Multiple scattering of electrons in axial channeling

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The classical theory of axial channeling of negative particles is developed. In the case of ultrarelativistic electrons it is applicable for energies above about 10 MeV. For the first time equations of the Fokker–Planck type are obtained which describe the passage of axially channeled negative particles through a thick single crystal. The calculation takes into account the diffusion in the transverse energy and angular momentum due to multiple scattering by electrons and thermal vibrations of nuclei. A distribution of particles in statistical equilibrium is considered for channeling, and also the redistribution of the flux of particles in the transverse plane. Diffusion coefficients and characteristic dechanneling lengths in Si and W are given, and also the dechanneling function obtained as the result of numerical solution of the kinetic equation.

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1. INTRODUCTION

It is well known that the motion of a particle in a single crystal under conditions of channeling is unstable. Multiple scattering due to inelastic scattering by electrons and thermal vibrations of the atoms even in an ideal crystal leads to an increase of the transverse energy of the particle and eventually to dechanneling. This process has been discussed previously for positively charged nonrelativistic heavy particles (protons, α particles, ions) and the respective kinetic equations of the Fokker-Planck type were obtained on the basis of classical mechanics for axial channeling[1,2] and planar channeling.[3] A similar equation for planar channeling of heavy particles was obtained by the methods of quantum mechanics,[4,5] and has been shown[5] to agree with the equations previously obtained[3] by the methods of classical mechanics. Since the quantum calculations require solution of the Schrödinger equation, they were carried out only for model cases of rectangular[4] and oscillator[5] potential wells. At the present time many papers have been devoted to the dechanneling of heavy particles, on the basis of classical mechanics and the kinetic equations (see the review by Gemmel[6]). Recently, for example, a comparison was made of the experimental results on axial dechanneling of H^+ and D^+ in Si and Ge with the numerical solution of the classical kinetic equation, and good agreement was demonstrated.[7]

The applicability of classical mechanics for description of channeling of both heavy particles and light particles (positrons and electrons) (in the latter case they must be ultrarelativistic) was justified by Lindhard.[8-10] Comparisons were also made of theoretical calculations of the yield of backscattered particles according to quantum mechanics and classical mechanics with the experimental data for positrons and electrons[11-14] and it was shown that the results of the classical theory agree with quantum mechanics and with experiment at energies $\gtrsim 1$ MeV for positrons and $\gtrsim 10$ MeV for electrons. The criterion for agreement used in those studies is the condition that the number of bound states in the potential well be much greater than unity. Multiple scattering was not taken into account here, and only trajectories in the averaged potential or the wave functions were calculated.

Modeling of trajectories of electrons with energy 20 MeV in MgO by the method of binary collisions with allowance for thermal vibrations of the nuclei also gave good agreement with experiment for the orientation dependence of the passage of electrons along a crystallographic axis.[15]

In this way it has been well established that quantum effects are observed in experiments on backscattering[11-14] and transmission[14,16] of electrons at energies ~1–10 MeV. Accordingly, the development of the quantum theory of channeling has been essentially limited to this energy range, and the discussion reduced to calculation of wave functions, energy levels, and the population of these levels in a thin crystal (see for example Ref. 17) without taking into account dechanneling, i.e., the redistribution of particles over the energy levels as the result of multiple scattering.

Recently as a result of the observation of the characteristic spontaneous x rays of light charged particles predicted previously in Ref. 18, considerable interest has arisen in the question of the dechanneling of these particles, in particular at high energies (in the 1-GeV region), where the intensity of the radiation is extremely high. The quantum nature of the motion of channeled particles, i.e., the discrete structure of the energy spectrum, is important in the radiation, generally speaking, up to higher energies than in backscattering. Individual transitions in axial channeling of electrons have been observed[19] at energies 1–3 MeV, and in planar channeling—at energies 56 MeV.[20] At the same energy 56 MeV in the axial case[20] and at 900 MeV in the planar case[21] individual transitions are not resolved.

Thus, classical mechanics can be used in description of backscattering, transmission, and radiation of electrons in the axial case beginning at energies ~10 MeV, and in the planar case at energies ~100 MeV (for positrons these limits are substantially lower). The kinetic equation describing planar dechanneling of nonrelativistic heavy particles is easily generalized to the case of relativistic particles, both positive and negative.[22] The equation for axial dechanneling is also easily generalized to the case of relativistic positive particles.

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The axial dechanneling of electrons, however, requires a new discussion in view of the two-dimensional nature of the bound transverse motion in the attractive field created by the average potential of an atomic string and the presence, in addition to the transverse energy, of an additional integral of the motion, namely the angular momentum.

In the present work on the basis of classical mechanics we obtain a kinetic equation of the Fokker-Planck type which describes the multiple scattering of axially channeled electrons (and also of other negative particles). The equation takes into account diffusion both in the transverse energy and in the angular momentum. The diffusion coefficients are expressed in terms of the rms angle for multiple scattering by electrons and thermal vibrations of the nuclei. Expressions are obtained for the initial distribution function and the distribution of the flux of channeled electrons in the transverse plane. The two-dimensional Fokker-Planck equation derived and the expressions obtained for the diffusion coefficients, the initial distribution function, and the flux distribution in the transverse plane differ from those known previously for the one-dimensional Fokker-Planck equation.[1-5] The theory is formulated in a closed form which permits numerical integration of the resulting equation. In contrast to the method of binary collisions, it permits a more general discussion also for large crystal thicknesses. We give values of the diffusion coefficients, estimates of the characteristic dechanneling lengths for Si and W crystals, and a calculation of the dechanneling function by numerical solution of the equations obtained.

The equation derived in the present work can be used for calculation of the real spectrum of radiation of electrons in axial channeling of ultrarelativistic particles. Up to the present time theoretical studies have not taken into account the influence of multiple scattering on the radiation spectrum of electrons in channeling. As was shown in the case of planar channeling of positrons,[23] it substantially affects the radiation spectrum. Formulas for the radiation of particles moving along specified trajectories exist in a number of papers[23-27] (see also the review articles[28,29]). In particular, the general nondipole case of radiation in planar channeling[23,25,27,28] and axial channeling[26,27] has been considered. In a previous study[27] formulas were obtained for the first time in a classical discussion for the dependence of the frequency of the individual harmonics of the radiation on the angle of emission of the photon and on the energy of the particle in the nondipole case. The criticism of this work by Baĭer *et al.*[30] is incorrect. The formulas in the second article of Ref. 27 for the planar case are valid in the general case, and those for the axial case are valid under certain conditions on the parameters of the problem: the trajectory of the particle must be close to circular, or for arbitrary eccentricities in the pre-dipole case $(E_\perp E)^{1/2}/mc^2 \lesssim 1$ ($E_\perp$ is the energy of the transverse motion).

2. KINETIC EQUATION

In channeling of relativistic particles the transverse motion is nonrelativistic, since the transverse energy takes on values of the order of magnitude of the averaged potential of the atomic string, which amounts to 100–1000 eV. The longitudinal motion with relativistic velocity is felt only in the increase of the particle mass and, naturally, in the Coulomb interaction cross section. Since the characteristic dechanneling lengths are much less than the stopping lengths, the channeled-particle flux can be described with sufficient accuracy by a distribution function only in the transverse motion variable, not taking into account changes in the energy of the channeled particles.

Multiple scattering by electrons and thermal vibrations of the atoms leads to diffusion of the particles in their transverse momenta. Using the expressions for the integrals of the motion in a field $U[r=(x^2+y^2)^{1/2}]$:

$$E_\perp = p_\perp^2/2M + U(r), \quad M_z = xp_y - yp_x,$$

where M_y is the angular momentum with respect to an axis z directed along an atomic string, $M = E/c^2$ is the mass of the relativistic electron, and $\mathbf{p}_\perp = \mathbf{i}p_x + \mathbf{j}p_y$ is the transverse momentum, we can find the corresponding diffusion coefficients in the space of $E_\perp$ and M_z. (For $E_\perp$ this is done in exactly the same way as in the discussion of dechanneling of positive particles.[1] The difference is only in the use in the present case of a different formula for the binding energy and momentum of an ultrarelativistic particle: $E \approx pc$.) As the result of simple calculations and averaging over an ensemble of electrons and atoms of the crystal which are undergoing thermal vibrations, we obtain the average and rms increments for an electron moving at distance r from the string:

$$\begin{aligned} &\overline{\Delta E_\perp} = {}^1/_2 E\overline{\Delta\theta^2}, \quad \overline{\Delta M_z} = 0, \\ &\overline{\Delta E_\perp^2} = E(E_\perp - U(r))\overline{\Delta\theta^2}, \\ &\overline{\Delta M_z^2} = {}^1/_2 p^2 r^2 \overline{\Delta\theta^2}, \quad \overline{\Delta E_\perp \Delta M_z} = {}^1/_2 M_z E\overline{\Delta\theta^2}, \end{aligned} \tag{1}$$

where $\overline{\Delta\theta^2}$ is the mean square scattering angle.

If the time of motion of the particle in a channel (the thickness of the crystal) is significantly greater than the characteristic period of its oscillation, then the expressions (1) must be averaged over the trajectories of the motion, assuming a uniform distribution of the particles in phase or, in other words, a distribution in statistical equilibrium,[1)] which was first introduced into channeling theory by Lindhard.[8]

Statistical equilibrium leads to the important phenomenon of redistribution of the channeled-particle flux over the transverse coordinates. The form of this distribution, which is established in a time of the order of the period of oscillation in the channel, is well known for positive particles.[31] In planar channeling of negative particles it does not change.[22]

We shall determine the form of this distribution for axially channeled negative particles. The equilibrium distribution is described by a function which depends only on the integrals of the motion. The probability of finding at a point (r, φ) a particle which has given values of $E_\perp$ and M_z is obviously determined by the following expression:

$$\begin{aligned} \frac{d^2W}{r\,dr\,d\varphi} &\propto \int \delta\left(U(r) + \frac{p_x^2+p_y^2}{2M} - E_\perp\right)\delta(yp_x + xp_y - M_z)\,dp_x\,dp_y \\ &= (2M)^{1/2} r^{-1}[E_\perp - U(r) - M_z^2/2Mr^2]. \end{aligned} \tag{2}$$

We see that it, naturally, does not depend on the angle φ. From Eq. (2) we find that the normalized distribution over the distance to the atomic string r has the form

$$\frac{dW}{dr}=\frac{1}{T}\left(\frac{2M}{E_\perp-U(r)-M_z^2/2Mr^2}\right)^{1/2}, \tag{3}$$

where the normalization factor

$$T=\int\left(\frac{2M}{E_\perp-U(r)-M_z^2/2Mr^2}\right)^{1/2}dr \tag{4}$$

coincides with the period of motion of the particle.

Thus, the statistical distribution (3) has the same form as for one-dimensional motion[3] with use of an effective potential $U(r)+M_z^2/2Mr^2$. This result is quite clear, since we are dealing with periodic motion. For a uniform distribution in phase, the probability of finding a particle at a given point is proportional to the time,

$$dW=\frac{dt}{T}=\frac{dr}{Tv_r}=\frac{dr}{T[E_\perp-U(r)-M_z^2/2Mr^2]^{1/2}}.$$

From this it follows also that averaging over the transverse coordinate r with use of a statistical distribution is equivalent to averaging over the period of motion.

The kinetic equation of the Fokker-Planck type will have the form

$$\frac{\partial F(E_\perp,M_z,t)}{\partial t}=\frac{\partial^2}{\partial E_\perp^2}\left[\left\langle\frac{\Delta E_\perp^2}{2\Delta t}\right\rangle F\right]+\frac{\partial^2}{\partial E_\perp\partial M_z}\left[\left\langle\frac{\Delta E_\perp\Delta M_z}{\Delta t}\right\rangle F\right]$$
$$+\frac{\partial^2}{\partial M_z^2}\left[\left\langle\frac{\Delta M_z^2}{2\Delta t}\right\rangle F\right]-\frac{\partial}{\partial E_\perp}\left[\left\langle\frac{\Delta E_\perp}{\Delta t}\right\rangle F\right]-\frac{\partial}{\partial M_z}\left[\left\langle\frac{\Delta M_z}{\Delta t}\right\rangle F\right], \tag{5}$$

where the quantities enclosed in the angle brackets are the coefficients of Eq. (1) averaged over the period of oscillation T of a particle with the given $E_\perp$ and M_z:

$$\langle X\rangle=\frac{2}{T}\int_{r_{min}}^{r_{max}}\frac{XM^{1/2}\,dr}{[2(E_\perp-M_z^2/2Mr^2-U(r))]^{1/2}}. \tag{6}$$

It is easy to show that they satisfy the relations

$$\left\langle\frac{\Delta E_\perp}{\Delta t}\right\rangle=\frac{1}{T}\left\{\frac{1}{2}\frac{\partial}{\partial E_\perp}\left[T\left\langle\frac{\Delta E_\perp^2}{\Delta t}\right\rangle\right]+\frac{1}{2}\frac{\partial}{\partial M_z}\left[T\left\langle\frac{\Delta E_\perp\Delta M_z}{\Delta t}\right\rangle\right]\right\}, \tag{7}$$

$$\left\langle\frac{\Delta M_z}{\Delta t}\right\rangle=\frac{1}{T}\left\{\frac{1}{2}\frac{\partial}{\partial M_z}\left[T\left\langle\frac{\Delta M_z^2}{\Delta t}\right\rangle\right]+\frac{1}{2}\frac{\partial}{\partial E_\perp}\left[T\left\langle\frac{\Delta M_z\Delta E_\perp}{\Delta t}\right\rangle\right]\right\}.$$

By means of these relations, Eq. (5) is transformed to the more compact form

$$\frac{\partial F}{\partial t}=\frac{\partial}{\partial E_\perp}\left[\left\langle\frac{\Delta E_\perp^2}{2\Delta t}\right\rangle T\frac{\partial}{\partial E_\perp}\frac{F}{T}\right]+\frac{\partial}{\partial E_\perp}\left[\left\langle\frac{\Delta E_\perp\Delta M_z}{2\Delta t}\right\rangle T\frac{\partial}{\partial M_z}\frac{F}{T}\right]$$
$$+\frac{\partial}{\partial M_z}\left[\left\langle\frac{\Delta M_z\Delta E_\perp}{2\Delta t}\right\rangle T\frac{\partial}{\partial E_\perp}\frac{F}{T}\right]+\frac{\partial}{\partial E_\perp}\left[\left\langle\frac{\Delta M_z^2}{2\Delta t}\right\rangle T\frac{\partial}{\partial M_z}\frac{F}{T}\right]. \tag{8}$$

A more rigorous proof of Eqs. (5)–(8) by means of the method developed in Refs. 1 and 3 is given in the Appendix.

3. INITIAL DISTRIBUTION

When a particle crosses the crystal boundary, on passage of a time of the order of several periods of oscillation there is established a distribution in statistical equilibrium, which must taken as the initial condition for the kinetic equation (8). If the incident beam has a distribution in transverse momentum $\partial^2N/\partial p_x\partial p_y$, then the initial distribution $F_0(E_\perp,M_z)$ can be found by means of the integral

$$F_0(E_\perp,M_z)=\int\frac{\partial^2N}{\partial p_x\partial p_y}\delta\left(E_\perp-\frac{p_x^2+p_y^2}{2M}-U(r)\right)\delta(xp_y-yp_x-M_z)\,dx\,dy\,dp_x\,dp_y. \tag{9}$$

For ideal collimation of beam incident on the crystal at an angle ψ_{in} to the string, we obtain

$$F_0(E_\perp,M_z)=4\Big/\left[\pi r_0^2\left(p_{\perp in}^2-\frac{M_z^2}{r_{in}^2}\right)^{1/2}\left|\frac{dU(r=r_{in})}{dr}\right|\right], \tag{10}$$

where F_0 has been normalized to one particle and one atomic string, $p_{\perp in}=p\psi_{in}$ is the initial transverse momentum, r_{in} is the initial point of incidence of the particle with the condition that it acquire a given transverse energy

$$E_\perp=U(r_{in})+p_{\perp in}/2M,$$

and πr_0^2 is the area per string.

Using the expression for the averaged potential of an atomic string obtained by Lindhard[8] by approximation of the Thomas-Fermi atomic potential (the so-called standard Lindhard potential),

$$U(r)=Ze^2d^{-1}\ln(1+3a^2/r^2), \tag{11}$$

where $a=0.45Z^{-1/3}$ Å is the Thomas-Fermi screening constant, d is the period of the atomic string, and Z is the atomic number of the target, it is easy to express the initial distribution (10) in explicit form in terms of $E_\perp$ and M_z:

$$F_0(E_\perp,M_z)=\frac{6a^2d}{Z\pi r_0^2e^2}\frac{D}{(D-1)^2}\left(\frac{p_{\perp in}^2 3a^2}{1-D}-M_z^2\right)^{-1/2},$$
$$D=\exp\frac{E_\perp-p_{\perp in}^2/2M}{Ze^2d^{-1}}. \tag{12}$$

4. DIFFUSION COEFFICIENTS

As can be seen from Eq. (1), all of the coefficients are expressed in terms of the increment of the transverse energy $\overline{\Delta E_\perp}=\frac{1}{2}E\overline{\Delta\theta^2}$. The increment of the transverse energy as the result of multiple scattering by electrons can be expressed in terms of the ionization loss[8]:

$$\overline{\Delta E_\perp}=\alpha\frac{m}{M}\overline{\frac{\Delta E}{E}},$$

where α is the fraction of energy loss in close collisions, which is approximately equal to 1/2 at high energies. The energy loss $\overline{\Delta E}$ in close collisions can approximately be considered proportional to the electron density:

$$\overline{\Delta E}=\frac{4\pi e^4NZL_e}{mv^2}n(r)\Delta t, \tag{13}$$

where $n(r)$ is the profile of the electron density in the channel, N is the density of atoms, v is the velocity of the particle, and L_e is the Coulomb logarithm with inclusion of relativistic corrections.[32] For the standard Lindhard potential (11) we obtain the following electron density profile:

$$n(r)=\left(\frac{r_0}{a}\right)^2\frac{3}{[3+(r/a)^2]^2}.$$

The averaging over the period of oscillation (over the impact parameters), in accordance with Eq. (3), can easily be carried out analytically or numerically. Analytically this averaging can be carried out only for certain simple potentials, for example, $1/r$, $1/r^2$, or r^2.

For multiple scattering by thermal vibrations the formulas are more complicated. In addition, since negative particles travel in the immediate vicinity of the atomic string, the Lindhard approximation,[8] which uses an expansion in $u_\perp/r$, where $u_\perp$ is the average amplitude of thermal vibrations, is generally speaking inapplicable. Recently Ohtsuki *et al.*[33] showed for nonrelativistic protons that at impact parameters $r \lesssim u_\perp$ the multiple scattering is determined by the same expression as in an amorphous material with appropriate allowance for the density distribution of the atoms of the string over the transverse coordinates, which is determined by the thermal vibrations.

In small-angle scattering, which we are considering, according to Bohr[34] the relativistic effects reduce to only an increase of the particle mass. This is clear also from the general postulate of channeling theory regarding the influence of relativity, which was discussed above. Thus, the nonrelativistic formulas for multiple scattering by nuclei are easily generalized. In the Ohtsuki approximation[33] we therefore have

$$\frac{\overline{\Delta E_\perp}}{\Delta t}=\frac{1}{2}E\frac{\overline{\Delta\theta_R^2}}{\Delta t}P(r),\quad \frac{\overline{\Delta\theta_R^2}}{\Delta t}=\frac{21^2}{E^2L_{rad}},$$
$$P(r)=(r_0^2/u_\perp^2)\exp(-r^2/u_\perp^2), \tag{14}$$

where $\overline{\Delta\theta_R^2}/\Delta t$ is the increment in the mean square angle of multiple scattering by nuclei in an amorphous material (E is in MeV) and L_{rad} is the radiation length; $P(r)$ is the distribution of atoms in the transverse plane in the presence of thermal vibrations.

5. MOTION OF UNBOUND ELECTRONS

Following the terminology proposed by Chadderton[35] for positive particles, we shall call unbound superbarrier electrons quasichanneled. This division is necessary as a result of the different nature of the motion of bound and unbound particles. Quasichanneled electrons moving in the field of many strings have only one integral of motion $E_\perp$. This situation is analogous to the axial channeling of positive particles.[1] The kinetic equation in this case has the form

$$\frac{\partial F}{\partial t}=\frac{\partial}{\partial E_\perp}\left[\left\langle\frac{\overline{\Delta E_\perp^2}}{2\Delta t}\right\rangle\frac{\partial F}{\partial E_\perp}\right]. \tag{15}$$

In statistical equilibrium, the positive particles are distributed uniformly in the transverse plane within the region of motion available $U(r)\leqslant E_\perp$. For negative quasichanneled particles the entire transverse plane is available, and their distribution is uniform in this plane; accordingly, averaging over the impact parameters r in calculation of the diffusion coefficient is carried out over all impact parameters:

$$\left\langle\frac{\overline{\Delta E_\perp^2}}{2\Delta t}\right\rangle=\frac{1}{\pi r_0^2}\int E(E_\perp-U(r))\,\overline{\Delta\theta^2}(r)\,d^2r. \tag{16}$$

For the same reason the area of the available region $S(E_\perp)$ does not enter into Eq. (15), since it is constant and equal to the entire area πr_0^2 per string.

It is clear from Eqs. (16) and (14) that Eq. (15) coincides with the diffusion equation for an amorphous medium if the transverse energy is measured from the minimum value of the potential of the atomic string with allowance for the thermal spread in the location of the nuclei (with accuracy to the electronic contribution to $\overline{\Delta\theta^2}$ at small r).

Initially only a small part of the beam is quasichanneled. For incidence of the beam parallel to an atomic string ($\psi_{in}=0$) this obviously is determined by the part of the transverse plane where the potential is produced not just by one atomic string, and amounts to 20–30%, depending on the crystallographic direction and type of lattice. Then, with increase of the penetration depth, there is an increase of the transverse energy and a transition of particles from the channeled part to the quasichanneled part. Accordingly, in solution of Eqs. (8) and (15) in the negative and positive regions of $E_\perp$ it is necessary to match the particle fluxes across the boundary $E_\perp=0$.

6. REDISTRIBUTION OF THE FLUX OF AXIALLY CHANNELED ELECTRONS

As was mentioned above, the flux of quasichanneled electrons is distributed uniformly in the transverse plane, as in the case of an amorphous medium (we neglect here the small variation ΔU of the potential in the region between strings, since as the result of the diffusion in the transverse energy or of the initial spread of the beam momentum the contribution of the region $\Delta E_\perp=\Delta U$ amounts to a small fraction $\sim\Delta U/|U_{min}|$). However, the channeled electrons are focused onto the atomic strings, and as a result the flux of particles in an electron beam is concentrated near the strings. This phenomenon is known as the flux-peaking effect or the redistribution of the flux of channeled particles in impact parameter. It is the consequence of a distribution in statistical equilibrium.[31]

Since in our case the distribution function of the channeled electrons is two-dimensional, we shall give an expression for the channeled-particle distribution flux in the transverse plane.

The distribution of particles in coordinate space (two-dimensional) is determined by the integral of the distribution function in phase space $f(\mathbf{p}_\perp, r, t)$:

$$d^2N/rdrd\varphi=\iint f dp_x dp_y.$$

Going over from the variables p_x, p_y to $E_\perp, M_z$ and replacing the function $f(\mathbf{p}_\perp, r, t)$ by $\Phi(E_\perp, M_z, r, t)$, and then using the relation between the distribution in phase space and the number of particles in the interval $dE_\perp dM_z$:

$$\Phi(E_\perp, M_z, r, t)v_r T/2=F(E_\perp, M_z, t),$$

we obtain (see also the Appendix)

$$\frac{d^2N}{rdrd\varphi}=\iint\frac{2F(E_\perp, M_z, t)\,dE_\perp dM_z}{T[E_\perp-U(r)-M_z^2/2Mr^2]^{1/2}}. \tag{17}$$

In view of the axial symmetry of the potential of the atomic string, the distribution (17) does not depend on the azimuthal angle φ. Substituting into Eq. (17) the initial distribution (9) and changing the order of integration, we obtain for ideal collimation of the incident beam a distribution of flux over r near the crystal surface after establishment of statistical equilibrium

$$\frac{dN}{dr}=\frac{(2M)^{1/2}}{\pi r_0^2}\int\frac{dx'dy'}{[E_\perp-U(r)-M_z^2/2Mr^2]^{1/2}T(E_\perp,M_z)}$$

$$E_\perp=U(r')+p_{\perp in}^2/2M,\quad M_z=x'p_{y,in}-y'p_{x,in}.$$

7. CONVERSION TO DIMENSIONLESS VARIABLES AND ESTIMATES OF THE DECHANNELING LENGTH

Equation (5) is more conveniently written for purposes of analysis in dimensionless variables, choosing the following scaling units: for energy $E_{\perp 0}=Ze^2/d$, for length $a=0.45Z^{-1/3}$ Å (the screening constant), for time (more precisely, for the penetration depth) $t_0=a\gamma/L_e$ (γ is the Lorentz factor), and also for the angular momentum

$$M_{z0}=(aU_a/v)\,(Ed/2Ze^2)^{1/2},$$

where U_a is the value of the string potential at a distance $r=a$. Here Eq. (5) can be written as follows:

$$\frac{\partial F}{\partial\tau}=\frac{\partial}{\partial\varepsilon}\left[D_{\varepsilon\varepsilon}T\frac{\partial}{\partial\varepsilon}\frac{F}{T}+D_{\varepsilon\mu}T\frac{\partial}{\partial\mu}\frac{F}{T}\right]+\frac{\partial}{\partial\mu}\left[D_{\mu\varepsilon}T\frac{\partial}{\partial\varepsilon}\frac{F}{T}+D_{\mu\mu}T\frac{\partial}{\partial\mu}\frac{F}{T}\right],\qquad(18)$$

where

$$\varepsilon=E_\perp/E_{\perp 0},\quad \mu=M_z/M_{z0},\quad \tau=t/t_0,$$
$$D_{\varepsilon\varepsilon}=\langle(\varepsilon-u)\,\overline{\Delta\varepsilon}/\Delta\tau\rangle,\quad D_{\mu\varepsilon}=D_{\varepsilon\mu}=\langle{}^1/_2\mu\overline{\Delta\varepsilon}/\Delta\tau\rangle,\qquad(19)$$
$$D_{\mu\mu}=\langle(r^2/u_a^2)\,\overline{(\Delta\varepsilon/\Delta\tau)}\rangle,\quad \overline{\Delta\varepsilon}/\Delta\tau=2\psi_1^{-2}\overline{\Delta\theta^2}/\Delta\tau,$$

and $\psi_1=(4Ze^2/pvd)^{1/2}$ is the Lindhard critical angle.

In Fig. 1 we have shown the increments of the transverse energy as the result of multiple scattering by electrons and by thermal vibrations, and the combined

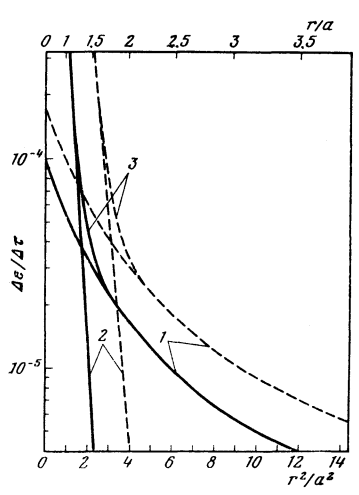

FIG. 1. Increment of transverse energy as a function of impact parameter for motion of an electron with energy 1 GeV in the ⟨100⟩ axial channel of silicon (solid lines) and tungsten (dashed lines) in dimensionless variables (the scaling unit of depth is respectively $t_0=20$ and 27 Å): 1—scattering by electrons, 2—scattering by thermal vibrations, 3—sum.

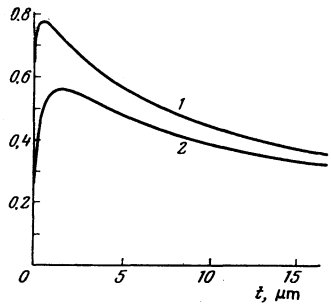

FIG. 2. Fraction of channeled electrons (with $E_\perp\leq 0$) as a function of the depth of their penetration in Si at $E=1$ GeV for two angles of incidence of the beam with respect to the ⟨111⟩ axis: 1—$0.2\psi_1$, 2—$0.35\psi_1$ (ψ_1 is the Lindhard critical angle).

increment (respectively curves 1, 2, and 3) as obtained from the formulas of Sec. 4. Proceeding from these data and the coefficients (19) of Eq. (18), we can obtain a crude estimate of the dechanneling length $x_{1/2}$ in which a large fraction of the beam is dechanneled, if we take into account that particles moving along helical trajectories far from an atomic string for $r\geq r_c\approx 2a$ experience scattering mainly by electrons and in general they determine the dechanneling length. Estimating the corresponding values of μ_c and ε_c, we obtain

$$x_{1/2}\sim\varepsilon_c/(\overline{\Delta\varepsilon}/\Delta t)_a,$$

where $(\overline{\Delta\varepsilon}/\Delta t)_a$ is determined by scattering by electrons with averaging over all $r\geq r_c$. For the ⟨100⟩ channel of Si and W we have respectively $x_{1/2}\sim 20$ and 100 μm at $E=1$ GeV. These approximate estimates are confirmed in a numerical solution of the kinetic equation (see below).

In Fig. 2 we have given the number of channeled particles (which have $E_\perp\leq 0$) as a function of the depth (the dechanneling function), obtained by numerical solution of the kinetic equation (18) for various angles of incidence of the beam.

As can be seen from the calculation, capture into bound states at small depths as the result of multiple scattering is characteristic of axial channeling of electrons. This is due to the fact that on entry into the crystal the particles are concentrated near $E_\perp=0$, i.e., near the potential barrier, since the interstring region of the channel where $U\approx 0$ has a substantial area. As a result of diffusion, part of these particles fall into the region of bound motion.

As can be seen from the estimates, which were confirmed by the numerical calculation, particles undergoing comparatively stable channeling move along a helical trajectory at some distance from the string (rosette motion), as a result of which the multiple scattering of these particles occurs mainly by electrons, and not by nuclei. The number of such particles amounts to a significant fraction of the beam (~50%).

We note that the characteristic length of scattering of the quasichanneled part of the beam is the same as in an amorphous medium (more precisely, as in a disoriented crystal), in contrast to the planar case, where

it is substantially smaller than in an amorphous medium, as a result of the fact that in the planar case the flux of quasichanneled electrons is concentrated between the atomic planes.

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APPENDIX

The kinetic equation for the distribution function $F(E_\perp, M_z, t)$ can be obtained, following the method developed in Refs. 1 and 3, by averaging the well known Fokker-Planck equation for particles moving in an external field and undergoing small-angle deflections in Coulomb collisions with the particles of the medium.

The Fokker-Planck equation for the distribution function in the phase space of the transverse motion $f(\mathbf{p}_\perp, \mathbf{r}, t)$ without allowance for the spread of the longitudinal momentum, which in the present case is negligible in comparison with the longitudinal momentum value, can be written in the form (see for example Ref. 36)

$$\frac{\partial f}{\partial t}+\frac{\mathbf{p}_\perp}{M}\nabla_r f-\nabla U\nabla_p f=\frac{\overline{\Delta p_z^2}}{2\Delta t}\left(\frac{\partial^2 f}{\partial p_x^2}+\frac{\partial^2 f}{\partial p_y^2}\right). \tag{A.1}$$

Here t is the time of the motion, which is proportional to the depth of penetration of the particle into the channel, and which must be small in comparison with the time of complete stopping (this enabled us to neglect the change in the longitudinal component of the momentum). Going over in Eq. (A.1) to the new variables

$$\begin{gathered}E_\perp=(p_x^2+p_y^2)/2M+U(r),\\ M_z=xp_y-yp_x,\quad r=(x^2+y^2)^{1/2},\quad \varphi=\operatorname{arctg}(y/x)\end{gathered} \tag{A.2}$$

and averaging the resulting equation over φ, we obtain

$$\frac{\partial f}{\partial t}+\left[\frac{2}{M}\left(E_\perp-\frac{M_z^2}{2Mr^2}-U(r)\right)\right]^{1/2}\frac{\partial f}{\partial r}=\frac{\overline{\Delta p_x^2}}{2\Delta t}\left\{r^2\frac{\partial^2 f}{\partial M_z^2}+\frac{\partial}{\partial E_\perp}\frac{2(E_\perp-U(r))}{M}\frac{\partial f}{\partial E_\perp}+\frac{1}{M}\frac{\partial}{\partial M_z}M_z\frac{\partial f}{\partial E_\perp}+\frac{1}{M}\frac{\partial}{\partial E_\perp}M\frac{\partial f}{\partial M_z}\quad\frac{\partial f}{\partial E_\perp}\right\}. \tag{A.3}$$

We divide Eq. (A.3) by

$$v_r=[2M^{-1}(E_\perp-M_z^2/2Mr^2-U(r))]^{1/2}$$

and replace the distribution function f by $\Phi(E_\perp, M_z, r, t)$ in accordance with the definition

$$f r dr dp_x dp_y=\Phi dr dM_z dE_\perp,$$

so that Φ represents the density of particles in the interval $dE_\perp dM_z dr$ and $\Phi=f/v_r$. Here Eq. (A.3) can be represented after simple manipulations in the form

$$\frac{\partial\Phi}{\partial t}+\frac{\partial}{\partial r}v_r\Phi=\frac{\partial^2}{\partial E_\perp^2}\left(\frac{\overline{\Delta p_x^2}}{\Delta t}\frac{E_\perp-U(r)}{M}\Phi\right)+\frac{\partial^2}{\partial M_z^2}\left(\frac{\overline{\Delta p_x^2}}{2\Delta t}r^2\Phi\right)+\frac{\partial^2}{\partial M_z\partial E_\perp}\left(\frac{\overline{\Delta p_x^2}}{\Delta t}\frac{M_z}{M}\Phi\right)-\frac{\partial}{\partial E_\perp}\left(\frac{1}{M}\frac{\overline{\Delta p_x^2}}{\Delta t}\Phi\right). \tag{A.4}$$

A uniform distribution of particles in the phase of oscillation means that $\Phi v_r T/2=F(E_\perp, M_z)$, where T is the period of oscillation in r and $F=\int\Phi dr$ is the total number of particles in the interval $dE_\perp dM_z$. This means that particles with a given $E_\perp, M_z$ are distributed over the radius r with a probability

$$dP=\frac{dr}{T[2M^{-1}(E_\perp-M_z^2/2Mr^2-U(r))]^{1/2}},\quad E_\perp-U(r)\geqslant\frac{M_z^2}{2Mr^2},\qquad dP=0,\quad E_\perp-U(r)<M_z^2/2Mr^2. \tag{A.5}$$

Integrating Eq. (A.4) over r with inclusion of these relations, we finally obtain

$$\begin{gathered}\frac{\partial F}{\partial t}=\frac{\partial^2}{\partial E_\perp^2}\left(\left\langle\frac{\overline{\Delta E_\perp^2}}{2\Delta t}\right\rangle F\right)+\frac{\partial^2}{\partial M_z^2}\left(\left\langle\frac{\overline{\Delta M_z^2}}{2\Delta t}\right\rangle F\right)+\frac{\partial^2}{\partial E_\perp\partial M_z}\left(\left\langle\frac{\overline{\Delta E_\perp\Delta M_z}}{\Delta t}\right\rangle F\right)-\frac{\partial}{\partial E_\perp}\left(\left\langle\frac{\overline{\Delta E_\perp}}{\Delta t}\right\rangle F\right),\\ \overline{\Delta E_\perp^2}/\Delta t=2(E_\perp-U(r))M^{-1}\overline{\Delta p_x^2}/\Delta t=(E_\perp-U(r))E\overline{\Delta\theta^2}/\Delta t,\\ \overline{\Delta M_z^2}/\Delta t=p^2r^2\overline{\Delta\theta^2}/2\Delta t=r^2\overline{\Delta p_x^2}/\Delta t,\\ \overline{\Delta E_\perp\Delta M_z}/\Delta t=M_zM^{-1}\overline{\Delta p_x^2}/\Delta t=M_zE\overline{\Delta\theta^2}/2\Delta t,\end{gathered} \tag{A.6}$$

and averaging over r is carried out as follows:

$$\langle X\rangle=2T^{-1}\int_{r_{\min}}^{r_{\max}}Xdr[2M^{-1}(E_\perp-M_z^2/2Mr^2-U(r))]^{-1/2}.$$

[1)]Here equilibrium can be understood in the sense that, as follows from Liouville's equation, any distribution $F_0(E_\perp, M_z)$ which depends only on integrals of the motion is stationary.

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