Lorentz group for two-dimensional integrable lattice systems

M. G. Tetel'man

Physicotechnical Research Institute, Gor'kiĭ State University

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It is shown that a continuous transformation group analogous to the Lorentz group is connected with integrable lattice systems. The generators of the Lorentz transformations (boosts) lead to transformation of an infinite series of pairwise commuting integrals of motion, which include the Hamiltonian of the system. In the continuum limit the lattice group breaks up into a Lorentz group and a group of transformations of higher integrals. An infinite-parametric expansion of the Lorentz lattice group, where an infinite set of boost generators corresponds to an infinite set of integrals of motion, is considered for a system of free fermions on the lattice.

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§1. INTRODUCTION

Factorized scattering theories and a number of models of quantum field theories and of statistical physics are completely integrable systems. Their characteristic feature is the presence of an infinite series of independent conservation laws, so that an exact solution can be obtained.

We define $S^{12}(\vartheta)$ as an operator acting in the tensor product of the first and second spaces, so that the matrix elements of $S^{12}(\vartheta)$ define a two-particle completely elastic scattering matrix of particles of species $i_{1,2}$ and $j_{1,2}; i_1+i_2 \to j_1+j_2$, while ϑ is the difference of the rapidities of the scattered particles:

$$[S^{12}(\vartheta)]_{i_1 i_2}^{j_1 j_2} = S_{i_1 i_2}^{j_1 j_2}(\vartheta). \tag{1}$$

For statistical-physics models, $S_{i_1 i_2}^{j_1 j_2}(\vartheta)$ is interpreted as the Boltzmann weight ascribed to the vertex of a two-dimensional lattice; i and j are fluctuating variables corresponding to the edges of the vertex, and ϑ is an anisotropy parameter associated with the angle at the vertex (see Fig. 1).

The conditions for full integrability are the Yang-Baxter relations. They are of the form

$$S^{12}(\vartheta_1)S^{13}(\vartheta_1+\vartheta_2)S^{23}(\vartheta_2) = S^{23}(\vartheta_2)S^{13}(\vartheta_1+\vartheta_2)S^{12}(\vartheta_1). \tag{2}$$

S^{23}, S^{13}, S^{12} act as unit operators in the spaces 1, 2 and 3, respectively. A particular case of relations (2) was first considered by Onsager.[1] Equations (2), formulated by Yang as conditions for the factorization of the multiparticle S matrix in scattering theory,[2] were used by Baxter as a basis for the solution of the eight-vertex model of statistical physics.[3]

The transfer matrix $T(\vartheta)$ acts in the tensor product of N spaces:

$$T(\vartheta) = \mathrm{Sp}_0[S^{01}(\vartheta)S^{02}(\vartheta)\dots S^{0N}(\vartheta)]. \tag{3}$$

The S matrices are multiplied as matrices in null space, and Sp_0 denotes a trace in null space (see Fig. 2). The product of transfer matrices is understood in the following sense:

$$T(\vartheta)T(\vartheta') = \mathrm{Sp}_0 \mathrm{Sp}_{0'} \prod_{n=1}^{N} [S^{0n}(\vartheta)S^{0'n}(\vartheta')]. \tag{4}$$

If the S matrices in (3) satisfy the Yang-Baxter relations (2), then transfer matrices with different parameters ϑ and ϑ' will commute.[4] This circumstance makes it possible to diagonalize the entire family of transfer matrices with the aid of the quantum method of the inverse problem, developed by Sklyanin, Takhtadzhyan, and Faddeev.[5]

We shall show that the Yang-Baxter relations lead to a certain group of transformations of a family of transfer matrices, which is the lattice analog of the Lorentz group.

Baxter,[6] Zamolodchikov,[4,7] and Stroganov[8] proposed independently unitarity relations for a transfer matrix, and used this relation to find the partition function for a number of models. In Sec. 3, with the Heisenberg XYZ model as the example, we demonstrate a method of constructing an excitation spectrum based on a solution of the unitarity condition.

§2. THE LORENTZ LATTICE GROUP

For the solutions of Eq. (2) at $\vartheta=0$ we use the usual initial condition: $S^{nm}(0)=P^{nm}$ is the permutation operator of the states of the nth and mth spaces and has the following properties

$$P^{nm}=P^{mn}; \quad P^{nm}P^{nm}=1; \quad P^{nm}S^{mk}=S^{nk}P^{nm}. \tag{5}$$

Denoting differentiation with respect to ϑ by a dot, we introduce the operator $\mathscr{H}^{nm}$, which acts nontrivially in the n and m spaces:

$$\mathscr{H}^{nm} = \dot{S}^{nm}(0)P^{nm}. \tag{6}$$

Differentiating Eq. (2) with respect to ϑ_2 and putting $\vartheta_2=0$, we obtain a differential equation for the S matrix:

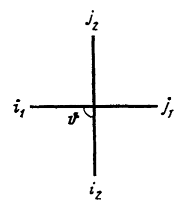

FIG. 1.

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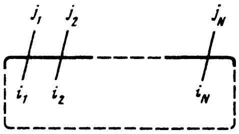

FIG. 2.

$$S^{12}(\vartheta)S^{13}(\vartheta)-S^{12}(\vartheta)S^{13}(\vartheta)=[S^{12}(\vartheta)S^{13}(\vartheta),\ \mathscr{H}^{23}]. \tag{7}$$

We now redesignate the spaces in which the operators in (7) act in the following manner: $1\to 0, 2\to n, 3\to n+1$. Multiplying (7) from the left by the product $\Pi\, S^{0m}(\vartheta)$ over $m<n$, and from the right by the same product over $m>n+1$, we obtain

$$\Big\{\prod_{m<n} S^{0m}(\vartheta)\Big\} S^{0n}(\vartheta)\Big\{\prod_{l>n} S^{0l}(\vartheta)\Big\}-\Big\{\prod_{m<n+1} S^{0m}(\vartheta)\Big\}\cdot$$

$$\cdot S^{0,n+1}(\vartheta)\Big\{\prod_{l>n+1} S^{0l}(\vartheta)\Big\}=\Big[\prod_{m=-\infty}^{\infty} S^{0m}(\vartheta),\mathscr{H}^{n,n+1}\Big]. \tag{8}$$

We multiply (8) by n and sum over n from $-\infty$ to $+\infty$. We find then that the transfer matrix satisfies the equation

$$\frac{dT(\vartheta)}{d\vartheta}=[G_1, T(\vartheta)],\quad G_1=-\sum_{n=-\infty}^{+\infty}(n+{}^1/_2)\mathscr{H}^{n,n+1}. \tag{9}$$

We propose here that the transfer matrix is bounded on a space of physical states for which the contribution of infinitely remote points is insignificant and the formal series in (9) converge. We note that Eq. (8) is invariant to changes of the normalization of the S matrix: $S^{nm}(\vartheta)\to\rho(\vartheta)S^{nm}(\vartheta)$, while Eq. (9) is not. The discarded boundary term violated this invariance. We normalize the transfer matrix in such a way that at maximum eigenvalue is independent of ϑ. Then G_1 causes the corresponding ground state to vanish. This guarantees nondegeneracy of the ground state with respect to ϑ.

By virtue of the initial condition for the S matrix, the transfer matrix at $\vartheta=0$ is the operator of translation by one step of the lattice:

$$T(0)=e^{iP}, \tag{10}$$

where P is the momentum operator. The Hamiltonian H_1 is defined as

$$H_1=-\frac{d\ln T(\vartheta)}{d\vartheta}\Big|_{\vartheta=0}. \tag{11}$$

It follows from (10) and (6) that

$$H_1=-\sum_n \mathscr{H}^{n,n+1}. \tag{12}$$

If $T(\vartheta)$ is taken to be the transfer matrix of the eight-vertex model, then, as shown by Baxter,[9] this yields the Hamiltonian of the Heisenberg quantum XYZ model.[10] It follows from (11) that the Hamiltonian commutes with the transfer matrix. This is easily seen also from (8) if one sums directly over n.

The expansion of the transfer matrix in powers of ϑ yields a set of commuting integrals I_n:

$$-i\ln T(\vartheta)=\sum_{n=0}^{\infty}\frac{(i\vartheta)^n}{n!}I_n,\quad I_0=P,\quad I_1=H_1. \tag{13}$$

Substituting (13) in (9) we obtain

$$[G_1, I_n]=iI_{n+1}, \tag{14}$$

$$[I_n, I_m]=0, \tag{15}$$

with $n,m=0,1,2,\ldots$. Here G_1 is the lattice analog of the generator of Lorentz transformations (boosts). It is seen from (9) and (12) that G_1 is connected with the density of the Hamiltonian in analogy with the continuum case. The only difference is that in the continuum theory the system (14) was closed at the second step, but for the lattice system G_1 generates an infinite chain of integrals. In the continuum limit I_{2n} goes over into P, and I_{2n+1} into H, while the algebra (14) contracts to the usual one.

The algebra of the higher integrals in the continuum limit is obtained by taking into account the next terms of the expansion in the lattice parameter.

The presence of continuum symmetry for the integrable system on an infinite lattice leads to degeneracy of the states. If a certain state is characterized by a set of excitation rapidities $\{\beta_j\}$, then the state with rapidities $\{\beta_j+\beta\}$ (j takes on the same set of values for both cases) is obtained from the preceding one with the aid of the boost operator $\exp(i\beta G_1)$, which will be unitary if the density of the Hamiltonian (6) is Hermitian. The Lorentz lattice group, just as the ordinary one, leads to a fully defined parametrization of the energy and momentum of the excitations in rapidity. In contrast to the continuum case, where the parametrization is expressed in single-period functions, the lattice group leads to a doubly periodic parametrization. The imaginary period, just as for the continuum case, as connected with the compactness of the Euclidean Lorentz group. The real period is a consequence of the lattice structure of the space. Since the excitation momentum is bounded from above by the reciprocal lattice parameter 1/8, it follows that there exist boosts that transform the momentum into an equivalent point in a neighboring Brillouin zone. The Lorentz group leads to identities for the correlation functions on the lattice, and this may facilitate the calculation.

Out of the densities of the higher integrals we can make up higher boosts and thus obtain a wider algebra than (14) and (15). We construct now an algebra with higher boosts for free fermions on a lattice.

We introduce the fermion creation and annihilation operators in a given site of a one-dimensional lattice, a_n^+ and a_n, which satisfy the anticommutation relation

$$\{a_n, a_m\}=\{a_n^+, a_m^+\}=0,\quad \{a_n, a_m^+\}=\delta_{nm} \tag{16}$$

(n and m are integers). We consider for simplicity the massless case. The series of Hamiltonians is defined as follows:

$$H_l=\frac{i}{2}\sum_{n=-\infty}^{\infty}(a_n^+a_{n+l}-a_{n+l}^+a_n) \tag{17}$$

(l are integers). The corresponding boosts are

$$G_l=-\frac{i}{2}\sum_{n=-\infty}^{\infty}\Big(n+\frac{l}{2}\Big)(a_n^+a_{n+l}-a_{n+l}^+a_n). \tag{18}$$

We note that

$$H_{-l}=-H_l,\ G_{-l}=-G_l. \tag{19}$$

Calculating the commutators directly, we can verify that the following relations are valid

$$[H_l, H_{l'}]=0, \quad [G_l, H_{l'}]=il'(H_{l+l'}-H_{l'-l}),$$
$$[G_l, G_{l'}]=i((l+l')G_{l-l'}-(l-l')G_{l+l'}) \tag{20}$$

(l and l' are integers). We define the transfer matrix in terms of the Hamiltonian H_l so as to satisfy Eqs. (9), (10), and (11):

$$-i\ln T(\vartheta)=P+\sum_{n=1}^{\infty} f_n(\vartheta)H_n. \tag{21}$$

The functions $f_n(\vartheta)$ satisfy the equations

$$f_n(\vartheta)=i[(n-1)f_{n-1}(\vartheta)-(n+1)f_{n+1}(\vartheta)], \quad n=1, 2, \ldots. \tag{22}$$

When account is taken of the initial conditions that follow from (10) and (11), the solution of (22) is

$$f_n(\vartheta)=\frac{1}{n}(\operatorname{th} i\vartheta)^n, \tag{23}$$

from which it follows that

$$-i\ln T(\vartheta)=P+\sum_{n=1}^{\infty}\frac{(\operatorname{th} i\vartheta)^n}{n}H_n. \tag{24}$$

The Jordan-Wigner transformation can be used to express the fermion operators in terms of the spin variables and to realize the algebra (20) for the spin chain. We take the transformation in the form

$$a_n^{\pm}={}^1/_2(\pm i)^n(\sigma_n^x\pm i\sigma_n^y)\prod_{m<n}\sigma_m^z, \tag{25}$$

where σ_n^x, σ_n^y and σ_n^z are the ordinary Pauli matrices corresponding to the site n. It is easy to verify that the Pauli-matrix algebra conforms to relations (16). In terms of the spin variables, H_1 yields the Hamiltonian of the XX-model:

$$H_1={}^1/_4\sum_n(\sigma_n^x\sigma_{n+1}^x+\sigma_n^y\sigma_{n+1}^y). \tag{26}$$

It is natural to assume that there exists an extension of the Lorentz lattice algebra (14) to higher boosts for an arbitrary integrable system, in analogy with a system of free fermions.

§3 CALCULATION OF THE SPECTRUM OF THE *XYZ* MODEL ON THE BASIS OF THE UNITARITY RELATION FOR THE *T* MATRIX

A one-dimensional fully anisotropic spin chain (the Heisenberg model) has an infinite set of conservation laws. The Hamiltonian of the model

$$H={}^1/_2\sum_n(J_x\sigma_n^x\sigma_{n+1}^x+J_y\sigma_n^y\sigma_{n+1}^y+J_z\sigma_n^z\sigma_{n+1}^z) \tag{27}$$

is the logarithmic derivative of the transfer matrix of the eight-vertex Baxter model,[9] $J_{x,y,z}$ are arbitrary constants, and $\sigma_n^{x,y,z}$ are the spin operators introduced in (25). Expressing (27) in terms of the fermion variables, we obtain with the aid of (25) the Hamiltonian of the Thirring lattice model, which is known to be equivalent to the sine-Gordon model in the continuum limit.[10,12]

The Baxter vertex matrix has the automorphism property for shifts of ϑ by half-periods ξ and ξ' (Ref. 13):

$$S^{12}(\vartheta+\xi)=\sigma_1^zS^{12}(\vartheta)\sigma_1^z=\sigma_2^zS^{12}(\vartheta)\sigma_2^z,$$
$$S^{12}(\vartheta+\xi')=\sigma_1^xS^{12}(\vartheta)\sigma_1^x=\sigma_2^xS^{12}(\vartheta)\sigma_2^x. \tag{28}$$

Here 2ξ is the real period and $2\xi'$ imaginary, with $\pi<\xi<\infty$. From the structure of the transfer matrix (3) it follows that the periods of $T(\vartheta)$ are ξ and ξ'. The Baxter matrix has a so-called crossing symmetry that yields at a certain normalization of the angle

$$S_{i_1i_2}^{j_1j_2}(\vartheta)=S_{i_1j_2}^{j_1i_2}(\pi-\vartheta)=S_{j_1i_2}^{i_1j_2}(\pi-\vartheta), \tag{29}$$

ξ and ξ' correspond to the given normalization of ϑ. The crossing symmetry (29) yields for the transfer matrix

$$T_{(i_1i_2\ldots i_N)}^{(j_1j_2\ldots j_N)}(\vartheta)=T_{(i_Ni_{N-1}\ldots i_1)}^{(j_Nj_{N-1}\ldots j_1)}(\pi-\vartheta). \tag{30}$$

We introduce an inversion operator Q such that (30) is rewritten in the form

$$T(\vartheta)=QT(\pi-\vartheta)Q. \tag{31}$$

The inversion reverses the sign of the momentum:

$$e^{iP}Q=Qe^{-iP}. \tag{32}$$

Let $|p\rangle$ be the eigenvalue of the transfer matrix corresponding to the momentum p:

$$T(\vartheta)|p\rangle=|p\rangle\Lambda_p(\vartheta), \quad \Lambda_p(0)=e^{ip}. \tag{33}$$

Equation (31) then leads to the relation

$$\Lambda_p(\vartheta)=\Lambda_{-p}(\pi-\vartheta). \tag{34}$$

In the limit of an infinite lattice, the transfer matrix satisfies the unitarity relation (see §1), which takes the form

$$T(\vartheta)T(\pi+\vartheta)=1. \tag{35}$$

The transfer matrix is constructed here out of unitarized S matrices, which satisfy the unitarity condition

$$S^{12}(\vartheta)S^{21}(-\vartheta)=1. \tag{36}$$

This renormalization of the transfer matrix corresponds to subtracting the energy of the ground state of the Hamiltonian. The relation (35) leads to a 2π periodicity of the transfer matrix on an infinite lattice. The real period of the transfer matrix on a finite lattice, ξ, is in the general case not equal to 2π. The change of the periods in the infinite-lattice limit is due to the change in the analytic properties of the eigenvalues of the transfer matrix $\Lambda(\vartheta)$ in this limit.

We consider now states with zero momentum on a finite lattice. The eigenvalues $\Lambda_0(\vartheta)$ corresponding to such states are meromorphic functions whose singularities can be obtained at points that are stationary under the crossing relation (34). Accurate to within the addition of periods, such points are

$$\vartheta_c={}^1/_2(\pi+\xi s), \quad s=0, \pm 1, \pm 2, \ldots. \tag{37}$$

In the limit of an infinite lattice, the singularities on the lines $\operatorname{Re}\vartheta=\vartheta_c$ condense and form cuts. The unitarity (35) begins to be satisfied everywhere with the exception of the cuts. As a result the eigenvalues of the transfer matrix, corresponding to states with zero momentum, can have singularities that are closest to

$\vartheta=0$ at the points

$$\vartheta_s=-\frac{s}{2}(\xi-\pi); \quad s=1,2,\ldots,s_{max}, \quad s_{max}=\left[\frac{\pi}{\xi-\pi}\right] \tag{38}$$

or

$$\vartheta=\pi/2. \tag{39}$$

The case (38) is possible at $\pi<\xi<2\pi$. Regarding the anisotropy constants in (27) as functions of the periods ξ and ξ' given by the Baxter parametrization, it can be seen that this case corresponds to attraction, while $\xi>2\pi$ corresponds to repulsion.

The solutions of the crossing conditions and the unitarity conditions will be sought in the class of meromorphic functions with periods 2π and ξ'. The function

$$f(\vartheta)=\frac{d\ln\Lambda(\vartheta)}{d\vartheta} \tag{40}$$

satisfies the equations

$$f(\vartheta)+f(\pi+\vartheta)=0, \tag{41}$$

$$f(\vartheta)+\bar{f}(\pi-\vartheta)=0, \tag{42}$$

$\bar{f}$ corresponds to a state with reversed momentum. Let $f(\vartheta)$ have l poles in the region $0<\mathrm{Re}\,\vartheta<\pi$ at the points a_α; $\alpha=1,2,\ldots,l$. Since the excitation energy is positive, $-f(0)>0$, we obtain the solution of (41) in the form

$$f(\vartheta)=\sum_{\alpha=1}^{l}\frac{1}{\mathrm{sn}[\lambda(\vartheta-\vartheta_\alpha)]} \quad \lambda=\frac{2K}{\pi}. \tag{43}$$

Here $f(\vartheta)$ is expressed in terms of the elliptic sine with modulus k, defined by the equation

$$\xi'/\pi=K'/K, \tag{44}$$

K and K' are complete elliptic interval of the first kind with moduli k and k' respectively.[14]

The state with reversed momenta corresponds to the set of poles $\{\bar{a}_\alpha\}$. Equation (42) leads to a connection between these sets

$$a_\alpha+\bar{a}_\beta=\pi \pmod{\xi'} \tag{45}$$

for certain α and β. It follows from (40) that

$$\ln\Lambda(\vartheta)=ip+\int_0^\vartheta f(\vartheta)\,d\vartheta. \tag{46}$$

Stipulating crossing for $\Lambda(\vartheta)$ and using (41), we obtain

$$p=\frac{i}{2}\int_0^\pi f(\vartheta)\,d\vartheta. \tag{47}$$

The fact that the momentum is real leads to the condition

$$\bar{a}_\beta=\bar{a}_\gamma \pmod{\xi'} \tag{48}$$

for certain β and γ. The bar corresponds to complex conjugation. Equation (48) means also that the energy is real.

We introduce new variables for the poles b_α:

$$b_\alpha=a_\alpha-\pi/2-\xi'/2. \tag{49}$$

The values of b_α satisfy equations that follow from (45) and (48):

$$b_\alpha+\bar{b}_\gamma=0, \quad \bar{b}_\alpha-\bar{b}_\beta=0 \pmod{\xi'}, \qquad \alpha,\beta,\gamma=1,2,\ldots,l. \tag{50}$$

Calculating the integrals (see Ref. 14) and taking (49) into account, we obtain after simple transformations

$$p=\frac{i}{2\lambda}\sum_{\alpha=1}^{l}\ln\left(\frac{1-k\,\mathrm{sn}(\lambda b_\alpha)}{1+k\,\mathrm{sn}(\lambda b_\alpha)}\right), \tag{51}$$

$$\ln\Lambda(\vartheta)=ip+\frac{1}{\lambda}\sum_{\alpha=1}^{l}\left\{\ln\left(\frac{1+k\,\mathrm{sn}[\lambda(\vartheta-b_\alpha)]}{1-k\,\mathrm{sn}(\lambda b_\alpha)}\right)-\ln\left(\frac{\mathrm{dn}[\lambda(\vartheta-b_\alpha)]}{\mathrm{dn}(\lambda b_\alpha)}\right)\right\}. \tag{52}$$

The state energy in terms of b_α is of the form

$$\mathscr{E}=\sum_{\alpha=1}^{l}k\frac{\mathrm{cn}(\lambda b_\alpha)}{\mathrm{dn}(\lambda b_\alpha)}. \tag{53}$$

Assume that there is one pole, $l=1$. It follows from (50) that

$$\mathrm{Re}\,b=0, \quad \mathrm{Im}\,b=-\mathrm{Im}\,\bar{b}. \tag{54}$$

We introduce the rapidity $\beta=\mathrm{Im}\,b$. Thus, the excitation spectrum is

$$p=\frac{i}{2\lambda}\ln\left(\frac{1-k\,\mathrm{sn}(\lambda i\beta)}{1+k\,\mathrm{sn}(\lambda i\beta)}\right), \quad \mathscr{E}=k\frac{\mathrm{cn}(\lambda i\beta)}{\mathrm{dn}(\lambda i\beta)}. \tag{55}$$

To change over to the continuum limit it is necessary to separate from p and $\mathscr{E}$ the lattice parameter a: $p\to pa$, $\mathscr{E}\to\mathscr{E}a$. As $a\to 0$, degeneracy $k=ma+o(a)$ sets in. In the limit, (55) leads to the ordinary spectrum

$$p=m\,\mathrm{sh}\,\beta, \quad \mathscr{E}=m\,\mathrm{ch}\,\beta, \quad m=\lim_{a\to 0}(k/a). \tag{56}$$

The spectrum of the two-particle excitation is determined by two poles b_1 and b_2. Besides solutions of type (54), in the case of attraction there are also others:

$$\mathrm{Re}\,b_1=-\mathrm{Re}\,b_2=b, \quad \mathrm{Im}\,b_1=\mathrm{Im}\,b_2=\beta. \tag{57}$$

$$\bar{\beta}=-\beta, \quad \bar{b}=b. \tag{58}$$

At $\beta=0$ the momentum $p=0$, and consequently b is determined by the condition (38)

$$b=\frac{\pi}{2}-s\frac{\xi-\pi}{2}-\frac{\xi'}{2}, \quad s=1,2,\ldots,s_{max}. \tag{59}$$

In the continuum limit we obtain an excitation spectrum of the type (56) with rapidity β and with mass

$$M_s=2m\sin\left(s\frac{\xi-\pi}{2}\right), \quad s=1,2,\ldots,s_{max}. \tag{60}$$

The spectrum of the states with arbitrary number of poles is constructed similarly. The spectrum of the low-lying excitations obtained in this manner coincides with the results of Ref. 15.

§4. CONCLUSION

We note that the spectrum of the transfer matrix has a universal character (if the unitarity and crossing relations are valid). The parameters of the spectrum are determined by the periods of the initial parametrization for the transfer matrix. This circumstance is connected with the presence of a continuous symmetry group for integrable systems on an infinite lattice. As a result, the problem of constructing integrable systems can be formulated as the problem of constructing the representation of the lattice Lorentz group.

It is natural to assume that an analogous generalization of the Lorentz group exists for integrable systems in a space with more than two dimensions.

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