Phase transition in the rotational band of a nonaxial nucleus

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A macroscopic theory is proposed for the phase transition occurring in the yrast band of a nonaxially deformed nucleus with spin $I > 50$. The transition is connected with the smooth variation of the nonaxial deformation, and is a consequence of the shell effects that occur in a nucleus when a subshell with a large single-particle angular momentum j near the Fermi surface begins to fill up. When the sign of the nonaxial deformation changes, the rotation of the nucleus about the axis with an intermediate moment of inertia becomes unstable. Therefore the transition through the critical point I_0 is accompanied by a 90° rotation of the angular-momentum vector relative to the body system of coordinates. The phase-transition parameter, which determines the character of the rotational spectrum of the nucleus, is the quantity $|\gamma|I_0^2$. The phase transition region is characterized by intersection of the bands, approach of the side bands to the yrast band, and a high degree of forbiddenness of the $E2$ transitions in the yrast band near the critical point. The indicated effects are computed for the nuclei of the rare-earth elements with neutron numbers N ranging from 92 to 96.

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§1. INTRODUCTION

1. The atomic-nucleus levels that possess the least energy for a given spin I form the lowest band, which is called the yrast band. This band has now been observed in certain nuclei up to quite large spins ($I \sim 30$). The states in the yrast band can be of either a single-particle or a collective nature.[1] In both cases the level energy depends on the level spin in approximately quadratic fashion.

The single-particle states in the yrast band are in fact many-quasiparticle excitations in which the angular momenta of the individual quasiparticles are aligned in the direction of the symmetry axis of the deformed nucleus. The number of quasiparticles and the energy of a quasiparticle in such an excitation are proportional to the spin of the excitation, which explains the quadratic dependence of the energy on I. The experimentally found coefficient of proportionality in this dependence is close to the value $\frac{1}{2}\mathcal{J}_s$, where $\mathcal{J}_s$ is the moment of inertia of the solid sphere. From the classical point of view, the noncollective states of the yrast band correspond to the rotation of the nucleus about the symmetry axis. The lifetime of the noncollective levels is of the order of nanoseconds, and these levels form the high-spin isomers observed in nuclei with almost filled neutron ($N = 82$) or proton ($Z = 50$) shells.

The collective levels of the yrast band are rotational states of the deformed nucleus. The ground state of a well-deformed nucleus is a prolate ellipsoid of revolution, and the roational band based on this state corresponds to rotation about an axis perpendicular to the symmetry axis of the nucleus. As the angular momentum increases, the ground-state-based band goes over at[1)] $I \sim 16$ into the yrast band, which consists of bands that are based on the excited states of the nucleus, and whose levels are for some reason the lowest for the spin region in question.

The deformation of the nucleus in the yrast band does not remain constant. It begins to vary appreciably from the instant when the centrifugal energy in the rotating nucleus becomes comparable to the shell energy. The deformation varies in such a way that the rotational energy of the nucleus is a minimum. Since the moment of inertia of an oblate nucleus is greatest when the nucleus rotates about its symmetry axis, the nucleus tries to go over from the prolate into the oblate state as the spin in the yrast band increases. Therefore, starting from spins $I \sim 30$–40, the nucleus is nonaxial, and the yrast band corresponds to the rotation of the nucleus about the axis with the greatest moment of inertia. Located above and parallel to the yrast band in the nonaxial nucleus is a system of side bands corresponding to the precessional motion. Such a structure of the spectrum in the region of the lowest rotational band of a deformed nucleus does not contradict the experimental data available at the present time, and, in particular, agrees with the fact that there are no isomeric states. In an axially deformed nucleus, the many-quasiparticle excitations with a large angular-momentum component along the symmetry axis of the nucleus would be isomeric states.

In recent years, the study of the yrast band of a rotating nucleus has become a rapidly developing field of nuclear physics. Significant progress was made in this field thanks to the use of reactions with heavy ions to excite the rotational states. Attractive in this direction is the unique information that can be obtained about the structure of the atomic nucleus by studying the yrast band. The latter circumstance is due to the effects of the Coriolis force in the rotating nucleus. This force plays a more important role in the nucleus than under terrestrial conditions, or even in molecular spectra. In particular, the effect of the Coriolis force on the nucleons in the subshell with the maximum single-particle angular momentum j near the Fermi surface allows us to explain the backbending of the moment of inertia in the ground-state-based band.[2]

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2. Another phenomenon connected with the nucleons in a subshell with a large j near the Fermi surface is the change in sign of the nonaxial deformation of the nucleus. Equilibrium-deformation computations performed in Refs. 3 and 4 with the aid of the Strutinsky shell-correction method for nuclei in the yrast band show that, in the spin region $50 < I < 60$, the shape of certain nuclei varies with increasing I in such a way that the nucleus becomes axial at some spin value I_0 and then nonaxial again at a higher spin value, with the nonaxial deformation parameter γ of opposite sign.

Let us illustrate the physical picture of this phenomenon for a model nucleus consisting of nucleons connected with a soft rotator. We write the Hamiltonian of the entire system in the form

$$H=\sum_k \frac{(I_k-J_k)^2}{2\mathscr{J}_k}+V(\beta,\gamma)+H_p, \tag{1}$$

where H_p is the Hamiltonian of the noninteracting nucleons that move in the self-consistent field of the nucleus, J is the total angular momentum of the nucleons, and V is the rotator's deformation energy which depends, besides on γ, on the axial deformation parameter β.

Let us consider extremely large spins for which the term $\sum_k J_k^2/2\mathscr{J}_k$ in the Hamiltonian (1) can be neglected and the rotator moments of inertia $\mathscr{J}_k$ can be considered to be the moments of inertia of a rigid body. They are determined by the geometrical dimensions of the rotator, which is an ellipsoid with semiaxes

$$\begin{gathered} R_1=R_0\left(1-\frac{\delta}{3}\cos\gamma-\frac{\delta}{\sqrt{3}}\sin\gamma\right),\\ R_2=R_0\left(1-\frac{\delta}{3}\cos\gamma+\frac{\delta}{\sqrt{3}}\sin\gamma\right),\quad R_3=R_0\left(1+\frac{2}{3}\delta\cos\gamma\right), \end{gathered} \tag{2}$$

where $R_0=1.2A^{1/3}$ fm (A is the number of nucleons in the nucleus) and $\delta=(45/16\pi)^{1/2}\beta$. In determining the equilibrium deformation in the yrast band for such large spins, we can consider the rotation to be classical.[2)]

The rotation–nucleon interaction is strongest for the nucleons located at the levels with $j\sim A^{1/3}$ near the Fermi surface. For the rare-earth elements these are the $i_{13/2}$ levels for the neutrons and the $h_{11/2}$ levels for the protons. These levels are distinguished by parity from the other states of the filled shell. Therefore, j for them is a good quantum number, since the admixture of states with other j as a result of the deformation and the rotation corresponds to transitions to a neighboring shell. Consequently, the subshell with the maximum j near the Fermi surface can be considered to be isolated. If we limit ourselves in the Hamiltonian (1) to only this subshell, then its single-particle part H_p will have the form

$$H_p=\sum_i h_j(i),$$

$$h_j=\varepsilon_j+\frac{\varkappa\delta}{6j(j+1)}\{(3j_{z'}^2-j^2)\cos\gamma+\sqrt{3}(j_{x'}^2-j_{y'}^2)\sin\gamma\},$$

where ε_j is the energy of the j-th level in the spherical nucleus $\varkappa$ is the quadrupole–quadrupole interaction constant and the $j_{k'}$ are the single-particle angular-momentum operators. The summation in H_p is over all the nucleons in the subshell.

We can assume that, for the spins I under consideration, the angular momentum of the nucleon is not coupled to the deformation, and is oriented along the direction of the axis of rotation of the nucleus. In this case the energy levels of the subshell form an equidistant spectrum with spacing[3)] $I/\mathscr{J}_1$. The deformation slightly distorts the alignment. Analyzing the deformation with the aid of perturbation theory, we find that the level characterized by the component ν of the angular momentum of the nucleon along the axis of rotation shifts, as γ varies, in proportion to the quantity[4)] $\gamma[3\nu^2-j(j+1)]$.

The total energy of the nucleons is obtained by summing the single-particle energies of all the filled levels in the subshell from $\nu=j$ to $\nu=\nu_0$. Performing simple calculations, we find the dependence of the energy of the nucleus on γ:

$$\begin{aligned} E_I(\gamma)=E_0&+\frac{1}{2}\beta^2\gamma^2C_\gamma+\frac{I^2}{2\mathscr{J}_\perp(1+b\gamma)}\\ &-\frac{j^2(1-\nu_0^2/j^2)I}{2\mathscr{J}_\perp(1+b\gamma)}+\gamma\frac{\varkappa\delta}{4\sqrt{3}}\nu_0\left(1-\frac{\nu_0^2}{j^2}\right), \end{aligned} \tag{3}$$

where E_0 is that part of the energy which does not depend on γ, C_γ is the coefficient of stiffness with respect to the γ deformation, the quantity b is proportional to δ, and the moment of the inertia $\mathscr{J}_\perp$ of the axial rotator about an axis perpendicular to the symmetry axis is equal to

$$\mathscr{J}_\perp=\mathscr{J}_s(1+\delta/3+5\delta^2/18). \tag{4}$$

Expression (3) allows us to determine the equilibrium nonaxial deformation of the nucleus. Noting that the Coriolis energy is small compared to the centrifugal energy, we obtain

$$\gamma=\alpha(I-I_0), \tag{5}$$

$$I_0=\left[\frac{1}{2}j\varkappa\mathscr{J}_s\frac{\nu_0}{j}\left(1-\frac{\nu_0^2}{j^2}\right)\right]^{1/2}. \tag{6}$$

As follows from the last formula, the critical spin I_0 depends on the population of the subshell with a large j near the Fermi surface. Calculations in the square-potential-well model show that $I_0\sim 90$ for nuclei containing 94–98 neutrons (isotopes of Er and Yb). In Mosel's calculations,[5] performed by the Hartree–Fock–Bogolyubov method with the Skyrme interaction, the nonaxiality of the Er^{164} nucleus disappears near $I=80$. The quantity α can be estimated after determining the stiffness C_γ from the energy of the γ vibrations in the adiabatic approximation. If we use for the mass parameter of these vibrations the empirical expression[6] $B_\gamma=1.5B_2$, where B_2 is the hydrodynamic value of this quantity, then $\alpha=2°$ on the average for the above-indicated isotopes of Er and Yb. In Ref. 4 this quantity is found to be equal to 3°.

The point I_0 in the yrast band coincides with the point of intersection of the bands corresponding (from the classical point of view) to the rotation of the nucleus about two mutually perpendicular axes: the y' axis for $\gamma<0$ ($I<I_0$) and the x' axis for $\gamma>0$ ($I>I_0$). Because of the instability of the rotation about the axis with the intermediate moment of inertia, the upper parts of these

bands cannot exist. In this sense we can speak of a "phase transition" in the yrast band. It is convenient to study this transition by measuring the dependence of the moment of inertia of the nucleus on the square of the rotation frequency. For $\alpha < \alpha_{cr} = 1/bI_0$ this dependence should have the characteristic s shape.

We should expect the interaction between the collective degrees of freedom of the deformed nucleus to be strong at the phase transition point. Below we shall limit ourselves to the consideration of only the rotational degrees of freedom, taking the nucleus to be a rigid nonaxial rotator with variable principal moments of inertia.

§2. NONAXIAL ROTATOR WITH LARGE ANGULAR MOMENTUM

Let us first discuss the method of describing the rotational states of the nonaxial nucleus with a large spin.

1. The group of operators for the asymmetric top consists of the operators of the angular-momentum components along the fixed (I_k) and moving ($I_{k'}$) axes, and of the operators D^J_{MK} that connect the physical quantities in the body and laboratory coordinate systems. This group can be realized in the basis of polynomials in four complex variables[7] ξ_1, ξ_2, τ, and ζ. In this representation the wave function of the top has the form

$$\Psi_{IMn}(\xi_1\xi_2\tau\zeta) = \frac{\xi_1^{I+M}\xi_2^{I-M}\tau^{2I}}{[(I+M)!(I-M)!(2I+1)!]^{1/2}}\varphi_{nI}(\zeta), \tag{7}$$

where M is the angular-momentum component along the z axis and φ_{nI} is the eigenfunction of the Hamiltonian ($\hbar = 1$)

$$H = {}^1/_2(A_1I_{x'}^2 + A_2I_{y'}^2 + A_3I_{z'}^2), \quad A_k = 1/\mathcal{J}_k. \tag{8}$$

Integration in the space of the functions (7) is defined with the aid of the invariant measure

$$dg = \frac{4|\tau|^2}{\pi^3(1+|\zeta|^2)^{2I+2}}\exp\left(-\sum_{i=1}^{2}|\xi_i|^2 - |\tau|^2\right)d^2\xi_1\, d^2\xi_2\, d^2\tau\, d^2\zeta. \tag{9}$$

The variables ξ_1, ξ_2, τ, and ζ are defined in the entire complex plane. The last variable is the sterographic projection of a point, specified by the polar angles ϑ and ϕ, of a sphere of radius $\frac{1}{2}$ on the plane passing through the southern pole:

$$\zeta = e^{i\phi}\operatorname{ctg}(\vartheta/2).$$

The angles ϑ and ϕ determine the orientation of the vector I in the moving coordinate system.

It is convenient to represent the function φ in the form

$$\varphi(\zeta) = \zeta^I\psi(\zeta) \tag{10}$$

and make a change of variable $\zeta = e^{iz}$ ($z = x + iy$), meaning conformal mapping of the plane of the complex variable ζ on the sheet $0 < x < 2\pi$. The operators for the angular-momentum components along the moving axes in the space of the functions ψ, which are trigonometric polynomials, have the form

$$I_{x'} = I\cos z - (\sin z)\frac{d}{dz}; \quad I_{y'} = -I\sin z - (\cos z)\frac{d}{dz}, \quad I_{z'} = -i\frac{d}{dz}. \tag{11}$$

The Hamiltonian (8) is invariant under rotation through 180° about any axis of the body coordinate system. The corresponding operators $R_{k'}(\pi)$ transform the function ψ in the following manner:

$$R_{x'}(\pi)\psi(z) = (-1)^I\psi(-z), \quad R_{y'}(\pi)\psi(z) = (-1)^I\psi(\pi - z), \quad R_{z'} = R_{x'}R_{y'}. \tag{12}$$

Below we shall need one more finite-rotation operator:

$$R_{z'}(\pi/2)\psi(z) = \psi(z + \pi/2).$$

2. Let us first consider the positive values of the nonaxiality parameter γ. In this case, as follows from the formulas (2), the rotator is extended along the z' axis, and its principal moments of inertia satisfy the relation $\mathcal{J}_1 > \mathcal{J}_2 > \mathcal{J}_3$. Substituting the expressions (11) for the angular-momentum component operators into the Hamiltonian (8), we obtain the ordinary differential equation

$$(\xi^2 + \sin^2 z)\frac{d^2\psi}{dz^2} - \left(I - \frac{1}{2}\right)\sin 2z\frac{d\psi}{dz} + \left[\epsilon + \frac{1}{2}I(I-1)\cos 2z\right]\psi = 0, \tag{13}$$

which depends on the spin 1 and on the parameter

$$\xi^2 = (A_3 - A_2)/(A_2 - A_1).$$

The rotational energy can be expressed in terms of the eigenvalues $\mathcal{E}$ of this equation as follows:

$$E = {}^1/_4(A_1 + A_2)I(I+1) + {}^1/_2(A_2 - A_1)\epsilon. \tag{14}$$

Equation (13) is invariant under the transformations (12). We can therefore introduce the quantum numbers r_1 and r_2, which assume the values ±1 and are the eigenvalues of the operators $R_{x'}$ and $R_{y'}$ respectively. In a nucleus with an even number of neutrons and protons, we can have only integer values of I. Moreover, the symmetry of the inner wave function leads to a situation in which all the rotational states of the nonaxial nucleus have $r_1 = r_2 = +1$. In all, for a given value of I there are $\frac{1}{2}+1$ states if I is even and $(I-1)/2$ if I is odd. We shall label these states by the subscript n. The lowest ($n = 0$) band of an even–even nucleus contains levels with even spins. Its wave function, which is a solution to Eq. (13), has the form

$$\psi_{0I}(z) = \sum_{m=0}^{I/2+1} A_m\cos 2mz.$$

Equation (13) is also invariant under the transformation $R_{z'}(\pi/2)$ with the simultaneous replacement of ξ^2 by $-(\xi^2+1)$ and $\mathcal{E}$ by $-\mathcal{E}$. This transformation (it is denoted hereafter by $\mathcal{R}$) is equivalent to the interchange of the x' and y' axes.

3. To determine the $E2$-transition probability for the region of the yrast band, we use the operator D^2_{MK}, found in Ref. 7, and the expressions (7), (9), and (10). After straightforward but quite tedious transformations, we obtain the following formula for the computation of the reduced probability for transition between levels with different values of the quantum numbers n and I:

$$B(E2, nI \to n'I') = \frac{5e^2}{16\pi(2I+1)}|\langle n'I'\|Q\|nI\rangle|^2, \tag{15}$$

$$\langle n'I'\|Q\|nI\rangle = (-1)^{I'-I}\left[(2I'+1)\frac{(2I'+1)!(I-I'+2)!(I'-I+2)!(I+I'-2)!}{(2I)!(I+I'+3)!}\right]^{1/2} \times \int \varphi^*_{n'I'}(\zeta)\, q_{I,I'-I}\varphi_{nI}(\zeta)\, dg_{I'}(\zeta). \tag{16}$$

In the last expression

$$dg_I(\zeta)=\frac{4d^2\zeta}{(1+|\zeta|^2)^{2I+2}},$$

and the operator $\hat{q}$ is connected with the quadrupole-moment components $q_{2\mu}$ in the body coordinate system ($q_{20}=q_0, q_{22}=q_{2,-2}=q_2$) in the following manner:

$$\hat{q}_{I\rho}=\sum_{\mu} q_{2\mu}[(2+\mu)!(2-\mu)!]^{1/2}\sum_{\nu}\frac{(-1)^{\nu}\zeta^{\rho+\mu+\nu}}{\nu!(\rho+\mu+\nu)!(2-\rho-\nu)!(2-\mu-\nu)!}\times\prod_{n=0}^{1-\rho-\nu}\left(2I-\nu-n-\zeta\frac{\partial}{\partial\zeta}\right)\frac{\partial^{\nu}}{\partial\zeta^{\nu}}. \quad (17)$$

It should be noted that the internal state of the nucleus was assumed to be fixed in the above-considered transitions.

4. Let us transform Eq. (13) with the aid of the substitution

$$\psi(z)=(\xi^2+\sin^2 z)^{(I-1/2)/2}u(z).$$

The differential equation for the function u has the form

$$u''(z)+g(\varepsilon, z)u(z)=0, \quad (18)$$

$$g(\varepsilon,z)=\frac{1}{4(\xi^2+\sin^2 z)^2}\left\{\sin^4 z-4\left[(1+\xi^2)\left(\varepsilon_0+\frac{1}{4}\right)-\varepsilon-\frac{\xi^2}{4}\right]\sin^2 z+4\varepsilon\xi^2\right\}, \quad (19)$$

where $\varepsilon=\mathcal{E}+\varepsilon_0/2$, $\varepsilon_0=I^2+I-1$. The spectrum of the rotational states with a given spin I is determined by eigenvalues ε contained, as simple classical estimates show, in the interval $0<\varepsilon<(1+\xi^2)I^2$. The minimum energy corresponds to rotation about the x' axis; the maximum energy, to rotation about the z' axis.

Equation (18) contains a large parameter ($\varepsilon_0\approx(I+\frac{1}{2})^2$), and this allows us to use the quasiclassical approximation to describe the rotation of a nonaxial nucleus in the case of large spins. The quasiclassical approximation breaks down at the points where the function $g(\varepsilon, z)$ vanishes. The zeros of this function play an important role in the construction of the quasiclassical solution, determining the form of the wave function and the eigenvalue spectrum. In all, the interval $0\leqslant x<\pi$ contains four such points, which are the sterographic projections of the turning points of the classical trajectories[5] on the plane of the complex variable ζ.

Two reversal points $B(\pi/2, y_B)$ and $B'(\pi/2, -y_B)$, where

$$y_B=\operatorname{arcgch}[2(s_+s_--\varepsilon+[(s_+^2-\varepsilon)(s_-^2-\varepsilon)]^{1/2})]^{1/2},$$
$$s_\pm=(I+{}^1/_2)(1+\xi^2)^{1/2}\pm\xi/2,$$

are always located on the straight line $x=\pi/2$. The positions of the two other points A and A', whose coordinates are determined from the equation

$$\sin z_0=[2(s_+s_--\varepsilon-[(s_+^2-\varepsilon)(s_-^2-\varepsilon)]^{1/2})]^{1/2},$$

depend on the magnitude of the parameter ξ and the eigenvalue ε. If[6] $\xi>1/(2I+1)$ then, in the region $0<\varepsilon<\varepsilon_0$, these reversal points are located on the real axis symmetrically with respect to the point $z=\pi/2$ ($z_0=x_A$ or $\pi-x_A$). As ε increases, the points A and A' approach the latter along the real axis if $\varepsilon>\varepsilon_0$, pass over to the straight line $x=\pi/2$ when $\varepsilon>\varepsilon_0$, and begin to move away from each other along this straight line as ε is increased further ($z_0=\pi/2\pm iy_A$).

The construction of the quasiclassical solution to Eq. (18) with four reversal points in the complex plane is described in Ref. 7. The energies of the rotational states of the nonaxial nucleus are determined from quantization conditions, which, depending on the disposition of the reversal points, are given by one of the equations:

$$\operatorname{ctg}\left(\int_0^{x_A}[g(\varepsilon,x)]^{1/2}dx+\phi_0\right)=\frac{1}{2}\exp\left(-2\int_{x_A}^{\pi/2}[|g(\varepsilon,x)|]^{1/2}dx\right),\quad 0<\varepsilon<\varepsilon_0, \quad (20)$$

$$\operatorname{tg}\left(\int_0^{\pi/2}[g(\varepsilon,x)]^{1/2}dx\right)=\frac{1}{2}\exp\left(-2\int_c^{y_A}\left[\left|g\left(\varepsilon,\frac{\pi}{2}+iy\right)\right|\right]^{1/2}dy\right),\quad \varepsilon_0<\varepsilon<\varepsilon_0(1+\xi^2), \quad (21)$$

where the phase ϕ_0 is equal to $\pi/4$ and $3\pi/4$ for even, and odd I, respectively.

The accuracy of the determination of the energies from the quantization conditions (20) and (21) deteriorates considerably when the eigenvalues fall within the interval $|\varepsilon-\varepsilon_0|<(I+\frac{1}{2})\xi$. In this interval the turning points A and A' are so close to each other that the regions around these points where the quasiclassical approximation is inapplicable overlap. Formally, the indicated energies correspond to the rotation of the nucleus about the axis with the intermediate moment of inertia $\mathcal{J}_2$. It will be shown below that these energies determine the phase-transition region.

In the limit of large ξ, we can obtain from the quantization condition (21) the rotational spectrum of a slightly nonaxial nucleus, a spectrum which consists of bands each with a definite angular-momentum component K along the symmetry axis. An exception is the yrast band with $K=0$, whose energy cannot be determined from Eq. (21). This circumstance is explained by the fact that, in the quasiclassical approximation, the energy in the band is expanded in powers of the parameter $(I/\xi K)^2$.

In the case of negative γ, the axis of greatest moment of inertia is the y' axis, i.e., $\mathcal{J}_2>\mathcal{J}_1>\mathcal{J}_3$. In order to determine the rotational spectrum of the nucleus, we should use in this case the transformation $\mathcal{R}$ (see Subsec. 2 of this section). By applying this transformation to Eq. (18), we can obtain quantization conditions similar to (20) and (21).

5. A cruder approximation is the oscillator approximation. It allows us to describe only that part of the rotational spectrum of a nonaxial nucleus which pertains to the region of the yrast band ($\varepsilon\sim 0$). If $\varepsilon/\varepsilon_0\ll 1$, then the region of small z, which corresponds to the classical trajectories near the point $\vartheta=\pi/2, \phi=0$, is decisive in Eq. (18). These trajectories correspond to the precession of the angular-momentum vector about the axis with the largest moment of inertia.

In the limit $|z|\ll 1$ Eq. (18) goes over into the Schrödinger equation for an oscillator with frequency

$$\omega_I=(2I+1)(1+\xi^2)^{1/2} \quad (22)$$

and "mass" $m=\frac{1}{2}\xi^2$. This is an equation of a complex

variable; therefore, the condition for the wave function to be bounded should be replaced by the condition for the norm of the wave function to be finite. The latter condition selects a solution in the form of Hermite polynomials. We present the final expression for the eigenfunctions of the Hamiltonian (8), which determine in accordance with the relations (7) and (10) the wave functions of the rotational states of the nucleus in the region of the yrast band:

$$\psi_{nI}(z)=\left[\frac{2^{2I}(I+1/2)}{\pi n!}\left(\frac{a}{2}\right)^{n}\frac{(1-a)^{1/2}}{(2-a)^{n+1}}\right]^{1/2}H_n(z\sqrt{m\omega_I})$$

$$\times\exp\left(-\frac{a}{2}m\omega_I z^2\right), \qquad (23)$$

where $a=[(1+\xi^2)^{1/2}-1]/(1+\xi^2)^{1/2}$. The energy of these states, which can be found from the expression (14), is equal to

$$E_{nI}=A_1I(I+1)/2+(I+1/2)\,[(A_3-A_1)(A_2-A_1)]^{1/2}(n+1/2). \qquad (24)$$

In the case of even–even nuclei the quantum number n assumes only even values.

Substituting the wave functions (23) in (10) and using (15)–(17) we obtain, to first order in I^{-1}, the reduced probabilities for $E2$ transitions in the yrast band[6]:

$$B(E2, nI\to nI-2)=5e^2Q_2^2/16\pi,$$

$$B(E2, nI\to n-1I-1)=\frac{5e^2}{16\pi}\frac{n}{I}\left(\sqrt{3}\frac{2-a}{2(1-a)^{1/2}}Q_0-\frac{a}{[2(1-a)]^{1/2}}Q_2\right)^2,$$

$$B(E2, nI\to n+1I-1)=\frac{5e^2}{16\pi}\frac{n+1}{I}\left(\frac{3^{1/2}a}{2(1-a)^{1/2}}Q_0-\frac{2-a}{[2(1-a)]^{1/2}}Q_2\right)^2,$$

where

$$Q_0=-\frac{1}{2}q_0+\left(\frac{3}{2}\right)^{1/2}q_2, \quad Q_2=-\left(\frac{3}{8}\right)^{1/2}q_0-\frac{1}{2}q_2$$

are respectively the quadrupole moment of the nucleus and the measure of its asymmetry with respect to the x' axis. These formulas show that an excitation occurring in the yrast band will induce a γ-quantum cascade corresponding to transitions in the same band.

To describe the region of the yrast band for negative values of γ, let us use the transformation $\mathcal{R}$. According to this transformation, the variable z should be replaced by $\pi/2+z$. Therefore, the vector I will precess about the $\vartheta=\pi/2, \phi=\pi/2$ direction, which coincides with the y' axis. In other words, the change in sign of γ causes the angular-momentum vector to turn through 90° in the body coordinate system. The wave function $\psi(z)$ for such a motion is obtained from the function (23) by replacing z by $\pi/2+z$ and ξ^2 by

$$\eta^2=(A_3-A_1)/(A_1-A_2), \qquad (25)$$

while the energy is obtained from the expression (24) by interchanging the subscripts for the x' and y' axes.

The oscillator approximation considered above is applicable when the angular-momentum component in the plane perpendicular to the axis of rotation is small. This requirement leads to the following condition:

$$I\gg(n+1/2)\,[(1+\xi^2)^{1/2}+1/(1+\xi^2)^{1/2}]. \qquad (26)$$

§3. THE ROTATIONAL SPECTRUM OF A NONAXIAL ROTATOR IN THE PHASE-TRANSITION REGION

The phase-transition region is characterized by approximately equal moments of inertia $\mathcal{J}_1$ and $\mathcal{J}_2$. We shall assume that the initially positive difference A_1-A_2 decreases, changes sign, and becomes negative.

1. Let us first consider the positive values of A_1-A_2. The small values of this quantity correspond to large values of the parameter ξ^2, for which values the turning points A and A' are close to the point $z=\pi/2$. The quantization condition (20) in this case gets simplified:

$$p\Phi(\lambda)+2\arctg(1/2\exp[-pX(\lambda)])=\pi(2n+1/2), \qquad (27)$$

$$p=(2I+1)/\xi, \qquad (28)$$

where the functions Φ and X are defined as follows:

$$\Phi(\lambda)=E(\sqrt{1-\lambda})-\lambda K(\sqrt{1-\lambda}), \quad X(\lambda)=E(\sqrt{\lambda})-(1-\lambda)K(\sqrt{\lambda}),$$

K and E being complete elliptic integrals of the first and second kinds. The roots

$$\lambda_n=|\varepsilon_n-\varepsilon_0|/(I+1/2)^2 \qquad (29)$$

of Eq. (27) that lie in the interval $0\leqslant\lambda\leqslant1$ determine the bottom part ($\varepsilon_n<\varepsilon_0$) of the spectrum of the rotator for a given spin I. The quantum number n assumes all integer values from zero to

$$n_0\leqslant p/2\pi-(\pi/4-\arctg(1/2))/\pi.$$

To determine the top part ($\varepsilon_n\gg\varepsilon_0$) of the spectrum, we should use the quantization condition (21), which in the region in question reduces to the equation

$$p\tilde{\Phi}(\lambda)-2\arctg(1/2\exp[-p\tilde{X}(\lambda)])=2\pi n; \qquad (30)$$

$$\tilde{\Phi}(\lambda)=\sqrt{1+\lambda}E(1/\sqrt{1+\lambda}), \quad \tilde{X}(\lambda)=\sqrt{1+\lambda}\{K(\sqrt{\lambda/(1+\lambda)})-E(\sqrt{\lambda/(1+\lambda)})\}.$$

The roots of this equation ($0<\lambda<\infty$) together with the expression (29) determine the energies of the levels with the quantum numbers $n_0'+1, n_0'+2, \ldots, I/2$, where the whole number

$$n_0'\geqslant p/2\pi-\arctg(1/2)/\pi.$$

Generally speaking, $n_0'<n_0$, which is explained by the breakdown of the quasiclassical approximation because of the drawing together of the turning points.

2. The drawing together of the turning points is a sign of the phase transition. In order to analyze this phenomenon, let us consider the classical trajectories of the system. They can be determined after averaging the Hamiltonian (8) over the coherent state, found in Ref. 7, of the nonaxial rotator. If the orientation of the vector I is specified by the polar angles ϑ and ϕ, the equation of the trajectories has the form

$$\cos^2\vartheta=(\varepsilon/\varepsilon_0-\sin^2\phi)/(\xi^2+\cos^2\phi).$$

There are two degenerate (with respect to the direction of the vector I) families of trajectories: around the north and south poles if $\varepsilon_0<\varepsilon<\varepsilon_0(1+\xi^2)$, and around the points $\phi=0$ and π on the equator if $0<\varepsilon<\varepsilon_0$. These two families are separated by two separatrices with energy $\varepsilon=\varepsilon_0$.

In the phase transition region, when ξ is large, the separatrices approach the equator, and the energy ε_g of

the lowest level approaches ε_0. Therefore, the classical trajectory corresponding to the lowest level with energy $\varepsilon_\ell < \varepsilon_0$ is located near the equator in the left or right hemisphere (Fig. 1). The turning points a and a' of the classical trajectories approach the point on the equator with $\phi = \pi/2$ as ξ increases. The turning points symmetrical to a and a' on the opposite side of the sphere approach the point $\phi = 3\pi/2$, which also lies on the equator.

The points A and A' are the sterographic projections of the turning points a and a', since, by definition,

$$e^{i\varphi} = e^{i\varphi}\operatorname{ctg}(\vartheta/2).$$

Consequently, the drawing together of the turning points A and A' is a consequence of the drawing together of the two degenerate classical trajectories. Because of this, tunneling from one trajectory to another becomes possible, as a result of which the motion of the angular-momentum vector ceases to be localized, and its end moves over the whole sphere near the equator. This circumstance leads to the intensification of the zero-point oscillations of the vector I, an intensification which is characteristic of the phase transition region.

In deriving the quantization conditions (20) and (21), we assumed that the reversal points A and A' are isolated points, i.e., that the regions around these points where the quasiclassical approximation is inapplicable do not overlap. Let us determine the condition of applicability of this approximation. To do this, we find from the equation

$$|g'(z)|/[2|g(z)|^{3/2}] = 1$$

the radius of the circle inside which the quasiclassical approximation is inapplicable, and compare it with the distance between the points A and A'. If these points are located on the real axis (i.e., if $\varepsilon < \varepsilon_0$), then they will be isolated when

$$p\{2[\lambda(1-\lambda)]^{1/2}(\arcsin\lambda^{1/2})^3\}^{1/2} > 1. \tag{31}$$

If, on the other hand, the points A and A' are located on the imaginary axis (i.e., if $\varepsilon > \varepsilon_0$), then the condition for them to be isolated has the form

$$p\{2[\lambda(1+\lambda)]^{1/2}(\operatorname{argsh}\lambda^{1/2})^3\}^{1/2} > 1. \tag{32}$$

The inequalities (31) and (32) together with Eqs. (27) and (30) allow us to determine the region of p values where the quasiclassical approximation breaks down because of the drawing together of the reversal points. It is natural that this region depends on the quantum number n. Thus, for the yrast band in the quasiclassical approximation breaks down for p values lying in the interval $0.28 < p < 1.90$.

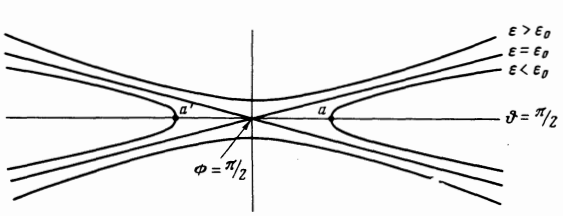

FIG. 1. Classical trajectories of a nonaxial rotator in the phase transition region near the point $\phi = \pi/2$ on the equator.

The breakdown of the quasiclassical approximation because of the drawing together of the turning points does not lead to any difficulties whatsoever, since, in the first place, the accuracy of the computations for sufficiently large I is fairly high even in the region where these points approach each other and, in the second place, we can use the quasiclassical approximation with parabolic cylinder functions, which approximation is free from the above-considered shortcoming.

3. As has already been said, the phase transition can be detected by detecting the intersection of the bands. Another sign of the phase transition is the decrease of the energy interval ΔE_I between the first side band ($n=1$) and the yrast band ($n=0$):

$$\Delta E_I = E_{1I} - E_{0I}$$

as the phase transition point I_0 is approached. It is convenient to refer this energy to the quantity $[(A_3 - A_1)(A_3 - A_2)]^{1/2}$. Then, according to the expressions (14) and (29), the dimensionless "excitation energy" will have the form

$$e_I(p) = \Delta E_I/[(A_3 - A_1)(A_3 - A_2)]^{1/2} = {}^1/_8 p|\lambda_1 - \lambda_0|. \tag{33}$$

The roots λ_0 and λ_1 are found from Eq. (27) or (30) as a function of the parameter p.

In the quasiclassical approximation the excitation energy e_I depends only on the parameter $p \sim 2I(|\gamma|)^{1/2}$, (28), which determines the proximity of a rotator with variable moments of inertia to the phase transition point $p = 0$. The function $e_I(p)$, (33), which is represented in Fig. 2 by the continuous curve, shows that the excitation energy decreases as the point $p = 0$ is approached. To determine e_I in the vicinity of this point, we can use perturbation theory, assuming the decrease of nonaxiality to be small[6]:

$$e_I(p) = 2 + \frac{11}{6144}p^4 + \ldots .$$

The corresponding curve is represented in Fig. 2 by the dotted line.

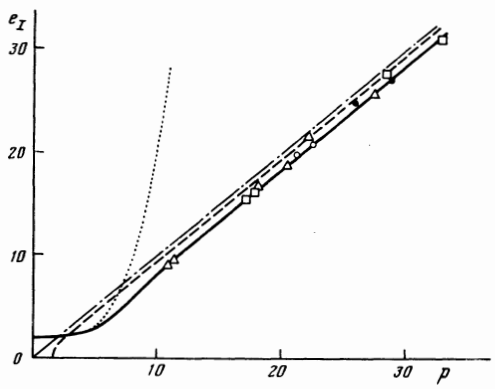

FIG. 2. The energy of the lowest excitation in the rotational spectrum of a nonaxial rotator. The curves, which correspond to different approximations, are described in the text. The points represent the results of the exact diagonalization of the Hamiltonian (8) for specific nuclei: ○) Sm^{154}, △) Er^{162}, □) Er^{164}, ●) Yb^{164}.

For sufficiently large p, the exponential terms in Eqs. (27) and (30) are small, and we can derive for the excitation energy the analytic expression

$$e_I(p)=\pi p/[2\mathbf{K}(\sqrt{1-\lambda_g})], \tag{34}$$

where λ_g is the root of the equation

$$\mathbf{E}(\sqrt{1-\lambda_g})-\lambda\mathbf{K}(\sqrt{1-\lambda_g})=\pi/2p. \tag{35}$$

In the limit $\varepsilon=0$ ($\lambda_g=1$) we obtain from the formula (34) the precession frequency (22):

$$e_I(p)=p.$$

The condition of applicability of this approximation is the inequality $p\gg 1$, which can be derived from Eq. (35). It is natural that it coincides with the general condition (26) in the limit of large ξ. The approximation (34) is close to the harmonic approximation, as follows from Fig. 2, in which the corresponding curves are represented by the dashed and dot-dash curves.

The excitation energy for negative values of the difference A_2-A_1 can be obtained by using the transformation $\mathscr{R}$, which is equivalent to the replacement of the quantity ξ by η, (25). The form of the dependence $e_I(p)$ will not change in the process. Thus, in the quasiclassical approximation the excitation energies to the left and right of the phase transition point should lie on the same curve.

Figure 2 shows the excitation energies, obtained through a numerical diagonalization of the Hamiltonian (8), for certain nuclei of the rare-earth elements in which the phase transition in question should be observed. In Table I we give the values, taken from Ref. 4, of the parameters α, I_0, and β (near I_0) necessary for the computation. The quantity p was computed from the formula

$$p^2=\frac{8\alpha}{\sqrt{3}}I_0^2|I-I_0|f\left(1-\frac{\mathscr{J}_\parallel}{\mathscr{J}_\perp}\right),\quad f(x)=\left\{\frac{2(1-x)^3}{x^2}[1-(1-x^2)^{1/2}]\right\}^{1/2},$$

into which enters, besides the moment of inertia $\mathscr{J}_\perp$ [see the formula (4)], the rigid-body moment of inertia $\mathscr{J}_\parallel$ about the symmetry axis of the axial rotator:

$$\mathscr{J}_\parallel=\mathscr{J}_s(1-\delta/3)^2.$$

We chose nuclei for which the deformation parameter β is approximately constant in the phase-transition region ($50<I<60$).

It can be seen from Fig. 2 that the quasiclassical approximation describes well the excitation energy near the phase transition point (the excitation energy was computed for spins $I=I_0\pm 1, I_0\pm 2$). The insignificant discrepancy for too large p is due to the inapplicability of the expression (33) in that region. It should be noted that, in spite of the slow variation of the nonaxial-deformation parameter near the transition point ($\alpha\ll 1$), the quantity $p>10$ for levels with spin $I_0\pm 1$. Consequently, we can use the harmonic approximation to describe these states.

TABLE I. Nucleus.

Nucleus	I_0	β	α	p	Θ^2
$_{62}Sm^{154}_{92}$	53	0.20	2.8°	22 *	$1.5\cdot10^{-6}$
$_{68}Er^{162}_{94}$	55	0.22	0.7°	11 *	$1.1\cdot10^{-3}$
$_{68}Er^{164}_{96}$	55	0.24	1.8°	17.5 *	$2.1\cdot10^{-5}$
$_{70}Yb^{162}_{92}$	52	0.11	7.3°	59 **	$1.0\cdot10^{-1}$
$_{70}Yb^{164}_{94}$	54	0.20	2.1°	29 **	$1.5\cdot10^{-1}$

*For spin I_0+1.
**For spin I_0+2.

4. Let us consider two levels in the yrast band between which the phase transition point I_0 is located. If their spins differ by two units, then the nonaxial deformation of the nucleus has opposite signs in these states, and the angular-momentum vector has, according to the foregoing, different orientations in the body coordinate system. Therefore, the $E2$ transitions between these levels will be weakened.

To estimate the degree of forbiddenness of the $E2$ transitions, let us compute the overlap integral for two functions (7) of the nonaxial rotator that correspond to deformation parameters (i.e., γ values) of opposite signs:

$$\Theta=\int\tilde{\Psi}^*_{I_0M_0}\Psi_{I_0M_0}dg=\frac{1}{2^{2I_0}}\int_{-\infty}^{\infty}\frac{dy}{(\mathrm{ch}\,y)^{2I_0+2}}\int_0^{2\pi}dx\,\bar{\psi}^*_{0I_0}(z)\Psi_{0I_0}(z),$$

where the function $\bar{\psi}_{0I0}=\mathscr{R}\psi_{0I0}$ corresponds to $\gamma<0$. Since the quantity p is large in both states, we can use the harmonic approximation (23) for these functions. Simple calculations lead to the following result:

$$\Theta^2=\frac{2[(1+\xi^2)(1+\eta^2)]^{1/4}}{(1+\xi^2)^{1/2}+(1+\eta^2)^{1/2}}\exp\left\{-\frac{\pi^2(I_0+1/2)}{4[(1+\xi^2)^{1/2}+(1+\eta^2)^{1/2}]}\right\}, \tag{36}$$

where the quantity η^2, which is given by the expression (25), differs from ξ^2 in that the x'- and y'-axis subscripts have been interchanged. If $\xi^2=\eta^2\gg 1$, i.e., if the transition point I_0 is located exactly halfway between the levels in question (the Sm^{154}, $Er^{162,164}$ nuclei), then the expression (36) gets simplified:

$$\Theta^2=\exp(-\pi^2p/16),$$

and the degree of forbiddenness is determined by the already known parameter p. Because of the large magnitude of this parameter, the overlap integral is small (see Table I), so that the lifetime of the upper level with spin I_0+1 can be of the same order of magnitude as the lifetime of the low-lying rotational states of the nucleus.

If the phase transition point is close to one of the levels of the yrast band, then the degree of forbiddenness can be estimated by considering the transition between an axial and a nonaxial state. In this case the overlap integral is equal to

$$\Theta_0^2=\left[\frac{(1+\xi^2)^{1/2}}{\pi(I_0+1/2)}\right]^{1/2}\approx\left(\frac{2}{\pi p}\right)^{1/2},$$

and the degree of forbiddenness, as follows from Table I (the $Yb^{162,164}$ nuclei), is not high. This result can be explained by the fact that in this case there is no preferred orientation for the angular-momentum vector in the axial state.

Finally, let us estimate the degree of forbiddenness resulting from a change in the magnitude (but not in

sign) of the nonaxial deformation. If the values of the nonaxial deformation for two neighboring states of the yrast band, to which correspond the parameters ξ_1 and ξ_2, differ by the amount $\Delta\gamma$, then the overlap integral is equal to

$$\Theta_\gamma^2=\frac{2[(1+\xi_1^2)(1+\xi_2^2)]^{1/4}}{(1+\xi_1^2)^{1/2}+(1+\xi_2^2)^{1/2}}\approx 1+k\Delta\gamma,\quad k\sim 1.$$

All the quantities Θ^2, Θ_0^2, and Θ_γ^2 found above determine the forbiddenness of the $E2$ transitions that stems from the differences among the rotational states of the nucleus. A change in the deformation of the nucleus caused by a change in the nuclear, self-consistent field leads to an increase in the degree of forbiddenness because of the difference among the internal states. But this effect, being a smooth function of the spin I, does not depend on the proximity to the phase transition point.

§4. CONCLUSION

As the angular momentum of the nucleus in the yrast band increases, the shape of the nucleus begins to change appreciably at spin values $I\sim 40$, when the orbital energy becomes comparable to the change that occurs in the rotational energy of the nucleus when its shape changes. Thus, a transition from the oblate to the prolate shape has been found to occur in the Te^{118} nuclei at $I\sim 30$ (Ref. 8) and Er^{159} nuclei at $I\sim 40$ (Ref. 9). This result was obtained from indirect data; for the measurement of the deformation in the yrast band for such spins is not possible at present. Therefore, it is extremely important for us to indicate the phenomena that are connected with a change in the deformation, and have a significant effect on the rotational spectrum of the nucleus in the region of the yrast band.

As an example of such phenomena, we can cite the above-described phase transition, which occurs when the sign of the nonaxial deformation of the nucleus is changed smoothly. This transition is a consequence of the shell effects that occur in nuclei in which the subshell with the largest single-particle angular momentum j near the Fermi surface has just began to fill up. The phase transition parameter, which determines the character of the rotation of the nucleus is the quantity $|\gamma|I_0^2$. Therefore, even when the equilibrium nonaxial deformation in the vicinity of the transition point I_0 is varied slowly, the rotational spectrum of the nucleus in the transition region changes markedly, since $I_0 \gg 1$. The phase transition region is characterized by the intersection of bands, the approach of the sides bands to the yrast band, and a high degree of forbiddenness of the $E2$ transitions in the yrast band in the vicinity of the point I_0.

In the paper we have developed a macroscopic theory of the phase transition on the basis of a model of a rigid rotator with variable principal moments of inertia. The question of the applicability of this model requires further investigation. It is well known that, in the adiabatic approximation, the amplitude of the zero-point γ vibrations for the ground state of a nucleus lies in the range from 8 to 19° (Ref. 6), i.e., exceeds the coefficient α in the formula (5) by a considerable factor.[5] We should expect the amplitude of the zero-point vibrations in a rotating nucleus to be smaller, since, according to the expression (3), the rigidity of the γ vibrations is appreciably greater because of the centrifugal term $I_0^2/\mathcal{J}$. Nevertheless, the γ-vibration-rotation interaction near the critical point I_0 can to some extent "smear out" the above-indicated effects. This will not occur in nuclei in which the γ vibrations are actually two-quasiparticle excitations [as, for example, in Yb^{172} (Ref. 10)].

It should also not be forgotten that the yrast band for so large spins is the envelope of a large number of intersecting bands with different single-particle structures. Therefore, the question of the adequacy of the rotator model (like the computation of the probability for transition between states with different deformations) comes under the purview of the microscopic theory. Finally, the question of the applicability of the phase-transition concept to the nucleus, which is, strictly speaking, not a macroscopic system, is a nontrivial one.

[1)] Here and below all numbers pertain to the nuclei of the rare-earth elements.

[2)] This assumption is inadmissible at the point I_0 because of the strong spin (I) fluctuations (see below).

[3)] It is assumed that the nucleus rotates about the axis, x', with the largest moment of inertia ($\gamma > 0$). The final result (5) and (6) will not change if the rotation is considered to be about an arbitrary axis in the $x'y'$ plane.

[4)] In a sufficiently narrow range of spins in the vicinity of I_0, the parameter γ is small, and the axial deformation β can be considered to be a constant. In this region $V\approx V_0+c\gamma^2$.

[5)] We have in mind the classical trajectories of the end of the vector **I** on a sphere of radius I.

[6)] Only such values of this parameter will be needed by us below. The small ξ values correspond to an almost axial rotator extended along the y' axis.

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