Electrons on a thin superfluid helium film

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The effect of deformation phenomena on the spectrum of an electron localized on a thin helium film is investigated. It is shown that this effect can take place for both the transverse and the longitudinal part of the electron spectrum. The question of stability of the surface electronic states on thin helium films is discussed. It is noted that an effective interaction is possible between the electron system on the helium film surface and a system of thermal vortical excitations in the film. The presence of this interaction may exert, under certain conditions, a substantial influence on the structure of a single electron dimple on the helium film.

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INTRODUCTION

One of the promising objects for the investigation of the properties of two-dimensional electron systems is a charged thin film of helium. The term thin film means that the localization of the electrons above the helium is due mainly to their interaction with the hard substrate and not to the attraction to the liquid surface of the film.

The behavior of an individual electron over a thin film of helium is not a new problem. Various characteristics that determine the behavior of an individual electron over a thin film were determined earlier.[1,2] In particular, it was found that over a thin film an electron is practically always auto-localized, since the electrostatic interaction with the subtrate is equivalent to the action of a clamping electric field of high intensity, which leads to the appearance of electron dimples on the surface of the helium. As a result, the electron mobility over the film should be low enough, as is confirmed by experiments.[3]

Many interesting details in the behavior of electrons over a thin film of helium remain unexplained, however. Most needed are refinements of the available information concerning the spectrum of an individual electron over the surface of a thin film. These refinements make it possible to observe the influence of deformation phenomena on the discrete part of the electron spectrum, corresponding to motion of the electron along a normal to the film surface, to introduce the concept of the minimum film thickness at which realization of electronic surface states is possible, and others.

Second, a new trend is developing, connected with the study of the interaction between the electrons and vortex pairs of fluctuation origin in the films. Recent investigations have shown that the production of such pairs in thin films of helium is possible. At a temperature $T_\lambda < T_\lambda^0$, where T_λ^0 is the temperature of the transition to the superfluid state of bulk helium, the dissociation of the vortex pair in the film should lead, according to Kosterlitz and Thouless,[4] to the loss of superfluidity of the helium film, as is indeed observed in experiment.[5] Assuming the real existence of vortex pairs of fluctuation origin, account must be taken of their presence in their description of various effects of deformation origin in helium films. In fact, as noted in a preceding paper,[6] an individual vortex pair interacts with a deformation of the film surface, particularly with the deformation of the surface in the vicinity of a charged dimple. This interaction, which has the sign of attraction, causes the electron dimple produced when a free electron is placed on the helium-film surface to attract to itself a definite number of vortex pairs, thereby forming a complicated charged vortex complex. The presence of an excess vortex-pair density in the vicinity of the dimple leads to additional local pressure on the helium surface, something that must be taken into account when describing the structure of the charged dimple on the helium film. The same considerations bring about the necessity for taking into account the finite density of the vortex pairs in the oscillations of the free surface of the helium film even in the neutral state. Thus, by observing the behavior of the electrons over a film at finite temperatures, it is possible, in principle, to obtain information on the parameters of the fluctuation vortex pairs—a fact of sufficient interest by itself.

Third, the collective phenomena that take place in a bound electron-ripplon system on the surface of a thin helium film are of considerable interest. In this problem one can expect a substantial increase in the critical density of the electrons (compared with the critical density of the electron over bulk helium), a qualitative change in the structure of the collective-oscillation spectra, and others.

A discussion of the foregoing new questions and possibilities in the problem of the properties of electrons over a thin helium film is the subject of the present paper. Principal attention will be paid to a description of the properties of an individual electron on a helium film. Collective phenomena in an electro-ripplon system on a thin helium film will be investigated separately.

1. SPECTRUM OF SURFACE ELECTRONS OVER A THIN FILM

A. The problem of the spectrum of a surface electron on a thin helium film will be considered by us assuming that the electron is self-localized. The cause of the autolocalization and the onset of a self-consistent deformation $\xi(r)$ of the helium surface in the vicinity of the localized electron is the strong electrostatic interaction of the electron with the dielectric substrate. At

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film thickness $d \leqslant 10^{-6}$ cm this interaction is equivalent to the presence of a strong clamping field $E_\perp \approx 10^4$ V/cm.

The initial equation for the electron over the thin film is of the form

$$-\frac{\hbar^2}{2m}\Delta\psi - \frac{\Lambda}{z+d}\psi = E\psi,$$

$$\psi|_{z=\xi(r)}=0, \quad \Lambda = \frac{e^2(\varepsilon_d-1)}{4(\varepsilon_d+1)}, \quad r=|\mathbf{r}|. \tag{1}$$

The plane $z=0$ coincides with the free surface of the film, d is the thickness of the film far from the dimple, ε_d is the dielectric constant of the substrate, m is the electron mass, and E is its energy. The substitution

$$\tilde{z} = z - \xi(r)$$

makes it possible to remove the perturbation $\xi(r)$ from the boundary condition for ψ on the free surface of the film. As for the potential energy, when written in terms of $\tilde{z}$ and r it can be represented in the form

$$V(\tilde{z},r) = \frac{\Lambda}{\tilde{z}+d+\xi(r)} = \frac{\Lambda}{\tilde{z}+d+\xi(0)} + \left[\frac{\Lambda}{\tilde{z}+d+\xi(r)} - \frac{\Lambda}{\tilde{z}+d+\xi(0)}\right]. \tag{2}$$

Recognizing that the motion of the electron along the helium surface is slower than in the z-direction, we average the second term in (2) over the variable $\tilde{z}$, using for simplicity the following approximation for the z-component of the wave function of the surface electron:

$$\psi(z,r) = f(z)\varphi(r), \quad f(z) = 2\gamma^{3/2} z e^{-\gamma z},$$

where γ^{-1} is the scale of the electron localization over the film. As a result, the energy $V(\tilde{z}, r)$ can be represented in the form

$$V(\tilde{z},r) \approx \frac{\Lambda}{\tilde{z}+\tilde{d}} + \langle V(r)\rangle, \quad \langle V(r)\rangle = \left[\frac{\Lambda}{d+\xi(r)} - \frac{\Lambda}{d+\xi(0)}\right], \tag{2a}$$

$$\tfrac{2}{3}\gamma\tilde{d} > 1, \quad \gamma R > 1, \quad \tilde{d} = d + \xi(0). \tag{2b}$$

The inequalities (2b), which are needed to be able to represent $V(\tilde{z}, r)$ in the form (2a), can be verified after determining the parameters γ and R, where R is the radius of the electron localization in the film plane. Thus, e.g., using the definition γ from (4), it is easy to find that the inequality $\gamma\tilde{d} > 1$ reduces to the requirement

$$(\hbar^2/2m\Lambda\tilde{d})^{1/3} < 1,$$

which is satisfied up to thicknesses $d \gtrsim 10^{-7}$ cm.

Similar estimates of the satisfaction of the inequality $\gamma R > 1$, obtained by determining γ from (4) and R from (7) and (9b), also verify that the initial assumptions (2b) are reasonable. Thus, the approximation (2a) for the potential $V(\tilde{z}, r)$ is self-consistent in the sense of satisfaction of the required inequalities.

Using the approximate equation (2a) for $V(\tilde{z}, r)$, we can separate the variables $\psi(r, z) = f(z)\varphi(r)$ in the initial Schrödinger equation (1):

$$f'' + \left(-\frac{1}{4} + \frac{\varkappa_l}{z+d^*}\right) f = 0, \tag{3}$$

$$z = \frac{\tilde{z}}{2a_0\varkappa_l}, \quad \varkappa_l = \frac{e^2}{4\hbar}\left(\frac{m}{2|\Delta_l|}\right)^{1/2}, \quad \Delta_l = -\frac{e^2}{32a_0\varkappa_l},$$

$$f(z)|_{z=0}=0, \quad f(\infty)\to 0, \quad d^* = d/2a_0\varkappa_l, \quad a_0 = \hbar^2/me^2,$$

$$-\frac{\hbar^2}{2m}\Delta_r\varphi + \langle V(r)\rangle\varphi = \lambda\varphi, \quad \varphi|_{r\to\infty}\to 0, \quad \varphi'|_{r\to 0}\to 0, \tag{3a}$$

Δ_l and λ are the energy eigenvalues of Eqs. (3) and (3a).

B. The approximate spectrum and the wave functions of Eq. (3) can be obtained by various methods. Thus, it is noted in Refs. 1 and 2 that the inequality $d \gg \gamma^{-1}$ is satisfied in a wide range of parameters. For this reason, to find the energy ground level and the localization scale of the electron over the film in the ground state we can expand the energy $V(z) = \Lambda/(z+d)$ in powers of z/d and reduce the problem (3) to a solution of the Airy equation. As a result, the localization length γ^{-1} and the first levels Δ_l are given by

$$\gamma^{-1} = \left(\frac{\hbar^2}{2mF}\right)^{1/3}, \quad \Delta_l = -\frac{\Lambda}{d} + \xi_l\left(\frac{\hbar^2}{2m}\right)^{1/3} F^{2/3}, \tag{4}$$

$F = \Lambda/d^2$, ξ_l are the zeros of the Airy function. The inequality $d \gg \gamma^{-1}$ used to determine Δ_l from (4) is satisfied in this case up to $d \gtrsim 10^{-7}$ cm.

Equation (3) for $f_l(z)$ has also an exact solution in terms of Kummer functions. If we use in this case the boundary condition indicated in (3), that the wave function on the surface of the helium film vanish, and assume the substrate to be metallic, then the spectrum of Eq. (3) for $f_l(z)$ can be written in the form[1]

$$\Delta_l \approx \frac{e^2}{32a_0}\left[l - \frac{3}{4} + \frac{1}{\pi}\left(\frac{2\tilde{d}}{a_0}\right)^{1/2}\right]^{-2}, \tag{4a}$$

$$\tilde{d} = d + \xi(0), \quad d \gg a_0, \quad l \gg 1.$$

It is appropriate to emphasize that according to (4a) a self-consistent deformation of the film surface in the vicinity of the localized electron influences the spectrum of the surface electron in the z-direction to the extent that $\xi(0)/d \neq 0$. In addition, the definitions (4) and (4a) of the ground-state energy correlate with each other accurate to the number ~1.

We note now that the boundary condition that the wave function of the electron vanish on the film surface is in fact idealized for our problem. This condition is accurate to ~5% already in the case of an electron over bulk helium. As for an electron over a thin film, where the effective clamping force is much stronger, the probability that the electron will penetrate in the interior of the film becomes quite noticeable. It might seem that this circumstance should lead to a tunneling breakthrough and to the penetration of the electron into the film. This process is hindered, however, by the absence of suitable (i.e., having negative energy) states in the film (if the substrate is not metallic). The details of the solution of the corresponding problem will be made clear in the Appendix. Here we summarize only the results obtained for $\varkappa_l^{(2)}$ when solving Eq. (3) with different boundary conditions, and gathered in Table I. Commenting on this table, it can be noted that the analytic definition (4a) of Δ_l, which is valid in the region $d \gg a_0$, is accurate enough compared with the results obtained by the more exact numerical search for the zeros of the Kummer function $U(a, b, x)$ listed in the second column of the table. However, allowance for the penetration of the electron wave function into the helium film alters noticeably the values of $\varkappa_l^{(2)}$, especially in the thickness region $d \sim 10^{-7}$ cm, as is evident from the third column of the table. Finally, the fourth column demonstrates the influence exerted on the spectrum by the image forces and the liquid surface on the film. This effect was calculated by perturbation theory. As for the variant of the problem with a metallic substrate, no discrete levels of finite depth appear in this case, i.e., the electron cannot be held on the film.

C. Changing over to Eq. (3a), which must be solved with the equation of mechanical equilibrium for ξ, we note first of all that it was investigated earlier[1] for the limiting case $\xi(0)/d \ll 1$ (the region of applicability of the results of Ref. 1 will be defined more accurately below). The transition into the region $\xi(0)/d \lesssim 1$ leads not only to a certain quantitative change in the values determined in Ref. 1, but also to the appearance of qualitatively new effects. To verify the validity of the last statement, we write down the equation for the mechanical equilibrium of the electron on a thin helium film:

$$\begin{gathered}\alpha\Delta\xi = F(r)\varphi^2 - f/(d+\xi)^3 + f/d^3 = P(r),\\ F(r) = eE_\perp + \Lambda/(d+\xi(r))^2, \quad \xi(0)/d \ll 1.\end{gathered} \tag{5}$$

Here Δ is the two-dimensional Laplace operator; α is the surface-tension coefficient; d is the equilibrium thickness of the helium film far from the dimple; $F(r)$ is the effective clamping force with account taken of the contributions from the external field $E_\perp$, from the image forces of the substrate having a dielectric constant ε_d [the actual form of this term is reconciled with the definition of $\langle V(r)\rangle$ in (2a)]; f is a constant that characterizes the van der Waals interaction of the helium film with the solid substrate of the given type. In the case of a glass substrate $f \approx 9.5 \times 10^{-15}$ g$\cdot$cm^2/sec^2.

In the limiting case $\xi(0)/d \ll 1$, linearization of Eq. (5) with respect to ξ leads to the equation

$$\Delta\xi - \tilde{\varkappa}^2\xi = F\varphi^2/\alpha, \tag{6}$$

where $\tilde{\varkappa}^2 = 3f/\alpha d^4$ is the effective capillary constant.

Consideration of the linear equation in (6) together with (3a) allows us to carry through to conclusion, in a self-consistent manner,[1] the problem of localization of an electron on a film. The final results are in this case the following (we have in mind the harmonic approximation)[2)]:

$$\varphi^2 = \frac{1}{\pi R_0^2}\exp\left(-\frac{r^2}{R_0^2}\right), \quad R_0^2 = \frac{2\pi\alpha\hbar^2}{mF^2},$$

$$\xi(r) = \begin{cases} \xi(0) + \dfrac{F}{4\pi\alpha}\left(\dfrac{r}{R_0}\right)^2, & r < R_0\\ -\dfrac{F}{2\pi\alpha}K_0(\tilde{\varkappa}r), & r > R_0\end{cases} \tag{7}$$

$$\xi(0) = -\frac{F}{2\pi\alpha}\ln\frac{1{,}5}{\tilde{\varkappa}R_0}, \quad \mathscr{E} = -\frac{F^2}{4\pi\alpha}\left[\ln\frac{1{,}5}{\tilde{\varkappa}R_0} - \frac{1}{2}\right],$$

$$\tilde{\varkappa}R_0 \ll 1, \quad F = \Lambda/d^2.$$

Here R_0 is the radius of localization of the electron in the dimple, $\mathscr{E}$ is the total energy of the dimple, and K_0 is a Macdonald function of zeroth order. For $d \lesssim 10^{-6}$ cm, a glass substrate, and $E_\perp \to 0$ we have $R_0 \approx 10^{-6}$ cm; $\tilde{\varkappa} \approx 1.7 \cdot 10^5$ cm^{-1}; $\tilde{\varkappa}R_0 \approx 0.17$, $\xi(0) \approx 5-10$ Å; $\mathscr{E} = -10$ K. The combination $\tilde{\varkappa}R_0$ is practically independent of d, and the values of $\xi(0)$ and $\mathscr{E}$ increase rapidly with decreasing film thickness (to the extent that $F \sim d^{-2}$).

TABLE I.

l	d, cm	$\varkappa_l^{(2)}$			
1	$5\cdot10^{-7}$	4.66	4.75	4.69	4.40
2		5.66	5.91	5.84	5.66
3		6.66	6.98	6.91	6.74
1	10^{-6}	6.43	6.29	6.24	5.78
2		7.43	7.52	7.47	7.13
3		8.43	8.64	8.59	8.30
1	$5\cdot10^{-6}$	14.2	12.7	12.7	10.8
2		15.2	14.1	14.1	12.8
3		16.2	15.4	15.4	14.3

Turning now to the conditions under which the definitions (7) are valid, it is easy to see that in addition to the necessary condition $\tilde{\varkappa}R \ll 1$, which according to the estimates given above is satisfied and does not depend on d, it is necessary also to satisfy the inequality

$$\xi(0)/d \ll 1, \tag{7a}$$

which was used to obtain Eq. (6). Taking into account the definition of $\xi(0)$ from (7), we easily verify that the inequality (7a) is violated if $\xi(0) \to d$. Thus, with decreasing film thickness, it is no longer correct to linearize the mechanical-equilibrium equation.

An approximate idea of the deformation phenomena that occur in the region $\xi(0) \lesssim d$ can be obtained in the following manner. We assume that, just as in the investigation of the limiting case $\xi(0) \ll d$, Eq. (3a) can be solved by using the oscillator approximation, i.e., $\langle V(r)\rangle$ can be represented in the form

$$\langle V(r)\rangle \approx \Lambda\xi''(0)r^2/2(d+\xi(0))^2, \quad d+\xi(0) > {}^1/_2\xi''(0)r^2.$$

As for the mechanical-equilibirium equation (5), we seek its solution under the assumption that the right-hand side of this equation is piecewise smooth:

$$P(r) \approx \frac{\Lambda\varphi_0^2}{(d+\xi(0))^2} - \frac{f}{(d+\xi(0))^3} + \frac{f}{d^3}, \quad r \leqslant R,$$
$$P(r) \approx 0, \quad r \geqslant R, \quad (\varphi_0^2 = 1/\pi R^2).$$

Taking these remarks into account and solving Eqs. (3a) and (5) we arrive at the following relations for $\xi''(0)$, $\xi(0)$, and R:

$$R^{-2} = m\omega\hbar^{-1}, \quad \omega^2 = \Lambda\xi''(0)/(d+\xi(0))^2 m,$$

$$2\alpha\xi''(0) \approx \frac{\Lambda\varphi_0^2}{(d+\xi(0))^2} - \frac{f}{(d+\xi(0))^3} + \frac{f}{d^3}, \tag{8}$$

$$\xi(0) \approx -\frac{1}{2\pi\alpha}\left\{\frac{\Lambda}{(d+\xi(0))^2} - \pi R^2 f\left[\frac{1}{(d+\xi(0))^3} - \frac{1}{d^3}\right]\right\}\ln\frac{1}{\tilde{\varkappa}R},$$

or, after simple transformations,

$$\frac{R^2}{(d+\xi(0))^2} \approx -\frac{\hbar^2}{m\Lambda\xi(0)}\ln\frac{1}{\tilde{\varkappa}R},$$

$$\xi(0) \approx -\frac{\Lambda}{2\pi\alpha(\alpha+\xi(0))^2}\left(1 - \delta\frac{d+\xi(0)}{d}\ln\frac{1}{\tilde{\varkappa}R}\right)\ln\frac{1}{\tilde{\varkappa}R}, \tag{8a}$$

$$\delta = \frac{3\pi\hbar^2 f}{m\Lambda^2}.$$

The solution of the system (8a) for R and $\xi(0)$ completes the investigation of the problem of determining the parameters of the electron dimple under the assumption that the inequalities $\tilde{\varkappa}R < 1$ and $\nabla\xi < 1$ and that the ratio of $\xi(0)$ to d is arbitrary but remains less than unity.

Proceeding to solve the system (8a), we note first that the parameter δ contained in the second equation of (8a) is small at typical values of f and ε_d. Thus, if $f\approx 10^{-14}$ erg and $\varepsilon_d \lesssim 10$, then $\delta\approx 10^{-1}$. The smallness of this parameter means that actually, even under conditions when the deformation of the helium surface is not small, the principal role in the mechanical equilibrium at the center of the electron dimple is played by the competition between the electron and Laplace pressures. As a result, and relation analogous to the definition (8a) of $\xi(0)$ can be obtained under the condition $\delta\ll 1$ from the definition (7) of $\xi(0)$, by replacing the expression F from (7) by

$$F=\Lambda/d^2\to\Lambda/(d+\xi(0))^2.$$

This circumstance, however, is not obvious beforehand, so that the foregoing analysis of the situation with $\xi(0)\lesssim d$ is necessary. In addition, it is perfectly possible that for certain solid substrates with relatively small values of ε_d the parameter δ may turn out to be not small and its presence cannot be neglected.

Taking into account the smallness of $\delta\ll 1$ and introducing the variable $x=\xi(0)/d$, we can rewrite the second equation of (8a) in the form

$$x(1-x)^2=Q,\quad Q=\frac{\Lambda}{2\pi\alpha d^3}\ln\frac{1}{\varkappa R}. \tag{9}$$

The left-hand side of this relation is bounded from above and has a maximum at $x=x_{max}=\frac{1}{3}$. Consequently the right-hand side of (9) should also be bounded from above by the requirement $Q\leq\frac{4}{27}$. In expanded form, this inequality represents a restriction on the film thickness:

$$d>d_{min},\quad d_{min}^3=\frac{27\Lambda}{8\pi\alpha}\ln\frac{1}{\varkappa R_{min}}. \tag{9a}$$

Numerically we have $d_{dim}\approx 10^{-7}$ cm.

Thus, within the framework of the approximation in which $\tilde{\varkappa}R<1$ and $\nabla\xi<1$ a stationary localized state of an electron on a thin film is possible only in the region $d>d_{min}$. When this inequality is violated, breakthrough takes place and the electron penetrates into the film, forming apparently a strongly deformed bubble.

The expression for R, which follows from (8a) and (9), is of the form

$$R^2=2\pi\alpha h^2(d+\xi(0))^4/m\Lambda^2. \tag{9b}$$

Obviously, this value of R is somewhat smaller than the R_0 defined in (7) [to the extent that $d+\xi(0)<d$]. Consequently, the inequality $\tilde{\varkappa}R<1$ used to obtain the results in (8a) actually holds true to the extent that the inequalitites $R<R_0$ and $\tilde{\varkappa}R_0<1$, discussed above, are satisfied.

The numerical scale of R in the thickness region $d\sim d_{min}$ is $R\sim d$. Consequently, the necessary inequalities $\gamma R>1$ from (2b) and $\Delta\xi<1$ [which can be written in the form $\xi(0)R^{-1}<1$] are actually satisfied at the applicability limit. As for the requirement $d+\xi(0)>\frac{1}{2}\xi''(0)R^2$ used to simplify the potential $\langle V(r)\rangle$, it reduces to the inequality $d^3>27\,\Lambda/8\pi\alpha$, which is satisfied so long as the inequality (9a) holds.

It should be noted, in concluding this section, that the quantity d used above is the total thickness of the film, including its liquid and solidified parts, which differ little in their dielectric properties. This thickness is somewhat larger than the thickness d of the superfluid part of the film, which "participates" in the hydrodynamic motions.

2. VORTEX PAIRS IN A HELIUM FILM AND THEIR INTERACTION WITH ELECTRONS

A. An interesting new formation produced in thin helium films at finite temperatures are vortex pairs of fluctuation origin. These excitations cannot stem from fluctuations in the bulk helium, since their energy is macroscopically large. In thin films, however, the situation changes, since the length of the vortex filament is now of the order of the thickness of the helium film, i.e., it is sufficiently small, and the logarithmic divergence and the self-energy due to the slow decrease of the velocity field around the vortex filament can be eliminated if the vortices are produced in pairs of opposite sign. As a result, the energy of an individual vortex pair is finite and independent of the area of the helium film:

$$W(L)=\frac{\rho_s d}{4\pi}\Gamma^2\ln\frac{L}{b}+W_0,\quad \Gamma=\frac{2\pi\hbar}{m_4}. \tag{10}$$

Here ρ_s is the bulk density of the liquid helium, d is the film thickness, L is the distance between the vortex axis, b is the radius of the core of the vertex filament, W_0 is the energy of the vortex core, Γ is the circulation quantum, and m_4 is the mass of the He^4 atom.

According to (4), the average value of L and the average density N for vortex pairs of fluctuation origin in a temperature region far enough from the Kosterlitz-Thouless phase-transition point are determined by the expressions

$$\begin{gathered}\langle L^2\rangle=\frac{\beta E_0-1}{\beta E_0-2}b^2,\quad \beta=T^{-1},\quad E_0=d\rho_s\Gamma^2/4\pi,\\ N=\frac{\pi}{b^2}\frac{1}{\beta E_0-1}\exp(-2\beta W_0),\quad \langle b^2N\rangle=\frac{\pi\exp(-2\beta W_0)}{\beta E_0-2}.\end{gathered} \tag{11}$$

The function $\langle b^2N\rangle$ diverges as $\beta E_0\to 2$. In this temperature region the vortex pairs begin to dissociate, and this is the cause of the Kosterlitz-Thouless phase transition in the helium film. The presence of this phase transition is revealed, e.g., by an abrupt change in the viscous properties of the film, and this viscosity change was indeed observed in experiment.

It should be noted that the definitions of $\langle L^2\rangle$ and N in (11) are somewhat ambiguous. If, e.g., we represent the vortex pairs as Bose quasiparticles with energy $W(p)$, momentum p, and velocity v:

$$\begin{gathered}W(p)=E_0\ln\frac{p}{p_0}+W_0,\quad p=\frac{d}{2}\rho_s\Gamma L,\quad p_0=\frac{d}{2}\rho_s\Gamma b,\\ v=\frac{\partial W}{\partial p}=\frac{\Gamma}{2\pi L}=\frac{d\rho_s\Gamma^2}{\pi p}\end{gathered} \tag{12}$$

[the Hamiltonian definitions of $W(p)$, p, and v for a linear vortex pair are known from classical hydrodynamics], the expression for N takes the form

$$N=\frac{1}{2\pi\hbar^2}\int n(p)p\,dp=\frac{p_0^2}{2\pi\hbar^2}\frac{\exp(-\beta W_0)}{\beta E_0-2}, \tag{13}$$

$$n(p)=\{\exp[\beta W(p)]-1\}^{-1},\quad |p|\gg p_0,\quad \beta W_0\gg 1.$$

Obviously, the pre-exponential factors in the definitions (11) and (13) of N are different, and in the case $b\sim a$, where a is the distance between the atoms in the liquid helium, the expression (13) for N exceeds (11) by $(d/a)^2$ times.[3)]

B. The lack of information on the vortex-pair-spectrum parameters W_0 and b, which cannot be calculated theoretically, makes it necessary to search for new experimental possibilities of investigating the interesting phenomenon of fluctuation vortex formation. In this sense, it may be useful to study the deformation phenomena that occur when electrons are located on the surface of a thin helium film. As noted in the Introduction, vortex pairs interact with an individual charged dimple and becomes attracted to it. The reason for this attraction is very simple and clear. According to its definition (10), the vortex-pair energy is proportional to the vortex-pair length. A decrease of this length leads to a decrease of the total vortex-pair energy. But it is precisely such a decrease of the vortex-pair length which takes place when the vortex pair lands in the region of the charged dimple. A quantitative measure of the resultant interaction is the definition of δW:

$$\delta W=\left(\frac{\rho_s}{4\pi}\Gamma^2\ln\frac{p}{p_0}+\frac{\partial W_0}{\partial d}\right)\xi(r),\quad \xi(r)<0, \tag{14}$$

in which the role of $\xi(r)$ is played by the self-consistent deformation of the film surface in the vicinity of the localized electron. Recognizing that the expected (i.e., the theoretically predicted) dimension L of an individual vortex pair is of the order of interatomic distances, and that the region of substantial deformation in the vicinity of the charged dimple has a scale $R\approx 10^{-6}$ cm, i.e., $R\gg L$, we can assume that a large number of vortex pairs, which is characterized by the local density $N(r)$ of the vortex pairs, should accumulate in the vicinity of the charged dimple. If the average vortex-pair density N is small enough, then $N(r)$ is defined by analogy with N (13):

$$N(r)=N\frac{\beta E_0-2}{\beta(E_0+\xi\,\partial E_0/\partial d)-2}\exp\left(-\beta\frac{\partial W_0}{\partial d}\xi\right). \tag{15}$$

For a film thickness $d\lesssim 10^{-6}$ cm and a glass substrate, when the estimates given in the comments on the definitions (7) yield $\xi(0)$ of the order of ~5–10 Å, which is a noticeable fraction of d, the local density of the vortex pairs in the vicinity of the electron dimple can exceed noticeably the average density N. This circumstance suggests that procedures in which electron dimples are used to investigate fluctuation vortex pairs are highly sensitive to the density of these pairs.

In the general case the local density of the vortex pairs in the vicinity of an electron dimple may turn out to be not small. In this situation, the problem of the density of vortex pairs in the vicinity of an electron dimple does not reduce to the determination of $N(r)$ (15) in a given field $\xi(r)$. A self-consistent solution must be obtained for the mechanical-equilibrium equation

$$\Delta\xi-\tilde{\varkappa}^2\xi=\alpha^{-1}P_{el}(r)+\alpha^{-1}P_v(r) \tag{16}$$

together with definitions of the electron $P_{el}(r)$ and vortex $P_v(r)$ pressures on the film surface.

Using the natural definition of $P_v(r)$, which is valid if $L\ll R$:

$$P_v=\frac{\partial\langle W\rangle}{\partial d},\quad \langle W\rangle=\frac{1}{(2\pi\hbar)^2}\int W(p,r)n(p,r)d^2\mathbf{p},$$
$$P_v=\left[\frac{\partial W_0}{\partial d}+\frac{\partial E_0}{\partial d}(\beta E_0-1)^{-1}\right]N(r), \tag{17}$$

and taking into account the definition $P_{el}=F\varphi^2(r)$, where the electron wave function $\varphi(r)$ satisfies Eq. (3a) and F is taken from (7), we arrive at a closed system of equations that determine in self-consistent manner the behavior of $\varphi(r)$, $\xi(r)$, and $N(r)$ is the vicinity of the electron dimple.

A consistent solution of the resultant nonlinear problem with respect to $\xi(r)$, $\varphi(r)$, and $N(r)$ has not yet been obtained. We present below an approximate analysis that should provide a qualitative description of the dimple properties in the presence of a finite vortex-pair density. We have in mind a representation of the deformation $\xi(r)$ in the vicinity of the center of the dimple in the form of the expansion

$$\xi(r)=\xi(0)+{}^1/_2\xi''(0)r^2+\ldots, \tag{18}$$

which contains two constants, $\xi(0)<0$ and $\xi''(0)>0$. According to (3a) and (18), the wave function of the ground state of the electron in the dimple has now an oscillatory form:

$$\varphi^2(r)=\frac{1}{\pi R^2}\exp\left(-\frac{r^2}{R^2}\right), \tag{19}$$

where the radius R of the electron localization in the dimple is connected with $\xi''(0)$ by a relation similar to (8):

$$\frac{1}{R^2}=\frac{m\omega}{\hbar},\quad \omega^2=F\xi''(0)m^{-1}. \tag{20}$$

Noting now that the right-hand side of the mechanical-equilibrium equation (16), with account taken of (18) and (19), is a sum of two exponential terms, we can obtain an exact inhomogeneous solution of this equation. Having an expression for $\xi(r)$ from (16), and determining with its aid the value $\xi(0)$ and $\xi''(0)$, we obtain the following relations which complete the system of transcendental equations needed to find the three parameters of the problem $[R,\ \xi(0),\ \text{and}\ L_1^{-2}=\frac{1}{2}\xi''(0)\beta\partial W_0/\partial d]$:

$$\xi(0)=-\frac{F}{4\pi\alpha}\exp\left(\frac{\tilde{\varkappa}^2R^2}{4}\right)\mathrm{Ei}\left(-\frac{\tilde{\varkappa}^2R^2}{4}\right)$$
$$-\tilde{\varkappa}^2L_1^2\frac{F^*}{4\pi\alpha}\exp\left(\frac{\varkappa^2L_1^2}{4}\right)\mathrm{Ei}\left(-\frac{\varkappa^2L_1^2}{4}\right)$$
$$F^*=\pi\tilde{\varkappa}^{-2}N\left[\frac{\partial W_0}{\partial d}+(\beta E_0-1)^{-1}\frac{\partial E_0}{\partial d}\right]\exp\left(-\beta\frac{\partial W_0}{\partial d}\xi(0)\right), \tag{21}$$
$$R^2=2\pi\alpha\hbar^2/mF^2\left(1+\frac{F^*}{F}\tilde{\varkappa}^2R^2\right),\quad \frac{R^2}{L_1^2}=\frac{1}{4}\beta\frac{\hbar^2}{mR^2}F^{-1}\frac{\partial W_0}{\partial d}.$$

In the limiting case of a weak influence of the vortex pressure on the electron-dimple parameters, when $F^*/F\to 0$, the expressions for $\xi(0)$ and R from (21) go over into the corresponding definitions of $\xi(0)$ and R

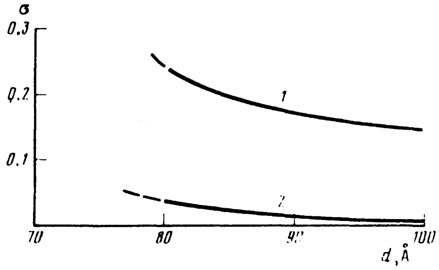

FIG. 1. Dependence of σ on d at a temperature $T = 1.5$ K under the assumption that $W_0 = \gamma E_0$: 1—$\gamma = 0.2$; 2—$\gamma = 0.3$.

from (7). Using this circumstance, we can represent R in the region $F^*\tilde{\varkappa}^2R^2/F \ll 1$ in the form

$$R^2 \approx R_0^2(1+\sigma)^{-1}, \quad \sigma = F_0^*\tilde{\varkappa}^2R_0^2/F < 1, \tag{22}$$

where R_0^2 is the value of R as $\sigma \to 0$ and F_0^* takes the form (21) with $\xi(0)$ from (7). In expanded form, recognizing that $\tilde{\varkappa}R_0 \approx 0.15$ [see the pertinent comments to (7)], σ is given by:

$$\begin{gathered}\sigma \approx 2.25\cdot 10^{-2}\frac{\pi N}{F\tilde{\varkappa}^2}\frac{\partial W_0}{\partial d}\cdot 10^{\Omega},\\ \Omega = \beta\frac{\partial W_0}{\partial d}\frac{F}{2\pi\alpha}, \quad \beta E_0 \gg 1.\end{gathered} \tag{22a}$$

The numerical value of σ depends essentially on the film thickness d. Assuming $W_0 = \gamma E_0$, where $\gamma \lesssim 1$ is a numerical coefficient, the value of σ is exponentially small in the region $d > 10^{-6}$ cm and becomes sufficiently noticeable in the region $d < 10^{-6}$ cm. The behavior of σ for $W_0 = \gamma E_0$, for different values of γ, and for a characteristic temperature $T = 1.5$ K is shown in Fig. 1. In the calculation of σ we took into account the fact that the thickness of the superfluid part of the helium film is approximately 10 Å smaller than the total film thickness, which includes additionally two or three layers of solidified helium.

The real behavior of σ can deviate noticeably from that shown in the figure, since the ratio $W_0 \sim E_0$ is only an assumption. In the absence of information on W_0, however, this assumption must be used for approximate estimates.

Thus, in the region $d < 10^{-6}$ cm one can expect noticeable devitations, of vortex origin, of R from R_0 to the extent that $\sigma \neq 0$. These deviations should have a characteristic (exponential) temperature dependence and are observed under conditions when the equilibrium thickness of the helium film is constant. The appearance of a tendency towards such a variation of R is described by Eqs. (21) and (22). As for significant changes in the structure of the electron dimples on account of the interaction with the vortex-pair density, they can arise in the region $d < 10^{-6}$ cm, i.e., in a situation wherein the structure of the electron dimple ceases to be described by relations (7), and conditions arise for the electron localization described by relations (8).

CONCLUSION

Let us summarize some of the results. The main purpose of this study was the description of the influence of the deformation phenomena on the parameters of the electronic spectrum of electrons localized on thin helium films. It was found that interesting singularities of deformation origin are present both in the transverse [see the definition (4a) of Δ_l] and in the longitudinal part of the electron spectrum. In the latter case this reduces to different degrees of localization of the electron as it moves over the film surface [see relations (7) and (9) and the pertinent comments]. The appearance of localization of this type can be detected by various methods, e.g., by measuring the cyclotron resonance. The shift of the cyclotron-resonance frequency can in this case be much larger than in the case of the known experiments on cyclotron resonance for electrons over bulk helium.[8] In the case of sufficiently thin helium films, a noticeable connection becomes possible between the electron and vortex components in the film, and this leads to a number of additional singularities in the behavior of the longitudinal electron spectrum as a function of the temperature and of the film thickness [see the definitions (21) and (22)].

APPENDIX

For a thin helium film ($d \ll 5\times 10^{-5}$ cm), the principal role is played by the potential of the interaction of the electron with the substrate and by the potential at which the electron enters the helium, which takes the form of a potential step $V_0 \sim 1$ eV. In this case the Schrödinger equation outside and inside the film has the same form as the Coulomb equation. After making a standard change of variables, the problem reduces to a solution of the following equations:

outside the film

$$\begin{gathered} xf_2''(x) + (2-x)f_2'(x) - a_2f_2(x) = 0,\\ x = 2\gamma_2(z+d), \quad \gamma_2^{-1} = 4a_0\varkappa_l^{(2)}, \quad a_2 = 1-\varkappa_l^{(2)},\\ \Delta_l = -\frac{e^2}{32a_0(\varkappa_l^{(2)})^2}, \quad xe^{-x/2}f_2(x)|_{x\to\infty} = 0;\end{gathered} \tag{A1}$$

inside the film

$$\begin{gathered} xf_1''(x) + (2-x)f_1'(x) - a_1f_1(x) = 0,\\ x = 2\gamma_1(z+d), \quad \gamma_1^{-1} = 4a_0\varkappa_l^{(1)}, \quad a_1 = 1-\varkappa_l^{(1)},\\ xf_1(x)|_{x\to 0} = 0, \quad \varkappa_l^{(1)} = \varkappa_l^{(2)}[1+32a_0V_0(\varkappa_l^{(2)})^2/e^2]^{-1/2}.\end{gathered} \tag{A2}$$

In addition, it must be stipulated that the logarithmic derivative of the wave function on the free film surface, located at the point $z = 0$, be continuous.

The only solution of (A1) satisfying the boundary condition at infinity is the confluent hypergeometric function $U(a_2, 2, x)$:

$$\begin{gathered} U(a,2,x) = \frac{1}{\Gamma(a+1)}\Big\{{}_1F_1(a,2,x)\ln x\\ + \sum_{k=0}^{\infty}\frac{(a)_kx^k}{(2)_kk!}[\psi(a+k) - \psi(1+k) - \psi(2+k)]\Big\},\\ (a)_k = a(a+1)\dots(a+k-1), \quad {}_1F_1(a,b,x) = \sum_{n=0}^{\infty}\frac{(a)_nx^n}{(b)_nn!},\end{gathered}$$

where $\psi(x)$ is the Euler psi function.

The second solution of Eq. (A1) does not satisfy the boundary condition at infinity. If $a_2 \neq -n$, $n = 0, 1, 2, \dots$, the second solution is the function ${}_1F_1(a_2, 2, x)$. In the case $a_2 = -n$, the functions $U(-n, 2, x)$ and ${}_1F_1(-n, 2, x)$

differ only by a constant factor, and the solution independent of $U(a_2, 2, x)$ is $\exp(x)U(2+n, 2, x)$. In both cases, the corresponding wave functions diverge at infinity.

The only solution of Eq. (A2), satisfying the boundary condition on a dielectric substrate, is ${}_1F_1(a_1, 2, x)$. Taking the foregoing into account and satisfying the continuity condition on the boundary of the free surface, we obtain the following equation for Δ_l:

$$\frac{\exp(\gamma_1 x)}{{}_1F_1(a_1,2,2\gamma_1 x)}\frac{d}{dx}(x\exp(-\gamma_1 x)\,{}_1F_1(a_1,2,2\gamma_1 x))|_{x=d}$$
$$=\frac{\exp(\gamma_2 x)}{U(a_2,2,2\gamma_2 x)}\frac{d}{dx}(x\exp(-\gamma_2 x)U(a_2,2,2\gamma_2 x))|_{x=d}. \qquad \text{(A3)}$$

The connection between the quantities Δ_l, γ_1, γ_2, a_1, a_2, $\varkappa_l^{(2)}$ was determined above.

Thus, the problem of finding the electron spectrum reduces to solution of the algebraic equation (A3). Table I lists the results of the solution of this equation in terms of $\varkappa_l^{(2)}$ in various approximations. The first and second columns give the results of the calculation of $\varkappa_l^{(2)}$ in an approximation corresponding to the vanishing of the wave function on the surface of the helium film. In the first column, $\varkappa_l$ was calculated with the aid of the asymptotic formula (4a) of the main text for the zeros of the function $U(a, b, x)$. In the second column this quantity was obtained numerically. The results of the numerical solution of Eq. (A3) are given in the third column. Finally, in the last column of the table are given the results of a numerical calculation of $\varkappa_l^{(2)}$, in which the potential of the interaction of the electrons with the image on the surface of the helium film was taken into account by perturbation theory.

As seen from the table, allowance for the penetration of the wave function of the electron into a helium film on a dielectric substrate alters the spectrum insignificantly. The main correction is introduced by allowance for the interaction of the electron with the image in the helium film. At a film thickness ~10^{-6} cm this correction is 15–20%.

In addition to a dielectric substrate, we can consider a variant with a metallic substrate. In this case the boundary condition for Eq. (A2) at $z=-d$ takes the form

$$(xe^{-x/2}f_1(x))'=0.$$

Solving the system (A1) and (A2) we can verify that the problem has a single eigenvalue corresponding to negative energies, $\Delta_l\to-\infty$ (the corresponding $\varkappa_l^{(2)}=0$). In this case the electron penetrates into the film, and there are no stationary solutions with finite negative energy.

The question of the stability of discrete electron levels on a thin helium film deposited on a dielectric substrate is quite fundamental. It is therefore meaningful to present additional arguments in favor of the existence of this stability. One of the leading considerations is the absence of negative discrete levels for a free electron located on an interface between liquid helium and a solid dielectric (the energy is reckoned from that of the electron in vacuum). In fact, on such a boundary the discrete part of the electron spectrum is of the form

$$\Delta_l=V_0-m\Lambda^2/2\hbar^2l^2,\quad l=1,2,3,\ldots \qquad \text{(A4)}$$

when account is taken of the energy scale $V_0\approx 1$ eV, even the lowest discrete level from (A4) is in a positive region at arbitrary $\varepsilon_d\gg 1$. At the same time, the discrete levels for an electron on a thin helium film lie in the negative region. Consequently, stable localization of the electrons should be realized precisely on these levels, as was in fact revealed by the concrete calculations given in the Appendix.

Additional evidence that, in an atomic scale, the electrons do not stick to a solid dielectric substrate is provided also by measurements[9] of the mobility of electrons in thin helium films on a dielectric substrate. The experiment shows that the mobility of the electrons in such films is only several times smaller than the corresponding mobility of electron bubbles inside liquid helium; this indicates most likely that the electrons in the helium film are in a bubble state. The energy of such a state is also positive compared with the energy of the electron in vacuum, and can therefore not be the most stable state in the problem of the spectrum of an electron on a thin helium film.

[1)]To our knowledge, a spectrum of the form (4a) for an electron over a helium film was first obtained by Gabovich, Il'chenko, and Pashitskiĭ.[7] We are grateful to them for the opportunity of becoming acquainted with their results prior to publication. They, however, did not take into account the deformation phenomena, and their final results for the electron spectrum do not contain the quantity $\xi(0)$.

[2)] What was actually used in Ref. 1 is not a harmonic approximation, but a somewhat more accurate variational procedure that yields for $R_0^{(2)}$ the expression $R_0^2\;\; 4\pi\alpha\hbar^2/mF^2$, i.e., a result twice as large as the harmonic-approximation value of R_0^2 from (7). Nonetheless, we adhere to the harmonic approximation, which admits of various generalizations.

[3)] The averaging in the definitions (11) and (13) of N was made with different statistical weights: $\exp[-2\beta W(L)]$ in (11) and $\exp[-\beta W(\rho)]$ in (13). This leads to additional numerical differences in the arguments of the exponentials and in the denominators of the corresponding expressions for N.

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