero we must add to the potential (21) a potential of the form $(R^2 - \rho^2)^{1/3}$ multiplied by an appropriate coefficient [see (21)]. As a result we obtain

$$\varphi(\rho)={}^4/_9\bar{n}[(R^2-\rho^2)^{3/2}-{}^3/_5R^2(R^2-\rho^2)^{1/2}],$$

or, taking the Fourier transform,

$$\varphi_{\mathbf{k}}=(8/3\pi)^{1/2}\cos(\mathbf{kr}_0)R^5[J_{5/2}(kR)/(kR)^{5/2}-{}^1/_3J_{3/2}(kR)/(kR)^{3/2}].$$

The energy is calculated as in the case of (28), and the result is

$$\mathscr{E}(0, R)=\bar{n}^2R^7/525\pi\sqrt{3}.$$

The total function $\mathscr{E}(\delta, R)$ is thus of the form

$$\mathscr{E}(\delta,R)=\frac{\bar{n}^2R^7(\delta)}{525\pi\sqrt{3}}\left(\tilde{R}^7-7\tilde{R}^2-\frac{7}{3}\tilde{R}^{-3}\right),\quad \tilde{R}\equiv\frac{R}{R(\delta)}.$$

It is easy to verify that the first and second derivatives of this expression with respect to R vanish at $\tilde{R} = 1$, and the expression itself coincides at $\tilde{R} = 1$ with the third term of (29). The last agreement was not affected by adjusting the coefficients and should be regarded as proof of the correctness of the function $\mathscr{E}(\delta, R)$ chosen by us.

6. OTHER TYPES OF SURFACE RESTRUCTURING

We have investigated above in detail the structure of a surface with hexagonal symmetry. Other structures are also possible in principle. We shall calculate their energies accurate to terms $\Delta\bar{n}^2$ and $\bar{n}^4$, and show that the energywise favoring of the hexagonal structure, proved earlier[2] with only the flexural nonlinearity taken into account, is preserved when the flexural and charge nonlinearities are taken into account simultaneously. As shown in Sec. 5, to calculate at the indicated accuracy the decrease of the surface energy upon restructuring, it suffices to use the charge density corresponding to a ZCR of zero size.

For a one-dimensional structure we have

$$n(\mathbf{r})=\bar{n}(1+\cos x),\ \mathscr{E}=-({}^1/_2\pi\Delta-{}^5/_{64}\pi^2\bar{n}^2)\bar{n}^2\approx-(1.57\Delta-0.771\bar{n}^2)\bar{n}^2.$$

In the case of quadratic symmetry

$$n(\mathbf{r})=\bar{n}\left[1+\frac{1}{2}(\cos x+\cos y)\right],$$

$$\mathscr{E}=-[{}^{1}/_{4}\pi\Delta-\pi^{2}({}^{51}/_{32}-\sqrt{2})\bar{n}^{2}]\bar{n}^{2}\approx-(0.785\Delta-1.77\bar{n}^{2})\bar{n}^{2}.$$

In addition to the already considered hexagonal structure, a structure of the type (5) with $\tilde{n}=-\bar{n}/3$ is possible; this corresponds to an interchange of the maxima and minima of $n(r)$. For such a structure

$$n(\mathbf{r})=\bar{n}\{1-{}^{1}/_{3}[\cos x+\cos(x/2+3^{1/2}y/2)+\cos(x/2-3^{1/2}y/2)]\},$$

$$\mathscr{E}=-\left[\frac{\pi}{6}\Delta-\left(1+\frac{\gamma}{12}\right)\frac{\pi^{2}}{18}\bar{n}^{2}\right]\bar{n}^{2}\approx-(0.524\Delta-0.592\bar{n}^{2})\bar{n}^{2}.$$

The energy of the hexagonal structure considered in the preceding section of the article is, according to (26),

$$\mathscr{E}=-[{}^{2}/_{3}\pi\Delta+{}^{4}/_{9}\pi^{2}(1-{}^{1}/_{6}\gamma)\bar{n}^{2}]\bar{n}^{2}\approx-(2.09\Delta+3.68\bar{n}^{2})\bar{n}^{2}$$

and lies lower than the energies of the other structures. Thus, the honeycomb arrangement of the ZCR is energy favored both at $\Delta\lesssim\bar{n}^{2}$, when simultaneous account must be taken of the flexural and charge nonlinearities, and at $\Delta\gg\bar{n}^{2}$, when the influence of the flexural nonlinearity can be neglected.

7. PERIOD OF STRUCTURE AND CONNECTIVITY OF CHARGE DISTRIBUTION

We have shown earlier[3] that the connectivity of the charge distribution at small transcriticalities makes it possible for the period of the structure to change with the field, thereby ensuring an energy minimum. To find this dependence at small Δ and $\bar{n}$ we must minimize, with respect to the wave vector k of a CDW of type (5), the principal term in the expression for the energy, a term equal to

$$\mathscr{E}(k)=\frac{3}{4}\left(\frac{2\pi}{k}-\frac{E^{2}}{k^{2}+1-2\pi\bar{n}^{2}}\right)\tilde{n}^{2}-6\pi^{2}\frac{k}{(k^{2}+1)^{2}}\bar{n}\tilde{n}^{3}+\frac{3}{2}\pi^{2}\left[(9-4\sqrt{3})\frac{k}{(k^{2}+1)^{2}}-\frac{9}{2}\frac{k^{4}}{(k^{2}+1)^{4}}\right]\tilde{n}^{4} \qquad (31)$$

[this expression goes over into (6) at $k=1$]. The period a of the structure is connected with the period k of the reciprocal lattice by the relation $ak=4\pi/\sqrt{3}$. Minimization of (31) with respect to k with substitution of the threshold value $\tilde{n}=2\bar{n}/3$ yields for k the expansion

$$k=\frac{4\pi}{\sqrt{3}}a^{-1}=1-\Delta-\frac{\pi}{3}\left(\frac{4}{\sqrt{3}}-1\right)\bar{n}^{2}\approx1-\Delta-1.37\bar{n}^{2}. \qquad (32)$$

Here, just as in (26), Δ and n^{2} enter in a combination different from δ [see (15)]. Expression (31) shows that the period of the structure increases with increasing field and with increasing average charge density, and the value of the period at the instability threshold is determined by $\bar{n}$. We note that there is no need to take into account the difference between k and unity when the size of the ZCR and the energy are calculated, since this difference leads to terms of order higher than accounted for here.

We attempt now to estimate the range of the clamping fields and charge densities in which the charge distribution remains connective, i.e., to find the values of E and $\bar{n}$ at which the ZCR are joined and the period of the structure becomes fixed (see Ref. 3).

The instant of joining is determined in principle by the competition between the growth of the ZCR dimensions and the growth of the lattice constant a. At small Δ and $\bar{n}$ their values are given by the asymptotic relations (23) and (32), but at the instant of joining the shape of the ZCR is certainly not circular, so that the use of the asymptotic form (23) is difficult.

We shall estimate the region of connectivity of the charge distribution in the following manner. We note that if we discard the higher harmonics in the charge density $n(\mathbf{r})$, leaving only the CDW (5), we get regions with negative charge density in those places where ZCR are located in the real charge distribution. At $\tilde{n}-2\bar{n}/3\ll\bar{n}$ these regions are circles of radius $R'^{2}=6(3\tilde{n}/2\bar{n}-1)$. Using for $\tilde{n}$ the expansion (25), we obtain $R'=R(\delta)$, i.e., in the limit as $\delta\to0$ the regions with negative charge density in the distribution $n'(\mathbf{r})=\bar{n}+\tilde{n}(\mathbf{r})$ coincide with the ZCR in the real distribution $n(\mathbf{r})$. Assuming that this correspondence is approximately preserved also at finite values of δ, when the shape of the ZCR differs from circular, we find that the joining of the ZCR and the fixing of the period of the structure take place at $\tilde{n}=\bar{n}$ [at $\tilde{n}/\bar{n}=1$ the joining regions $n'(\mathbf{r})<0$ have at that instant a triangular shape (see Fig. 2)]. This value of $\tilde{n}$ is reached, according to the expansion (25), at

$$\delta=\delta_{f}=9/10\pi^{2}\approx0.09. \qquad (33)$$

The region where a connective charge distribution exists can be characterized by two quantities: the field E_f at which the ZCR are joined in the limit as $\bar{n}\to0$ (this field was introduced in Ref. 3), and the density $\bar{n}_f$ at which the ZCR are joined directly at the threshold (at $E=E_c$). From (33) and (15) we have

$$E_{f}\approx1.045E_{c},\quad \bar{n}_{f}\approx0.19. \qquad (34)$$

This result, obtained by extrapolating the expansions in the small parameters Δ and $\bar{n}$, is only an estimate.[1)] It follows from it that the range of Δ and $\bar{n}$ in which the charge distribution is connective is perfectly noticeable, albeit narrow. In particular, by the instant of joining of the ZCR the wave vector k decreases by 10% at $E=E_f$ and $\bar{n}\to0$, and by 5% at $E=E_c$ and $\bar{n}=n_f$, compared with the value $k=1$ at $E=E_c$ and $\bar{n}\to0$. The corresponding increase of the period of the structure can be noted in experiment under conditions of a sufficiently large area of the experimental cell.

8. LIMIT OF WEAKLY INTERACTING DIMPLES

So far we have considered clamping fields that are close to the critical E_c, i.e., $\Delta\ll1$. We consider now, neglecting the flexural nonlinearity, the region of fields close to $E_0=4.0612=1.15E_c$ ($\Delta\approx0.31$), at which the existence of a single multielectron dimple becomes energywise favored.[4]

Let us investigate the field dependence of the charge distribution in a state corresponding to an absolute minimum of the functional (2). Such states will be called stable for brevity. Since the dimple energy is proportional to the square of its charge, at $E>E_0$ all the electrons present on the surface should be gathered into a dimple of finite radius. Assume that the transition to this state proceeds continuously as the field E increases to the value E_c (the calculation that follows will confirm this assumption). In the considered limit, the charge distribution should then constitute a widely

spaced (with period $a \gg 1$) triangular lattice of weakly interacting dimples whose charge equals $Q = \sqrt{3}na^2/4$ and increases with increase of the period.

The surface energy consists of the energies of the individual dimples in the clamping field and the dimple-interaction energy. According to our calculations,[4] the energy of a single dimple with charge Q, at a small deviation of the field from E_0, is linear in E and is equal to

$$0.4331Q^2(E_0-E).$$

In the calculation of the interaction energy of dimples with large distances between them the dimples can be regarded as point charges, and the interaction pure Coulomb. In a triangular lattice, the energy per charge, according to Ref. 8, is

$$-2.107Q^2/a.$$

The total surface energy per unit area is

$$\mathscr{E} = [0.4331(E_0-E) - 2.107/a]\sqrt{3}a^2\bar{n}^2/2.$$

Minimization with respect to a yields

$$\mathscr{E} = -2.22\bar{n}^2(E_0-E)^{-1}, \quad a = 2.43(E_0-E)^{-1}. \tag{35}$$

We see that the period of the structure indeed diverges as $E \to E_0$, i.e., the assumption that the transition to the single-dimple limit is continuous is confirmed.

The foregoing reasoning leads to two conclusions. First, the field region in which a stable periodic structure exists is bounded from below and above. The behavior of the energy and of the period of the structure near the boundaries E_c and E_0 of this region are described by Eqs. (20), (32), and (35), respectively.

Second, the divergence of the period of the structure at finite dimensions of the individual dimples[4] means that at a certain $E < E_0$ the connectivity of the charge distribution must break. Since a stable state of the system for which this statement is valid is indeed realized so long as the distribution is connective, i.e., up to the joining of the ZCR, we have for the field E_f corresponding to this instant the rigorous inequality

$$E_c < E_f < E_0.$$

We have by the same token obtained a proof that does not follow in principle from the rough estimate (34), that the field E_f indeed exists.

After breaking up the charge distribution into individual dimples, it becomes impossible to obtain an absolute minimum of the energy, and various metastable states, i.e., corresponding to local energy minima, should be realized.[2)] The surface configuration is determined in this case not only by the value of the field, but also by the prior history. It appears that as the field is gradually increased the structure undergoes a number of jumplike restructurings due to the coalescence of neighboring dimples. As a result, the period of the structure on the whole will increase, but in accordance with a law different from (35). When the field is decreased, the reverse takes place, but the successive jumplike breakups of the dimples will take place only in fields weaker than the value E_D at which a single dimple becomes unstable.

9. EXPERIMENTAL SITUATION

At the present time a hexagonal restructuring of the surface and individual dimples are reliably observed by optical methods,[7,9,10] and the interferograms of the surface make it possible to measure directly the shape of the restructured surface. We now discuss the possibility of observing the effects discussed in the present article.

The near-threshold hysteresis investigated in Sec. 5 has apparently already been observed by Leiderer and Ebner,[7] although the accuracy of the measurements is as yet insufficient for comparison with the calculation. At the ratio of the charge density to the saturation value $n/n_{max} = 0.08$, indicated in Ref. 7, the calculated hysteresis (30) amounts to 0.13%, as against 0.5% in Ref. 7. We note that measurement of the hysteresis calls for a very high homogeneity of the field, which has not been attained so far.

The much stronger temperature-dependent hysteresis observed in Ref. 9 is due apparently to some extraneous mechanism outside the scope of the model considered by us (Sec. 1) and stabilizes the surface structure at $T \gtrsim 3.7$ K. In particular, this mechanism makes viable the strip-like periodic structure that is certainly not favored energywise (see Sec. 6).

The growth of the ZCR with increasing field, and their shapes, are difficult to observe directly, since the refracting ability of the surface at the ZCR boundary is low.[4] An important result would be an experimental determination of the field E_f at which the ZCR are joined; this can be observed by determining the period and also, probably by determining the change of the spectrum of the long-wave plasma oscillations of the surface electrons and of the capillary waves because of the restriction of the region of oscillations of the electrons by the individual dimples.

It would be of interest to observe the hysteresis phenomena that occur in fields exceeding E_f and are due to the onset of instability of the structure with respect to joining of neighboring dimples as the field is gradually increased, and to the breakup of each dimple into several in the reverse process (see Sec. 8). Among the first and foremost problems is, in particular, the measurement of the field E_D, calculated in Ref. 4 and corresponding to the stability limit of a single dimple. This field exceeds E_e by 4.4% and can be readily measured.

We note that to observe these effects it is necessary to experiment with low charge densities, to prevent the dimples with the ever increasing charge from vanishing into the helium.

It is also possible to investigate in the same field range the behavior of the system when the field is changed abruptly, and also when the field is weakened for a brief instant and then returned to its previous value. In the latter case the electrons, which have high mobility at low temperatures, become capable of jump-

ing from dimple to dimple, but the shape of the surface will change little. When such a "connectivity" of the charge density sets in, it becomes possible to vary smoothly the period of the structure, in analogy with its variation in the region $E < E_f$.

In conclusion, we consider the general question of the applicability of the results of the theoretical analysis of an infinite charged surface in a homogeneous clamping field to experimental facilities of finite size. The condition that the distance between the surface and the field-producing electron be large compared with the period of the structure is usually satisfied and is furthermore of no importance in principle, since the interaction with the electrodes can be accounted for in the calculation.

Since some of the investigated phenomena take place at very small changes of the clamping field, the latter must satisfy very stringent intensity-homogeneity requirements. In experiments with low charge density the field must also be highly homogeneous in direction, otherwise the entire charge will be gathered in the region where the electron energy is a minimum.

Our last remark concerns the experimental observation of a smooth change of the period as a function of the field. In contrast to an infinite surface, the period can be changed in practice only by vanishing and appearance of dimples on the boundary of a charged region, and in the general case entails the surmounting of an energy barrier of the order of the dimple energy. This leads to a unique edge effects that makes measurement of the period difficult. Its influence does not weaken with increasing surface area, and depends only on the width and the character of the boundary region. A strong edge effect can make a smooth variation of the period utterly impossible.

[1)]It should be noted that the maximum attainable charge density on the surface is $\bar{n}_{max} = 1/\sqrt{2\pi} \approx 0.41$, so that when estimating $\bar{n}_f$ it's apparently necessary to take into account the flexural nonlinearity more rigorously. It may turn out in particular that there is no joining of ZCR directly at $E = E_c$ all the way to the value $\bar{n} = \bar{n}_f$ at which the charges vanish in the helium after the surface stability is lost (see Ref. 7).

[2)]The temperature effects that lead to electron exchange between dimples have been estimated in Ref. 4 to be negligibly small, so that metastable states are as viable as the stable ones.

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