Scattering theory for composite particles

D. A. Kirzhnits and F. M. Pen'kov

P. N. Lebedev Physics Institute, Academy of Sciences of the USSR
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A method is proposed for describing the scattering and bound states of two particles (which may be either elementary or composite) whose interaction consists of short- and long-range terms with very different ranges. The method is a generalization of Landau and Smorodinskiĭ's theory of proton-proton scattering, which describes the combined effect of Coulomb and nuclear forces, to include forces of any nature and the case of composite particles. As an example, the problem of elastic proton-deuteron scattering at low energies is solved by reducing it to the analogous neutron-deuteron scattering problem.

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1. INTRODUCTION

It is well known that inclusion of the long-range Coulomb interaction greatly complicates the solution of the three-body problem by the standard method using the Faddeev integral equations, which then no longer have Fredholm kernels. It is only very recently that general procedures for overcoming this difficulty have been found, but they lead to complicated equations and require extensive numerical calculations (see Ref 1).

In the present paper, we draw attention to a simple possibility of radically simplifying the solution of many problems of this type by reducing them to analogous problems without Coulomb interaction. Such problems include the description of the scattering and the bound states of two charged complexes that are each bound by short-range forces. The simplest example of this kind—the scattering of the proton (p) on the deuteron (d)—will serve as an illustration of the approach, which derives from Landau and Smorodinskiĭ's well-known theory of pp scattering.[2]

This approach consists of a generalization of the two-particle problem with Coulomb interaction to the case of composite particles (the usual approach is to generalize the nuclear three-body problem to the case when a Coulomb interaction is included). Such a program can be effectively realized because the deuteron radius[1)] $\varkappa^{-1}=(2E_d)^{-1/2}\approx 4$ fm (E_d is the deuteron binding energy) is small compared with the characteristic length of the Coulomb interaction α/r, which is the Bohr radius $a_0=1/|\alpha|\approx 43$ fm of the system. Because of this, the deuteron participates to a considerable extent in the Coulomb interaction as a single entity, and its internal structure is manifested in a comparatively small correction. As a result of this, there is also a comparatively simple connection between the characteristics of pd and nd scattering (n is the neutron).

In fact, the approach applies to a fairly large class of problems in not only nuclear but also atomic physics, and also solid-state physics when it is necessary to describe the interaction of two complexes representing the sum of a long-range term V_L (range R_L) and short-range term V_S (range R_S). The Coulomb interaction, above all, will play the part of V_L, for which the range R_L in the inequalities given below can be replaced by a_0. Other possibilities are the van der Waals interaction of atoms, molecules, and excitons, the interaction of the magnetic moments of nucleons, and their nuclear attraction. The term V_S may correspond to the nuclear interaction of charged complexes, the exchange repulsion of an electron or ion by the electron shell of an ion, and the repulsive hard core in the interaction of nucleons.

In the general case, the terms of such a composite interaction interfere so strongly that the solution of the problem cannot be expressed in terms of the solutions of the Schrödinger equations with the potentials V_L and V_S separately. Such a solution is possible if the ranges of the forces are very different,

$$R_S \ll R_L, \tag{1.1}$$

when the direct interference between the potentials V_L and V_S due to their simultaneous action in the same region of space is small and each of them acts mainly in its "own" domain.

This last assertion is obvious for V_S, which by definition is small in the far region. With regard to V_L, in the near region it is small compared with either V_S or with the kinetic energy R_S^{-2}. Thus, for the Coulomb interaction its ratio to the kinetic energy is of order $R_S/a_0 \ll 1$; the same is true for nuclear attraction, which in the far region is of order of the kinetic energy (shallow levels), increasing with decreasing distance more slowly than its square.

The smallness of the direct interference of the potentials V_L and V_S (which it is sufficient to take into account by perturbation theory in V_L in the short-range region) does not mean that the interference effects are small altogether. There is also indirect interference, since the long-range forces determine the probability of the complexes approaching to distances at which the short-range forces come into play. If the condition (1.1) is satisfied, such effects can be expressed entirely in terms of the solution Ψ_L of the Schrödinger equation for the potential V_L, namely, in terms of $A^2=|\Psi_L(0)|^2$, the ratio of the probabilities of finding the complexes at coincident positions in the presence of V_L and in its absence.

These considerations essentially provide the physical basis of the Landau–Smorodinskiĭ theory, which leads to the relation

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$$A^2 \operatorname{ctg}(\delta-\delta_L)-\operatorname{ctg}\delta_S=K. \qquad (1.2)$$

Here, δ, δ_L, and δ_S are, respectively, the phase shifts for the potentials V_L+V_S, V_L, and V_S (here and in what follows, the indices identifying the moments, spins, etc., are omitted), and K describes the direct interference. The indirect interference corresponds to the deviation of the factor A^2 from unity: If $A^2\to 0$, then $\delta=\delta_L$ (no short-range interaction), and if $A^2=1$ and $K=0$, there is no interference at all, and $\delta=\delta_L+\delta_S$.

The main qualitative conclusion of this paper is that if the condition (1.1) is satisfied, then formula (1.2) is universal and valid for describing the scattering of both elementary and composite particles. The structure of the latter affects only K, whereas the left-hand side of (1.2) is determined by the long-range interaction of the complexes as complete entities. Similar conclusions can also be drawn concerning bound states, which we describe below as well as scattering, using the formalism of Jost functions.

We shall consider only elastic scattering, for which only the case of low energies ($kR_S \ll 1$, kR_L arbitrary; k is the momentum of the relative motion) is nontrivial. But if $kR_S \gtrsim 1$, then $kR_L \gg 1$, and the long-range effects are small and can be described simply by means of perturbation theory.[2)] For same reason, the long-range forces are important in the case of inelastic scattering (to which we shall devote a separate paper) only in the immediate vicinity of the threshold for the disintegration of the colliding complexes.

2. JOST FUNCTION

There exists a function that, possessing simple mathematical properties, nevertheless contains many-sided information about a quantum-mechanical system (see, for example, Ref. 3). In this section, we consider the two-body problem with central potential V_L+V_S ($V_L=\alpha v_L$, in which α is the coupling constant) which decreases sufficiently rapidly at infinity; here and below we shall consider only the s state. Let $\varphi(r)$ be the radial solution of the Schrödinger equation

$$\left(\frac{d^2}{dr^2}+k^2-2(V_S+\alpha v_L)\right)\varphi=0$$

that tends in the limit $r\to\infty$ to the asymptotic behavior $\exp(ikr)$. By definition, the Jost function is $u(k)\equiv\varphi(0)$.

It has the following formal properties: a) $u(-k)=u^*(k)$; b) $u(k)\to 1$ on a large circle in the half-plane Im $k\geq 0$; c) $u(k)$ is analytic with respect to k (together with the function φ) in the same region; d) $u(k)$ is an entire function of α. If the potential decreases with the distance faster than the Yukawa potential, the region of analyticity of $u(k)$ extends to the entire complex plane of k; otherwise, singularities of u arise in the lower half-plane (dynamic singularities).

The Jost function contains the following physical information. Its phase is equal to the phase shift with opposite sign,

$$u(k)=|u(k)|\exp(-i\delta(k)), \qquad (2.1)$$

and its modulus determines the quantity A introduced in Sec. 1:

$$|u(k)|=1/A=1/|\Psi(0)|. \qquad (2.2)$$

The zeros of the Jost function in the half-plane Im $k>0$ lie on the imaginary half-axis and determine the energies $E_n=-\varkappa_n^2/2$ of the bound states:

$$u(i\varkappa_n)=0 \qquad (2.3)$$

(the zeros in the half-plane Im $k<0$ are associated with virtual states and resonances).

Below, we also need the wave function $\xi=\varphi e^{i\delta}$, which has the asymptotic behavior

$$\xi\to\exp(i(kr+\delta)) \quad (r\to\infty). \qquad \xi\to|u(k)| \quad (r\to 0).$$

and also the regular, $\chi=\operatorname{Im}\xi$, and irregular, $\tilde\chi=\operatorname{Re}\xi$, solutions of the Schrödinger equation with asymptotic behavior

$$\chi\to\sin(kr+\delta) \quad (r\to\infty). \qquad \chi\to kr/|u(k)| \quad (r\to 0).$$
$$\tilde\chi\to\cos(kr+\delta) \quad (r\to\infty). \qquad \tilde\chi\to|u(k)| \quad (r\to 0).$$

The function χ is a real wave function of the considered system.

The Jost function is determined by the equation[3)]

$$d\ln u(k)/d\alpha=\frac{2}{k}\int_0^\infty dr v_L(\tilde\chi\chi+i\chi^2), \qquad (2.4)$$

which describes its evolution with variation of the coupling constant α (see Ref. 4) with initial condition $u=u_S$ at $\alpha=0$ (u_S is the Jost function for the potential V_S). Equation (2.4) follows from the asymptotic behaviors of the functions χ and $\tilde\chi$ just given and the relation

$$\int_0^\infty dr v_L\chi_1\chi_2=\frac{1}{2}\left(\chi_1\frac{d\chi_2'}{d\alpha}-\chi_1\frac{d\chi_2}{d\alpha}\right)\Big|_0^\infty, \qquad (2.5)$$

where $\chi_{1,2}$ are arbitrary solutions of the Schrödinger equation, and the prime denotes the derivative with respect to r. The relation (2.5) itself is obtained by differentiating the Schrödinger equation for χ_2 with respect to α, multiplying the result from the left by χ_1, integrating over r from 0 to infinity, and applying Green's theorem.

Equation (2.4) contains the laws of evolution with varying α of the phase shift, the quantity A, and the energies of the bound states. Taking the imaginary and real parts of (2.4), we have in accordance with (2.1) and (2.2)

$$d\delta(k)/d\alpha=-\frac{2}{k}\int_0^\infty dr v_L\chi^2,$$

$$d\ln A/d\alpha=-\frac{2}{k}\int_0^\infty dr v_L\tilde\chi\chi.$$

The first of these equations has already been widely used (see Ref. 4). Further, using the relation

$$2(\tilde\chi\chi+i\chi^2)=i(|\varphi|^2-\varphi^2u^*/u),$$

we continue (2.4) to the point $k=i\varkappa_n$ [see (2.3)]. This leads to the well-known equation

$$dE_n/d\alpha=\int_0^\infty dr v_L\chi_n^2, \qquad (2.6)$$

where

$$\chi_n=(iu^*(k)/u'(k))^{1/2}\varphi|_{\varkappa_n} \tag{2.7}$$

is the normalized wave function of a bound state.[3]

3. JOST FUNCTION FOR COMBINED INTERACTION

We continue our study of the two-particle problem, having as our aim the explicit finding of the Jost function when the condition (1.1) is satisfied. We shall not restrict ourselves to the region of small momenta but attempt to obtain a general expression that in the limit of large k goes over into the result of applying perturbation theory in V_L.

We separate from the function $u(k)$ the Jost function $u_L(k)$ for the potential V_L:

$$u=u_L w. \tag{3.1}$$

The quantity w, which goes over into u_S when $\alpha=0$, describes not only the effects of the potential V_S but also the interference of V_S and V_L. It follows from (2.4) that

$$\frac{d\ln w}{d\alpha}=\frac{2}{k}\int_0^\infty dr v_L(\bar\chi\chi+i\chi^2-\bar\chi_L\chi_L-i\chi_L^2), \tag{3.2}$$

where χ_L and $\bar\chi_L$ are the corresponding wave functions for the potential V_L.

First, we shall take into account only the effects of the indirect interference, which corresponds to the following mixing of the wave functions ($\bar\delta=\delta-\delta_L$):

$$\chi=\chi_L\cos\bar\delta+\bar\chi_L\sin\bar\delta,\qquad \bar\chi=-\chi_L\sin\bar\delta+\bar\chi_L\cos\bar\delta,$$

which leads to the correct phase shift. Denoting the corresponding approximate expression for w by w_0, we obtain from (3.2) and (2.5)

$$\frac{dw_0}{d\alpha}=-\operatorname{Im} w_0\frac{2}{k}\int_0^\infty dr v_L\xi_L^2=-\frac{\operatorname{Im} w_0}{k}\left(\xi_L'\frac{d\bar\xi_L}{d\alpha}-\xi_L\frac{d\bar\xi_L'}{d\alpha}\right)\Big|_0^\infty.$$

Using the asymptotic behaviors given in Sec. 2, we can write the expression in the brackets in the form

$$i\frac{d}{d\alpha}\ln|u_L|^2-\frac{1}{k|u_L|^2}\frac{d}{d\alpha}(\varphi_L'(0)/u_L).$$

Finally, integration of the obtained equation using the boundary condition at $\alpha=0$ gives

$$w_0=u_S+\frac{\operatorname{Im} u_S}{k}\Sigma(k), \tag{3.3}$$

$$\Sigma(k)=\varphi_L'(0)/u_L-ik. \tag{3.4}$$

The function $\Sigma(k)$ tends to zero on a large circle in the half-plane $\operatorname{Im} k>0$, has $\operatorname{Im}\Sigma=k(1/|u_L|^2-1)$, and is analytic together with φ_L and u_L in this region everywhere except the points $k=i\varkappa_{nL}$, which correspond to the bound states in the field V_L. At these points, $\Sigma\to-2\varkappa_{nL}|u_L|^2$, which follows from the expression for the Wronskian $\varphi_L'\varphi_L^*-\varphi_L\varphi_L^{*\prime}=2ik$ corresponding to the point $r=0$. In view of all this, we can use the Cauchy formula for the contour C shown in Fig. 1:

$$\Sigma(k)=\frac{1}{\pi i}\int_C\frac{z\,dz}{z^2-k^2-i\varepsilon}\Sigma(z).$$

This gives

$$\Sigma(k)=\frac{2}{\pi}\int_0^\infty\frac{dzz^2}{z^2-k^2-i\varepsilon}\left(\frac{1}{|u_L|^2}-1\right)-P(k), \tag{3.5}$$

$$P(k)=4i\sum_n(\varkappa_{nL})^2/[(k^2+\varkappa_{nL}^2)u_L^*(i\varkappa_{nL})u_L'(i\varkappa_{nL})].$$

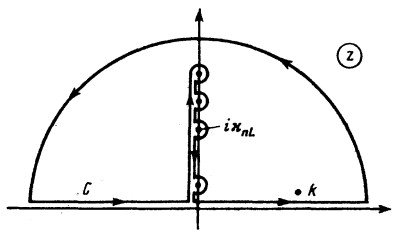

FIG. 1.

Note that the analyticity of the Jost function in α (property d) in Sec. 2 makes it possible to find the function Σ first in the region of α in which there are no bound states and $P=0$ and then continue the result analytically to the required region. This approach also leads to (3.5).

It remains to describe the effects of the direct interference between the interactions, which, as we have already noted in Sec. 1, corresponds to a manifestation of the interaction V_L in the near region that can be taken into account with sufficient accuracy in the first order in α. Below, we shall see the importance of these effects, which ensure correct description of the region of momenta $kR_S\gtrsim1$, and for the Coulomb interaction eliminate the inherent logarithmic divergence. It can be shown that the effects of the direct interference can be taken into account by replaceing (3.3) by the expression

$$w=u_S(1+I)+\operatorname{Im}[u_S(1+I)]\Sigma(k)/k, \tag{3.6}$$

where I, which is proportional to α, will be calculated below.

We denote the right-hand side of (3.2) by $S(w)$ [and the right-hand side of the analogous equation for w_0 by $S_0(w_0)$]. Then

$$\Phi(w)=\Phi_0(w_0)=u_S,$$

$$\Phi_0(w_0)=w_0\exp\left(-\int_0^\alpha d\alpha S_0(w_0)\right),$$

$$\Phi(w)=w\exp\left(-\int_0^\alpha d\alpha S(w)\right)=\Phi_0(w)\exp\left(\int_0^\alpha d\alpha(S_0-S)\right).$$

Hence, bearing in mind that w_0 is the inverse of the function Φ_0, we readily arrive at (3.6), in which

$$I=\int_0^\alpha d\alpha(S(w)-S_0(w))\to\ln(w/w_0)\to(w-w_0)/u_S|_{\alpha\to0}.$$

To first order in α and noting that χ and $\bar\chi$ go over at $\alpha=0$ into χ_S and $\bar\chi_S$, the wave functions for the potential V_S, we find

$$w=u_S\left[1+\frac{2}{k}\int_0^\infty drV_L(\bar\chi_S\chi_S+i\chi_S^2-e^{ikr}\sin(kr))\right].$$

In the same approximation, it follows from (2.4), (3.3), and (3.5) that

$$\Sigma=-2\int_0^\infty drV_Le^{2ikr}$$

and

$$w_0=u_S-\operatorname{Im} u_S\frac{2}{k}\int_0^\infty drV_Le^{2ikr}.$$

Using also the readily verified identity

$$u_s[\exp[i(kr+\delta_s)]\sin(kr+\delta_s)-e^{ikr}\sin(kr)]+\operatorname{Im} u_s e^{2ikr}=0,$$

we arrive at the expression

$$I=\frac{2}{k}\int_0^\infty dr V_L(\chi_s\chi_s+i\chi_s^2-\exp[i(kr+\delta_s)]\sin(kr+\delta_s)). \tag{3.7}$$

As is necessary, only the region of the short-range interaction (the near region) contributes to (3.7). The relations (3.1), (3.5), (3.6), and (3.7) completely solve the problem we have posed,[4] if the potential V_L decreases sufficiently rapidly at infinity (the special case of the Coulomb potential will be considered below).

4. SCATTERING AND BOUND STATES. THE COULOMB CASE

The most important physical information about the scattering and bound states is already contained in the function w. For scattering, this can be seen by comparing the expressions (2.1) and (3.1), since (up to the sign) the phase of the function w is equal to the phase shift difference $\delta-\delta_L$. With regard to the bound states, which are determined by the zeros of the function u_L when the interaction V_S is absent, they are associated with the zeros of the function w when V_S is present. This is due to the appearance of poles of w at points at which $u_L=0$ [the term P in (3.5)], and it is completely necessary if the bound levels are to be shifted under the influence of the interaction V_S.

Accordingly, using the formula $\cot(\delta-\delta_L)=-\operatorname{Re}w/\operatorname{Im}w$ and (3.6) and (3.7) to describe the scattering, and noting that $I\ll 1$, we obtain

$$|u_L|^{-2}\operatorname{ctg}(\delta-\delta_L)-\operatorname{ctg}\delta_s=[L(k)-\operatorname{Re}\Sigma(k)]/k, \tag{4.1}$$

$$L=\frac{2}{\sin^2\delta_s}\int_0^\infty dr V_L(\chi_s^2-\sin^2(kr+\delta_s)). \tag{4.2}$$

The structure of (4.1) is fully analogous to the Landau–Smorodinskiĭ formula (1.2). In the limit $k\to 0$, introducing

$$a_s^{-1}=-k\operatorname{ctg}\delta_s|_{k=0},\quad \tilde a_s^{-1}=-k\operatorname{ctg}(\delta-\delta_L)/|u_L|^2|_{k=0}$$

(a_S is the scattering length associated with the potential V_S), we find

$$\tilde a^{-1}-a_s^{-1}=-L(0)+\operatorname{Re}\Sigma(0). \tag{4.3}$$

As we have already noted, the energies $E_n=-\varkappa_n^2/2$ of the bound states are determined by the equation $w(i\varkappa_n)=0$. Introducing the amplitude of scattering on the potential V_S, which has the form $f_S=-\operatorname{Im}u_S/ku_S$, we find

$$f_s^{-1}(i\varkappa_n)+L(i\varkappa_n)=\Sigma(i\varkappa_n). \tag{4.4}$$

We consider first deep levels near the levels E_{nS} in the field V_S. Then perturbation theory with respect to V_L is valid, and we readily arrive at the obvious formula

$$E_n=E_{nS}+\int_0^\infty dr V_L\chi_{nS}^2. \tag{4.5}$$

The second term derives from L, i.e., ultimately from I [see (3.7)]. In the opposite limiting case, when the levels are near $E_{nL}=-\varkappa_{nL}^2/2$ and their energy is small on the scale of V_S, the arguments on the left-hand side of (4.4) can be replaced by zero and (4.3) used. This gives, when (3.5) is used,

$$E_n=E_{nL}(1-4i\tilde a/[u'(i\varkappa_{nL})u'(i\varkappa_{nL})]). \tag{4.6}$$

We now consider the most important class of problems of the type discussed in this paper, which is when the Coulomb interaction $1/(ra_0)$ plays the part of V_L. In this case, the procedure described above is not strictly valid, since the long-range Coulomb interaction makes it impossible to use the formalism of Jost functions fully. Nevertheless, the quantity w, which describes the short-range interaction V_S and its interference with the Coulomb interaction, preserves all its properties.[5] In particular, one can extract from this function in the usual manner information about the scattering and bound states (see the beginning of this section).

With regard to the actual determination of w, this can be based on Eq. (3.2), which leads to the expression (3.6), in which Σ can be found directly by explicit solution of the Schrödinger equation for the Coulomb interaction. To this end, it is necessary to rewrite (3.4) in the form

$$\bar\chi'(0)/\bar\chi(0)+ik(\bar\chi(0)^{-2}-1),$$

where $\bar\chi$ is the irregular solution of this equation, and replace the argument 0 for the time being by the small quantity ε. The result is

$$\Sigma(k)=\frac{2}{a_0}\left[\psi\left(1+\frac{i}{ka_0}\right)+2C+\ln(-2ik\varepsilon)\right], \tag{4.7}$$

where C is Euler's constant, and

$$\psi(x)=\Gamma'(x)/\Gamma(x)=-C+(x-1)\sum_{n=1}^\infty[n(n+x-1)]^{-1}.$$

Note that the same result is obtained by using (3.5) (in the region of repulsion and with subsequent analytic continuation with respect to a_0), in which $|u_L|^{-2}$ must be replaced by (see Sec. 1)

$$A^2=|\Psi(0)|^2=\frac{2\pi}{ka_0}\left[\exp\left(\frac{2\pi}{ka_0}\right)-1\right]^{-1}. \tag{4.8}$$

In the case of the Coulomb interaction, the integral I (3.7) also diverges. Truncating it at the same ε as above, we obtain

$$I=-\frac{2}{ka_0}(l(k)+\ln(2\varepsilon/|a_0|)),$$
$$l(k)=\int_0^\infty dr\ln\left(\frac{2r}{|a_0|}\right)\frac{d}{dr}[\bar\chi\chi_S+i\chi_S^2-e^{i(kr+\delta_S)}\sin(kr+\delta_S)]. \tag{4.9}$$

This gives

$$w=u_s(1-2l/ka_0)+\operatorname{Im}[u_s(1-2l/ka_0)]\frac{2}{ka_0}(\psi(1+i/ka_0)+\ln(-ik|a_0|)+2C), \tag{4.10}$$

which does not contain divergences.

From (4.10) it is easy to obtain relations that determine the phase shift and energies of the bound states ($l_1=\operatorname{Im}l/\sin^2\delta_S$):

$$A^2\operatorname{ctg}(\delta-\delta_L)-\operatorname{ctg}\delta_s=-\frac{2}{ka_0}(\operatorname{Re}\psi(1+i/ka_0)+\ln(k|a_0|)+l_1+2C), \tag{4.11}$$

$$f_s^{-1}(i\varkappa_n)=\frac{2}{a_0}\left(\psi\left(1+\frac{1}{\varkappa_n a_0}\right)+\ln(\varkappa_n|a_0|)+l_1(i\varkappa_n)+2C\right). \tag{4.12}$$

The first of these is identical to the Landau–Smorodinskiĭ formula[2] as improved by Jackson and Blatt[5] (see also the detailed investigation in Ref. 6 of the Coulomb corrections to the scattering lengths). Equation (4.12) was obtained and investigated by Popov, Kudryavtsev, Lisin, and Mur.[7]

5. SCATTERING AND BOUND STATES OF COMPLEXES

In this section, which occupies a central position in the paper, the results obtained above are generalized to the case of the interaction of composite particles (complexes), in contrast to structureless particles. The radius R_0 of the composite particles determines the distance between them at which the interaction V_S comes into play. Being, as a rule, greater than R_S, R_0 occurs in the condition which replaces (1.1),

$$R_0 \ll R_L. \tag{5.1}$$

It is convenient to introduce the coordinates $\mathbf{r}$ (the vector joining the centers of mass of the complexes) and $\boldsymbol{\rho}$ (the set of interval coordinates of the complexes referred to the centers of mass); the symbol $\int d\boldsymbol{\rho}$ denotes an integral over all the interval coordinates. If the distance between the complexes is large compared with R_0, the $\boldsymbol{\rho}$ dependence of the total wave function can be separated in the form of $\Phi(\boldsymbol{\rho})$, the product of the wave functions of the complexes. This function is normalized by the condition

$$\int d\boldsymbol{\rho} |\Phi|^2 = 1.$$

Under the same conditions, the interaction $V_L = \alpha v_L$ (it depends in the general case on $\mathbf{r}$ and $\boldsymbol{\rho}$) goes over into $V_L^0 = \alpha v_L^0$, which depends only on $\mathbf{r}$ and represents the long-range interaction of the complexes as single entities. In what follows, the superscript 0 will be added to quantities which refer to the interaction V_L^0.

Restricting ourselves in this paper to elastic scattering and being interested only in the corresponding phase shift and the energies of the bound states, we do not need to consider the problem of the Jost function of the complexes in its entirety. For what follows, it is sufficient that there exists a function $u_S(k)$ corresponding to the interaction V_S such that

$$u_S = |u_S| \exp(-i\delta_S), \quad u_S(i\varkappa_{nS}) = 0 \tag{5.2}$$

[see (2.1) and (2.3)] and also a function that directly generalizes (2.4):

$$\frac{d \ln u}{d\alpha} = \frac{2}{k} \int d\boldsymbol{\rho} \int_0^\infty dr v_L (\bar{\chi}\chi + i\bar{\chi}^2). \tag{5.3}$$

The boundary condition for (5.3) has the form $u = u_S$ at $\alpha = 0$,

$$\chi \to \Phi(\boldsymbol{\rho}) \sin(kr+\delta), \quad \bar{\chi} \to \Phi(\boldsymbol{\rho}) \cos(kr+\delta) \quad \text{as} \quad r \to \infty.$$

In the Appendix, it is shown that relations of the type (5.2) with the replacements $u_S \to u$, $\delta_S \to \delta$, $\varkappa_{nS} \to \varkappa_n$ are valid for the solution of (5.3). Therefore, physical information can be extracted from the function $u(k)$ in the usual manner.

Turning to the solution of Eq. (5.3) when the condition (5.1) is satisfied, we shall proceed in exactly the same way as above in considering the two-particle problem. Introducing the function w [see (3.1)], we find for it an equation that differs in form from (3.2) only by the presence of the additional integration over $\boldsymbol{\rho}$ on the right-hand side. This equation can also be solved in two stages. In the first, the direct interference of the interactions is ignored (as in Sec. 3, this reduces to a mixing of the wave functions), and also the interaction V_L is replaced by V_L^0, the long-range interaction of the complexes as single entities. This stage correspond to the equation

$$\frac{d \ln w_0}{d\alpha} = \frac{2}{k} \int_0^\infty dr \int d\boldsymbol{\rho} v_L^0 (\bar{\chi}^0 \chi^0 + i\chi^{02} - \bar{\chi}_L^0 \chi_L^0 - i\chi_L^{02}), \tag{5.4}$$

where $\bar{\chi}^0, \chi^0$ and $\bar{\chi}_L^0, \chi_L^0$ are, respectively, the solutions of the Schrödinger equations with the potentials $V_S + V_L^0$ and V_L^0, containing in the limit $r \to \infty$ the factor $\Phi(\boldsymbol{\rho})$. The explicit solution of (5.4) has a form analogous to (3.2):

$$w_0 = u_S + \operatorname{Im} u_S \Sigma^0(k)/k, \tag{5.5}$$

where u_S is the Jost function for the interaction V_S corresponding to (5.2), and the function Σ^0 is determined by (3.5) with V_L replaced by V_L^0 (in particular, u_L is the Jost function for the two-particle potential V_L^0).

In the second stage, we take into account the effects of the direct interference and the effects of the difference between V_L and V_L^0. The effects of both kinds are associated with the region of short distances between the complexes, and therefore they can be considered in the lowest order in α. As in Sec. 3, the problem reduces to making the substitution

$$u_S \to u_S (1 + I), \tag{5.6}$$

where $I = (w - w_0)/u_S$ as $\alpha \to 0$, in (5.5). In this limit, it follows from (5.5) that

$$w_0 = u_S - \operatorname{Im} u_S \frac{2}{k} \int_0^\infty dr V_L^0 e^{2ikr},$$

and from the equation for w that

$$w = u_S \left[1 + \frac{2}{k} \int_0^\infty dr \int d\boldsymbol{\rho} V_L (\bar{\chi}_S \chi_S + i\chi_S^2 - e^{ikr} \sin(kr) |\Phi(\rho)|^2) \right].$$

Hence, using the identity given in Sec. 3, we obtain

$$I = \frac{2}{k} \int_0^\infty dr \int d\boldsymbol{\rho} [V_L^0 (\bar{\chi}_S \chi_S + i\chi_S^2 - \exp(i(kr+\delta_S)) \sin(kr+\delta_S) |\Phi(\rho)|^2 + (V_L - V_L^0)(\bar{\chi}_S \chi_S + i\chi_S^2 - \exp(ikr) \sin(kr) |\Phi(\rho)|^2)]. \tag{5.7}$$

Here, $\bar{\chi}_S$ and χ_S are the wave functions for the potential V_S, which in the limit $r \to \infty$ have, respectively, the asymptotic behaviors

$$\Phi(\rho) \cos(kr+\delta_S) \text{ and } \Phi(\rho) \sin(kr+\delta_S).$$

The relations (5.5)–(5.7) solve the problem we have posed of describing the interaction of the complexes. Summarizing our investigation, we see that when the condition (5.1) is satisfied, the composite nature of the particles is manifested directly only in the direct interference effects, leading to the appearance of the purely structural second term in (5.7). Of course, it is assumed that the formulas which are transferred from

the two-particle problem now contain the quantities that refer to the interaction of the complexes: u_S, χ_S, etc.

To conclude this section, we consider the description of scattering and bound states of the complexes. The formulas (4.1) and (4.4) obtained above remain fully valid if u_L and Σ correspond to the interaction V_L^0, and for $L(k)$ we use the expression

$$L=\frac{2}{\sin^2\delta_s}\int_0^\infty dr\int d\rho[V_L^0(\chi_s^2-\sin^2(kr+\delta_s)|\Phi(\rho)|^2] +(V_L-V_L^0)(\chi_s^2-\sin^2 kr|\Phi(\rho)|^2)]. \quad (5.8)$$

The modification of these formulas in the case of the Coulomb interaction will be considered in the following section in connection with proton–deuteron scattering.

6. PROTON-DEUTERON SCATTERING

Our above approach can be illustrated by the example of the scattering of a proton on a light nucleus; the description of this process presupposes that we know the corresponding characteristics for the scattering of a neutron on the nucleus. The condition (5.1) restricts the applicability of our approach to light nuclei with $Z \ll 10$. Below, we shall consider the simplest problem of this kind—proton–deuteron scattering at zero energy.

As before, we use the coordinates $\mathbf{r}$ (the vector joining p and the center of mass of the d) and $\boldsymbol{\rho}$ (the vector joining the components of the d). In these coordinates,

$$V_L=(|\mathbf{r}-\boldsymbol{\rho}/2|a_0)^{-1} \text{ and } V_L^0=(ra_0)^{-1}.$$

The wave function Φ is identical to the d wave function, which has the well-known form

$$(\varkappa/2\pi)^{1/2}e^{-\varkappa\rho}/\rho.$$

Transformations exactly like those in Sec. 4 again lead to formula (4.11), which determines the phase shift, in which l_1 now has the form

$$l_1=\int_0^\infty dr\int d\rho\left\{\ln\left(\frac{2r}{|a_0|}\right)\frac{d}{dr}[\chi_s^2-\sin^2(kr+\delta_s)|\Phi(\rho)|^2] +\left(\frac{1}{r}-\frac{1}{|\mathbf{r}-\boldsymbol{\rho}/2|}\right)(\chi_s^2-\sin^2 kr|\Phi(\rho)|^2)\right\}\Big/\sin^2\delta_s. \quad (6.1)$$

In the limit $k\to 0$, we obtain a formula similar to (4.3):

$$\tilde{a}^{-1}-a_s^{-1}=2a_0^{-1}(l_1(0)+2C), \quad (6.2)$$

where

$$\tilde{a}^{-1}=-\lim_{k\to 0}kA^2\operatorname{ctg}(\delta-\delta_L)$$

(it is this quantity that is extracted experimentally from pd scattering experiments),

$$a_S^{-1}=-\lim_{k\to 0}k\operatorname{ctg}\delta_s$$

is the reciprocal nd-scattering length, and

$$l_1(0)=\frac{1}{a_s^2}\int_0^\infty dr\int d\rho\left\{\ln\left(\frac{2r}{|a_0|}\right)\frac{d}{dr}[r^2\Psi_s^2(0)-(r-a_s)^2|\Phi(\rho)|^2] +\left(\frac{1}{r}-\frac{1}{|\mathbf{r}-\boldsymbol{\rho}/2|}\right)r^2[\Psi_s^2(0)-|\Phi(\rho)|^2]\right\}. \quad (6.3)$$

Here we have introduced for convenience the total wave function $\Psi_S(k)=\chi_S/kr$ of the nd system.

Although the relations (6.2) and (6.3) in principle solve the problem of describing pd scattering as $k\to 0$, we shall not here use the cumbersome data on the wave function Ψ_S, particularly since the experimental data for the scattering lengths themselves have an appreciable spread. For the purpose of an estimate, we shall take a simplified expression for $\Psi_S(0)$ and assume that it is equal to its asymptotic behavior $\Phi(\rho)(1-a_S/r)$ at r greater than the deuteron radius $\varkappa^{-1}$ (i.e., for $r \geq \xi\varkappa^{-1}$, $\xi \gtrsim 1$) and replace $\Psi_S(0)$ as a function of r by a constant at short distances:

$$\Psi_s(0)=\Phi(\rho)(1-a_s/r) \quad (r>\xi/\varkappa),$$
$$\Psi_s(0)=\Phi(\rho)(1-\varkappa a_s/\xi) \quad (r<\xi/\varkappa). \quad (6.4)$$

The reasonableness of this approximation was verified for the example of pp scattering: For matching radius ξr_0 ($r_0=2.3$ fm is the radius of the potential well) with $0.8<\xi<3$, the value of $\tilde{a}$ varies in the range from -7.8 to -14.3 fm; the experimental value is -7.8 fm.

Substitution of (6.4) in (6.2) and (6.3) shows that the second term in (6.3), which describes the dipole, quadrupole, etc., effects, makes a negligible contribution on account of the small numerical coefficients, and we ultimately obtain the simple formula (an elementary derivation of this formula is contained in our note Ref. 9)

$$\tilde{a}^{-1}=a_s^{-1}(1-2\xi/\varkappa a_0)+2a_0^{-1}(\ln(2\xi/\varkappa a_0)+2C-1/2). \quad (6.5)$$

As ξ varies from 1 to 3, formula (6.5) gives for the quartet state, for which $a_S=6.4$ F, a value of $\tilde{\alpha}$ in the interval 12.3–14.4 fm (the experimental value[8] is 10.2–13.2 fm). For the doublet state, $a_S=0.7$ fm, $\tilde{a}=0.9$–1.7 fm in accordance with Eq. (6.5), and $\tilde{a}=1.1$–1.5 fm according to the experiment of Ref. 8. We see that for all its simplifications, our approximation gives results that are entirely reasonable.

In following papers, we hope to apply the above approach to other problems, in particular, the problem of electron scattering on a heavy ion.

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APPENDIX

Using a direct generalization of the proofs of relations (2.1) and (2.3) (see the end of Sec. 2), we can show that these relations also remain valid for the Jost function of the complexes determined by (5.2) and (5.3). As regards the first of these relations, the matter reduces to taking the imaginary part of (5.3),

$$\frac{d\operatorname{Im}\ln u}{d\alpha}=\frac{2}{k}\int d\rho\int_0^\infty dr v_L\chi^2, \quad (A.1)$$

and generalizing the relation (2.5) to the case of complexes [this reduces to adding integrals over ρ on both sides of (2.5)], after which the right-hand side of (A.1) can be represented in the form $-d\delta/d\alpha$, which corresponds to (2.1).

To prove the validity of (2.3), it is necessary to con-

tinue (5.3) analytically to the neighborhood of the point $k = i\varkappa_n$. It is then necessary to introduce the solution $\varphi(\mathbf{r}, \boldsymbol{\rho})$ of the Schrödinger equation having the asymptotic behavior $\Phi(\boldsymbol{\rho})\exp(ikr)$ as $r \to \infty$. In terms of this function we can express the contents of the brackets in (5.3):

$$\chi\chi + i\chi^2 = i(|\varphi|^2 - \varphi^2 u^*/u)/2.$$

Finally, direct generalization of the arguments that lead lead to (2.7) (see Ref. 3) show that this relation is also valid for complexes. Ultimately, we obtain the well-known equation [see (2.6)]

$$\frac{dE_n}{d\alpha} = -\varkappa_n \frac{d\varkappa_n}{d\alpha} = \int d\rho \int_0^\infty dr v_L \chi_n^2,$$

which confirms the validity of Eq. (2.3), used in its derivation.

[1)] Here and below, Planck's constant and the reduced mass of the system are taken equal to unity.

[2)] We do not consider here the particular problem of the interference of the Coulomb interaction and the nuclear interaction in high-energy physics.

[3)] The combination $\bar{\chi}\chi + i\chi^2$ in (2.4) is not fortuitous; for this quantity is not subject to exponential growth on a large circle with Im $k > 0$ in accordance with property b) (see above).

[4)] The presence in (3.6) of $u_S(k) \cdot u_S(-k)$ may raise doubts, since, as was noted in Sec. 2, this quantity may have dynamical singularities for Im $k > 0$. However, these singularities are in the region $|k| R_S \gtrsim 1$, in which perturbation theory with respect to V_L is valid, and in such an approximation w no longer contains u_S^* (see above).

[5)] This can be seen by introducing $\zeta(r) = \exp(ikr)\psi/\varphi_L$ and defining w as $\zeta(0)$. The Schrödinger equation for ζ has a short-range effective potential, and this brings us back to the usual formulation of the problem of the properties of the Jost function.

[1]In: Problema neskol'kikh tel v yadernoĭ fizike (Few-Body Problems in Nuclear Physics), D4-80-271, JINR, Dubna (1980).

[2]L. D. Landau and Ya. A. Smorodinskiĭ, Zh. Eksp. Teor. Fiz. **14**, 269 (1944).

[3]M. L. Goldberger and K. M. Watson, Collision Theory, New York (1964) [Russian translation published by Mir, Moscow (1967)]; V. de Alfaro and T. Regge, Potential Scattering, North-Holland, Amsterdam (1965) [Russian translation published by Mir, Moscow (1966)].

[4]D. A. Kirzhnits, F. Yu. Kryuchkov, and N. Zh. Takibaev, Fiz. Elem. Chastits At. Yadra **10**, 741 (1979) [Sov. J. Part. Nucl. **10**, 289 (1979)].

[5]J. Jackson and J. Blatt, Rev. Mod. Phys. **22**, 77 (1950).

[6]A. E. Kudryavtsev, V. D. Mur, and V. S. Popov, Preprint ITÉF-180 [in Russian] (1980).

[7]V. S. Popov, A. E. Kudryavtsev, V. I. Lisin, and V. D. Mur, Zh. Eksp. Teor. Fiz. **80**, 1271 (1981) [Sov. Phys. JETP **53**, 650 (1981)].

[8]J. Van Oers and K. Brockman, Nucl. Phys, **A92**, 561 (1967).

[9]D. A. Kirzhnits and F. M. Pen'kov, Phys. Lett. (1982) (in press).

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