Fluctuations of the director of a nematic liquid crystal in a cell of finite thickness

B. Ya. Zel'dovich and N. V. Tabiryan[1)]

Institute of Problems of Mechanics, Academy of Sciences, USSR
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Scattering of light by thermal fluctuations of the director orientation in nematic liquid crystals is discussed. In contrast to the traditional formulation of the problem, we calculate the transverse correlation of the polarized component in a light wave after passage through the cell in the near zone, right at the exit from the cell. It is shown that for planar orientation of the nematic liquid crystal, the strongest correlation decrease occurs for a wave of the extraordinary type, obliquely incident on the cell. Explicit expressions for the correlation of the fields are obtained in the single-constant approximation.

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1. INTRODUCTION

At present liquid-crystal (LC) cells are widely used for making devices to transform information or to regulate optical transparencies, and in many other devices. The recently discovered gigantic optical nonlinearity of nematic liquid crystals (NLC)[1] enables us to count on the realization of a whole series of nonlinear optical effects in NLC. But as is well known, the LC mesophase gives a quite strong scattering of light, and with it a deterioration of the transverse coherence of a light wave passing through a cell. An experimental investigation of the transverse incoherence introduced by a NLC cell was made, for example, in a paper of Akopyan and others.[2] Such investigations in principle make it possible to obtain valuable information about the fluctuational properties of NLC. Unfortunately, the theory of light scattering in NLC by fluctuations of the director has so far been based on the traditional formulation of the problem,[3,4] in which the differential scattering (extinction) coefficient is calculated in the far zone for an infinite medium (see also Ref. 5).

The present paper treats the specific features of director fluctuations in thin cells of NLC. Furthermore, the formulation of the electrodynamic problem of light scattering corresponds to calculation of the degree of coherence at two points of the cross section for the field of the light wave in the near zone, i.e., directly after the cell. We consider only that polarized component of the field of the transmitted wave which coincides with the incident wave.

2. FLUCTUATIONS OF THE DIRECTOR IN A CELL WITH PLANAR ORIENTATION

We take the free energy included in a cell of NLC, occupying the region $0 \leqslant x \leqslant L$, in the standard form

$$F = {}^1/_2 \int [K_{11}(\operatorname{div}\mathbf{n})^2 + K_{22}(\mathbf{n}\operatorname{rot}\mathbf{n})^2 + K_{33}[\mathbf{n}\times\operatorname{rot}\mathbf{n}]^2]\,dx\,d^2\rho. \tag{1}$$

Here $\mathbf{n}(\mathbf{r})$ is the unit vector of the NLC director, and K_{11}, K_{22}, K_{33} are the Frank constants. We suppose that the normal to the cell is directed along the x axis, so that $\mathbf{r} = (x, \rho) \equiv (x, y, z)$.

We first consider a planar orientation and shall suppose that the director is rigidly pinned at the boundaries (see Fig. 1):

$$\mathbf{n}(x=0,\ y,\ z) = \mathbf{n}(x=L,\ y,\ z) = \mathbf{n}^0 \equiv \mathbf{e}_z. \tag{2}$$

We shall suppose that the fluctuational departures $\delta\mathbf{n}(\mathbf{r})$ of the director from the unperturbed direction of planar orientation are small, and that by virtue of the normalization condition, $\delta\mathbf{n}\cdot\mathbf{n}^0 \approx 0$; that is,

$$\delta\mathbf{n}(\mathbf{r}) = \{\delta n_x(\mathbf{r}),\ \delta n_y(\mathbf{r}),\ 0\}.$$

In investigation of fluctuations in a cell of finite thickness, the expansion usually used for the quantities $\delta n_x(\mathbf{r})$ and $\delta n_y(\mathbf{r})$ is in functions of the form

$$\mathscr{L}_{mq}(x, \rho) = \sin(m\pi x/L)\,e^{i\mathbf{q}\rho}. \tag{3a}$$

An advantage of these functions is that they satisfy the boundary conditions (2) at the walls of the vessel. Unfortunately, expansion in such functions does not permit diagonalization of the expression (1) for the free energy if $K_{11} \neq K_{22}$ and $q_y \neq 0$. The function $\mathscr{L}_{mq}$ of (3a) can be represented as the two terms

$$\mathscr{L}_{mq}(x, \rho) = 1/2i\{\exp(i\mathbf{k}_1\mathbf{r}) - \exp(i\mathbf{k}_2\mathbf{r})\},$$
$$\mathbf{k}_1 = (m\pi/L,\ \mathbf{q}),\quad \mathbf{k}_2 = (-m\pi/L,\ \mathbf{q}). \tag{3b}$$

For an infinite medium, the exponential functions $\delta n \propto e^{i\mathbf{k}\cdot\mathbf{r}}$ make it possible to diagonalize the expression (1), but when $K_{11} \neq K_{22}$ the eigenvalues and the corresponding combinations $\delta n_x, \delta n_y$ depend explicitly on the mutual orientation of the vectors $\mathbf{n}^0$ and $\mathbf{k}$ (see Ref. 3). In particular, the combination $(\delta n_x, \delta n_y)$ corresponding to one of the two eigenvalues for $\mathbf{k}_1$ no longer coincides with the combination corresponding to the same eigenvalue for $\mathbf{k}_2$.

For this reason, in order to find functions that diagonalize the expression (1), it is necessary to solve the

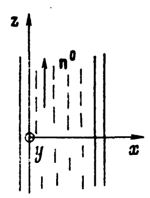

FIG. 1. Cell containing NLC, uniformly oriented along the z axis, $\mathbf{n}^0 = \mathbf{e}_z$. The x axis is perpendicular to the cell plates; the y axis is in the plane of the plates and perpendicular to the plane of the figure.

eigenvalue of the eigenfunctions problem. On setting

$$\mathbf{n}(\mathbf{r})=\mathbf{n}^0+\mathbf{e}_x\delta n_x(\mathbf{r})+\mathbf{e}_y\delta n_y(\mathbf{r})+O(\delta n^2)$$

and retaining in (1) only terms of the second order in δn, we get

$$\delta F=\int \delta n_i(\mathbf{r})\mathscr{L}_{ik}\delta n_k(\mathbf{r})\,d^3\mathbf{r},$$
$$\mathscr{L}_{xx}=-K_{11}\frac{\partial^2}{\partial x^2}-K_{22}\frac{\partial^2}{\partial y^2}-K_{33}\frac{\partial^2}{\partial z^2},$$
$$\mathscr{L}_{xy}=(K_{22}-K_{11})\frac{\partial^2}{\partial x\partial y},\tag{4}$$
$$\mathscr{L}_{yy}=-K_{22}\frac{\partial^2}{\partial x^2}-K_{11}\frac{\partial^2}{\partial y^2}-K_{33}\frac{\partial^2}{\partial z^2}.$$

The expression (4) for δF has the form of a mean value of the operator $\mathscr{L}$ acting in the space of the functions $(\delta n_x, \delta n_y)$. It is easy to verify that this operator, under the boundary conditions (2), is self-adjoint. Therefore the system of its eigenfunctions $\delta\mathbf{n}(\mathbf{r})$,

$$\hat{\mathscr{L}}\delta\mathbf{n}(\mathbf{r})=\Lambda\delta\mathbf{n}(\mathbf{r})$$

has the property of completeness and orthogonality. If, moreover, we normalize the eigenfunction $\delta\mathbf{n}^{(\alpha)}$, with the eigenvalue $\Lambda^{(\alpha)}$, by the condition

$$\int \delta\mathbf{n}^{(\alpha)*}\delta\mathbf{n}^{(\beta)}d^3\mathbf{r}=\delta_{\alpha\beta},\tag{5}$$

then the expression for F takes the form

$$F=\sum_\alpha \Lambda^{(\alpha)}|C_\alpha|^2,\tag{6a}$$

$$\delta\mathbf{n}(\mathbf{r})=\sum_\alpha C_\alpha\delta\mathbf{n}^{(\alpha)}(\mathbf{r}).\tag{6b}$$

In thermodynamic equilibrium, by the equipartition theorem,

$$\langle C_\alpha^* C_\beta\rangle=(\Lambda^{(\alpha)})^{-1}k_BT\delta_{\alpha\beta},$$

so that

$$\langle\delta n_i(\mathbf{r}_1)\delta n_k(\mathbf{r}_2)\rangle=k_BT\sum_\alpha \delta n_i^{(\alpha)*}(\mathbf{r}_1)\delta n_k^{(\alpha)}(\mathbf{r}_2)(\Lambda^{(\alpha)})^{-1}.\tag{7}$$

An important property of the operator $\hat{\mathscr{L}}$ of (4) is its invariance to a shift along all three coordinates x, y, z; this follows from the uniformity of the unperturbed state of a nematic. Only the boundary conditions distinguish the two fixed planes $x=0$ and $x=L$. Therefore we shall seek an eigenfunction of the operator $\hat{\mathscr{L}}$ in the form

$$\delta\mathbf{n}(\mathbf{r})=e^{i\mathbf{q}\boldsymbol{\rho}}\sum_{j=1}^{N}(\mathbf{e}_xA_j+\mathbf{e}_yB_j)e^{i\mu_jx}.\tag{8}$$

From the expression (4) for $\hat{\mathscr{L}}$, it follows that the values of A_j, B_j, and μ_j satisfy the system of equations

$$\begin{gathered}A(-K_{11}\mu^2-K_{22}q_y^2-K_{33}q_z^2+\Lambda)-B(K_{11}-K_{22})q_y\mu=0,\\ -A(K_{11}-K_{22})q_y\mu+B(-K_{22}\mu^2-K_{11}q_y^2-K_{33}q_z^2+\Lambda)=0\end{gathered}\tag{9}$$

for each of the values of j. Setting the determinant of this system equal to zero gives a biquadratic equation for μ; its roots are conveniently written in the form

$$\mu_{1,3}=\pm\left(\frac{\Lambda-K_{11}q_y^2-K_{33}q_z^2}{K_{11}}\right)^{1/2},\quad \mu_{2,4}=\pm\left(\frac{\Lambda-K_{22}q_y^2-K_{33}q_z^2}{K_{22}}\right)^{1/2}.\tag{10}$$

Thus the parameter N of (8) is equal to four. The appearance of four terms in the sum (8) can be explained by the following considerations. When $K_{11}=K_{22}$, there are two exponential structures of the form (3b) for each of the two forms of perturbation, δn_x and δn_y. It is clear that passage to the general case $K_{11}\neq K_{22}$ can change the specific exponential arguments but not the number of exponential structures. Furthermore, from equations (9) the following relations follow:

$$q_yA_j=\mu_jB_j\quad(j=1,3),\qquad q_yB_j=-\mu_jA_j\quad(j=2,4).\tag{11}$$

Substitution of the expression (8) in the boundary conditions (2) gives

$$\sum_{j=1}^4 A_j=0;\quad \sum_{j=1}^4 B_j=0;\quad \sum_{j=1}^4 A_je^{i\mu_jL}=0;\quad \sum_{j=1}^4 B_je^{i\mu_jL}=0,\tag{12}$$

so that, altogether, (11) and (12) contain eight linear homogeneous equations with eight unknowns. On setting the determinant of this system equal to zero, we get the following equation for the eigenvalues Λ of the operator $\mathscr{L}$:

$$2q_y^2\mu_1\mu_2(1-\cos\mu_1L\cos\mu_2L)+(q_y^4+\mu_1^2\mu_2^2)\sin\mu_1L\sin\mu_2L=0;\tag{13}$$

we recall that $\mu_1(\Lambda)$ and $\mu_2(\Lambda)$ are determined by the explicit expressions (10).

If we write the dissipation function in the form

$$R=\frac{1}{2}\gamma\int\left(\frac{\partial\delta\mathbf{n}}{\partial t}\right)^2d^3\mathbf{r},\tag{14}$$

then the mode with eigenvalue Λ decays with retention of the spatial structure, according to the exponential law

$$M_\Lambda(\mathbf{r},t)\propto M_\Lambda(\mathbf{r})e^{-\Lambda t/\gamma}.$$

In writing (14), we have neglected the hydrodynamic mechanism of relaxation (for more details, see our earlier paper[6]).

The characteristic time of hydrodynamic relaxation is $\tau_2\sim\rho/\eta q^2$ where $\rho\sim1$ g/cm^3 and $\eta\sim10^{-2}$ P. The time of orientational relaxation is $\tau_1\sim\gamma/Kq^2$, where $K\sim10^{-6}$ dyn and $\gamma\sim10^{-2}$P. Therefore, independently of the value of q, we have the estimate $\tau_2/\tau_1\sim10^{-2}$; that is, the hydrodynamic variables follow the orientation quasistatically. It is for this reason that the influence of hydrodynamics can be neglected here.

In the general case $K_{11}\neq K_{22}$, the solution of equation (13) is quite complicated. When $K_{11}=K_{22}$, it follows from (10) that $\mu_1=\mu_2$, so that equation (13) reduces to

$$(q_y^2+\mu^2)^2\sin^2\mu L=0.$$

Hence follows

$$\mu_m=mp,\quad p=\pi/L,\quad \Lambda_{m,q}=K_{11}(q_y^2+m^2p^2)+K_{33}q_z^2,\tag{15}$$

and the eigenfunctions have the form (3a) separately for δn_x and δn_y.

Thus when $K_{11}=K_{22}$, the spatial-temporal correlators of the fluctuations of the components of the director ($n_0=e_z$) in a cell of thickness L in the planar orientation are

$$\langle\delta n_i(\mathbf{r}_1,t)\delta n_k(\mathbf{r}_2,t+\tau)\rangle=(\delta_{ik}-n_i^0n_k^0)$$
$$\times\frac{k_BT}{2\pi^2K\ L}\int d^2q\sum_{m=1}^\infty\exp[i\mathbf{q}(\boldsymbol{\rho}_1-\boldsymbol{\rho}_2)]\sin mpx_1\sin mpx_2\Lambda_{m,q}^{-1}\exp(-\Lambda_{m,q}|\tau|/\gamma).\tag{16}$$

For the single-time correlator (i.e., when $\tau=0$), it is possible to carry out an explicit integration over $d^2\mathbf{q}$. We give the specific result for the case $K_{11}=K_{22}=K_{33}$:

$$\langle\delta n_i(x_1,0)\delta n_k(x_2,\rho)\rangle=(\delta_{ik}-n_i^0n_k^0)\frac{k_BT}{2K_{11}L}\times\sum_{m=1}^{\infty}iH_0^{(1)}(imp|\rho|)\sin mpx_1\sin mpx_2, \tag{17}$$

where $H_0^{(1)}$ is a Hankel function.

We note that for $|\rho_1-\rho_2|\to0$ the expression (17) diverges. Furthermore, there is logarithmic divergence even of an individual term with $m=$const, since

$$H_0^{(1)}(imp|\rho|)\propto\ln(mp|\rho|)\quad\text{as}\quad|\rho|\to0.$$

This divergence is cut off when $|\rho_1-\rho_2|$ is of the order of several molecular dimensions, where the continuum theory of LC and the director concept cease to operate. Actually, however, in our problem the cutting off of this logarithm occurs at still larger radii (see below). The expression for $K_{11}=K_{22}\neq K_{33}$ differs from (17) by the stretching $z'\to z(K_{11}/K_{33})^{1/2}$.

Above, we considered a cell with planar orientation. The case of homeotropic orientation of the cell is somewhat simpler mathematically, since here both vectors, the unperturbed director $\mathbf{n}^0$ and the normal $\mathbf{e}_x$ to the planes that bound the cell, coincide: $\mathbf{n}^0=\mathbf{e}_x$; as a result, there is an axis of symmetry of the problem. The eigenfunctions here have a spatial variation of the form (3a); the condition $|\mathbf{n}^0+\delta\mathbf{n}|=1$ gives $\delta n_x=0$; and in the most general case, $K_{11}\neq K_{22}\neq K_{33}$, we have

$$\langle\delta n_i(x_1,0,t)\delta n_k(x_2,\rho,t+\tau)\rangle=\frac{k_BT}{2\pi^2L}\left\{\sum_m\int d^2q\left(\delta_{ik}^{(2)}-\frac{q_iq_k}{q^2}\right)\Lambda_2^{-1}\exp\left[-\frac{\Lambda_2|\tau|}{\gamma}\right]+\sum_m\int d^2q\frac{q_iq_k}{q^2}\Lambda_1^{-1}\exp\left[-\frac{\Lambda_1|\tau|}{\gamma}\right]\right\}; \tag{18}$$

$$\Lambda_1=K_{11}q^2+K_{33}m^2p^2,\qquad\Lambda_2=K_{22}q^2+K_{33}m^2p^2,$$

where i,k run through the values y,z and where $\delta_{ik}^{(2)}$ is the two-dimensional Kronecker symbol. We note that when $K_{11}=K_{22}$, any two components of the vector $\delta\mathbf{n}$ orthogonal to each other are uncorrelated for arbitrary x_1,x_2,ρ,τ.

3. PHASE FLUCTUATIONS OF RADIATION TRANSMITTED THROUGH A LC CELL

In the geometric-optics approximation and for not too great optical anisotropy of the nematic, the advance of phase of a beam propagated at angle α to the normal is

$$\delta\varphi(\rho_0)=\varepsilon_a\frac{\omega}{c}\frac{1}{n_e\cos\alpha}(\mathbf{n}^0\mathbf{e})(\mathbf{e}\mathbf{h}(\rho_0)). \tag{19}$$

The two-dimensional vector $\mathbf{h}=(h_x,h_y)$ (such that $\mathbf{h}\cdot\mathbf{n}^0=0$), of dimensions length, describes the integral of the director fluctuation $\delta\mathbf{n}$ along the beam:

$$h_i(\rho_0)=\int_0^L\delta n_i(x,\rho_0+\mathbf{t}x)\,dx. \tag{20}$$

Here $\mathbf{e}$ is the unit vector of polarization of the wave; $\varepsilon_a=\varepsilon_\parallel-\varepsilon_\perp$ is the anisotropy of the optical polarizability; the vector $\mathbf{t}$, with components t_y and t_z, describes the inclination of the beam, and $|\mathbf{t}|=\tan\alpha$ (see Fig. 2).

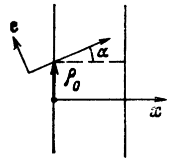

FIG. 2. Calculation of the phase fluctuation of radiation transmitted through a LC cell. A beam with polarization vector $\mathbf{e}$ is propagated at angle α to the normal to the cell plates. The two-dimensional vector $\mathbf{p}_0$ lies in the yz plane and describes the position of the beam with respect to the center of coordinates.

We shall first discuss the dependence of the polarization factor on the angle of inclination α (we measure the angle within the medium, i.e., with allowance for refraction). For definiteness, let $\mathbf{t}=(t_y,t_z)=(0,\tan\alpha)$, i.e., let the wave vector lie in the xz plane, formed by the director $\mathbf{n}^0=\mathbf{e}_z$ and the normal $\mathbf{e}_x$ to the cell. Then the ordinary wave, with polarization unit vector $\mathbf{e}_y$, undergoes no phase fluctuation at all. This assertion is correct for any orientation of the NLC in the cell: planar, oblique, homeotropic, and generally inhomogeneous. It was discussed earlier[7,8]; Ref. 7 contains a calculation and estimates.

For the extraordinary wave, when $\varepsilon_a\lesssim1$ we can set $\mathbf{e}=\mathbf{e}_z\cos\alpha-\mathbf{e}_x\sin\alpha$, and only the x component of the vector $\mathbf{h}$ operates in the expression (19); the dependence of $\delta\varphi$ on the angle α has the form

$$\delta\varphi(\rho_0)=-\varepsilon_a\frac{\omega}{cn_e}\sin\alpha h_x(\rho_0). \tag{21a}$$

Thus we arrive at the important conclusion that the strong phase fluctuations (21), caused by thermal fluctuations of the director, show up only in a wave of the extraordinary type, and furthermore only for oblique incidence, $\alpha\neq0$. This requirement was found necessary also for appearance of the gigantic nonlinearity of NLC.[1] This coincidence is not accidental; it is due to the general relation between fluctuations and susceptibility (a form of the fluctuation-dissipation theorem).

What is usually measured experimentally is not an exactly local value of the phase, but a value averaged over some area of the cross section. If we denote by $f(\rho)$ the normalized weighting function,

$$\int f(\rho)\,d\rho_y\,d\rho_z=1,$$

we can write

$$H(\rho)=\int h_x(\rho')f(\rho-\rho')d^2\rho'. \tag{22}$$

Correspondingly, the recorded phase fluctuations are given by the expression

$$\langle\delta\varphi(\rho_1)\delta\varphi(\rho_2)\rangle=\left(\frac{\varepsilon_a\omega}{cn_e}\right)^2\sin^2\alpha\langle H(\rho_1)H(\rho_2)\rangle. \tag{21b}$$

In the expression (20), (22) for $H(\rho)$, we must substitute δn_i in the form of an expansion in the functions (3a). Then the integration over $d^2\rho'$ and over dx gives

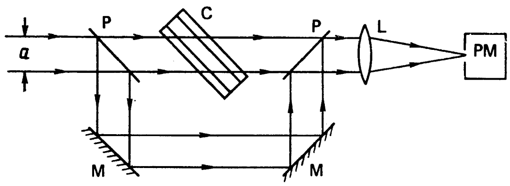

FIG. 3. Experimental scheme for heterodyne measurement of phase, averaged over the cross section: P, semitransparent plates; M, mirrors; L, lens; C, cell of NLC, oriented obliquely in relation to the incident light beam: PM, photomultiplier.

a factor of the form

$$\exp(i\mathbf{q}\rho_0)\tilde{f}(\mathbf{q})G(m,\ \mathbf{qt}),$$

$$\tilde{f}(\mathbf{q})=\int f(\rho)e^{-i\mathbf{q}\rho}d^2\rho,\quad \tilde{f}(0)=1, \tag{23}$$

$$G(m,\mathbf{qt})=\frac{mp}{(mp)^2-(\mathbf{qt})^2}(1-\cos m\pi e^{i\mathbf{qt}L}).$$

The correlator of the fluctuations of the values of $H(\rho_1)$ and of $H(\rho_2)$ can be obtained on the basis of the expression (16). In the single-constant approximation $K_{ii}=K$, we get

$$\langle H(\rho_1)H(\rho_2)\rangle=k_BT/2\pi^2KL$$

$$\times\sum_m\int d^2\mathbf{q}\exp\{i\mathbf{q}(\rho_1-\rho_2)\}|\tilde{f}(\mathbf{q})|^2|G(m,\mathbf{qt})|^2(q^2+m^2p^2)^{-1}. \tag{24}$$

We shall further consider several different experimental schemes.

A. Let the light beam have width a much greater than than the cell thickness, and let the phase averaged over an area S of the cross section be measured by the heterodyne method: see Fig. 3. Then $\tilde{f}(\mathbf{q})$ is a very narrow function of its agrument $[(\Delta q)^2\sim S^{-1}]$, and the integral over $d^2\mathbf{q}$ gives

$$|\tilde{f}(\mathbf{q})|^2d^2\mathbf{q}=(2\pi)^2\int f^2(\rho)d^2\rho=\frac{(2\pi)^2}{S_{\text{eff}}}.$$

When $a\gg L$, all the other factors in (24) are smoother, and we may take their value at $q=0$; furthermore, we may set $\rho_1=\rho_2$. As a result

$$\langle H^2\rangle=\frac{k_BT}{kLS_{\text{eff}}}\frac{8}{p^4}\sum_m\frac{1}{m^4}\sin^2\frac{m\pi}{2}. \tag{25}$$

The principal contribution to this sum comes from the mode with $q=0, m=1$ (the exact value of the sum in (25) is $\pi^4/96=1.014678$). As was to be expected, with increase of the averaging area S_{eff} (i.e., of the transverse dimensions of the beam) there is an increase of the number of independent contributions to the phase fluctuation, and therefore

$$\langle(\delta\varphi)^2\rangle\propto\langle H^2\rangle\propto S_{\text{eff}}^{-1}.$$

B. We now consider phase fluctuations of a narrow beam, with transverse dimension $a\ll L$. Then in the main region of integration in (24), the value of $|\tilde{f}(\mathbf{q})|^2$ may be considered to be unity, and furthermore $\rho_1=\rho_2$. We furthermore consider the case of comparatively small angles, $|\mathbf{t}|\ll 1$. The value of the function $G(m,\mathbf{qt}=0)$ is

$$|G(m,0)|^2=\frac{4\sin^2(m\pi/2)}{m^2p^2},$$

and the function $G(m,\mathbf{qt})$ itself falls off significantly at

$$\mathbf{qt}=q_z|\mathbf{t}|\propto mp=m\pi/L.$$

It can be shown that here also the main contribution comes from the term with $m=1$. If we suppose that $t=0$, then the integral over $d^2\mathbf{q}$ in the expression (24) diverges logarithmically at large q_z. Therefore we shall suppose that $|\mathbf{t}|\ll 1$ and shall calculate the integral over $d^2\mathbf{q}$ in (24) with logarithmic accuracy, i.e., we shall replace $q_{z\,\max}$ by $\pi/L|\mathbf{t}|$. Restricting ourselves to terms with $m=1$, we get

$$\langle H^2\rangle=\frac{4k_BT}{\pi^3k}L\ln\frac{1}{|\mathbf{t}|}. \tag{26}$$

For quite small $|\mathbf{t}|$ the bounding of the range of integration is connected with inaccuracy of the function $f(\rho)$ (so that $q_{\max}\sim\Delta\rho^{-1}$); in particular, because of diffraction it is known that $\Delta\rho\gtrsim(L/k)^{1/2}$, where $k=2\pi n/\lambda$ is the wave number of the light.

C. Finally, we consider the correlation of the phase fluctuations of fine beams, passing at a distance $|\rho|\gg L|\mathbf{t}|$. Furthermore, we shall again suppose that $|\mathbf{t}|\ll 1$. In this case also, the principal contribution to the correlator comes from the terms with $m=1$. The factor $G(m=1,\mathbf{qt})$ can again be approximated by its value at $|\mathbf{t}|\to 0$, and then the expression for $\langle H(\rho_1)H(\rho_1)\rangle$ is conveniently obtained directly from (17):

$$\langle H(\rho_1)H(\rho_2)\rangle=\frac{k_BT}{K}\frac{4L}{\pi^3}K_0\left(\frac{\pi|\rho_1-\rho_2|}{L}\right), \tag{27}$$

where $K_0(Z)=\frac{1}{2}i\pi H_0^{(1)}(iZ)$ is a modified Bessel function. For $\pi|\rho_1-\rho_2|\gtrsim L$ we have

$$K_0\left(\frac{\pi|\rho_1-\rho_2|}{L}\right)\approx\left(\frac{L}{2|\rho_1-\rho_2|}\right)^{1/2}\exp\left(-\frac{\pi|\rho_1-\rho_2|}{L}\right), \tag{28}$$

so that the phase-fluctuation correlation falls exponentially. For $|\rho_1-\rho_2|\to 0$, the expression (28) reduces to (26) on replacement of $|\rho_1-\rho_2|$ by $L|\mathbf{t}|/\pi$.

4. DISCUSSION OF RESULTS

The formulas obtained in this paper permit calculation of the mean square fluctuation and also of the spatial and temporal correlation of the phase of radiation transmitted through a LC cell. For planar orientation of the NLC, the strongest decrease of the correlation occurs for propagation of a wave of the extraordinary type obliquely incident on the cell. If, furthermore, the diameter a of the beam is much larger than the thickness L of the cell, then the mean square phase fluctuation $\langle(\delta\varphi)^2\rangle$ varies with the angle of inclination α (Fig. 2) as $\sin^2\alpha$ and with the cell thickness as L^3. For LC parameters characteristic of MBBA ($K=5.8\times10^{-7}$ erg/cm, $\varepsilon_a\approx 0.7$), cell thickness $L=10^{-2}$ cm, $a=5L$, $S_{\text{eff}}=\pi a^2/4$, wavelength $\lambda=0.4$ μm, and $\alpha=45°$, at $T=300$ K, we get from formulas (21) and (25) $\langle(\delta\varphi)^2\rangle^{1/2}\approx 0.08$ rad.

In the case considered in Sec. B, i.e., propagation of a fine beam ($a\ll L$) at small angles with respect to the normal to the cell, $\alpha\ll 1$, the variation of $\langle(\delta\varphi)^2\rangle$ with L and α is determined by the expression

$$\langle(\delta\varphi)^2\rangle\propto -L\alpha^2\ln\alpha.$$

For $\alpha=0.1$ and the values taken above for the other parameters, we get

$$\langle(\delta\varphi)^2\rangle^{1/2}\approx 0.09.$$

This means that the mean value of the transmitted field, measured by interference with the unperturbed reference beam, is

$$\langle E\rangle=\exp(-0.5\langle(\delta\varphi)^2\rangle)\approx 0.996.$$

And finally, we get for the mutual phase correlation of two beams, separated by a distance $|\rho_1-\rho_2|=L/\pi$ and propagated at angle $\alpha\approx 0.1$ (Case C),

$$\langle\delta\varphi(\rho_1)\delta\varphi(\rho_2)\rangle=1.5\cdot 10^{-2}.$$

The degree of correlation of the field is determined by the expression

$$\langle\exp\{i\delta\varphi(\rho_1)-i\delta\varphi(\rho_2)\}\rangle=\exp\{-\langle(\delta\varphi)^2\rangle+\langle\delta\varphi(\rho_1)\delta\varphi(\rho_2)\rangle\}\approx 0.992.$$

Here, as in cases B and C, we were compelled to restrict ourselves to consideration of small angles of inclination α of the beams. For large angles α ($\alpha\sim\pi/4$), for which consideration of the phenomenon has the greatest importance, it is necessary to do numerical calculations for the specific experiment.

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APPENDIX

EXPLICIT FORM OF THE EIGENFUNCTIONS FOR A CELL WITH PLANAR ORIENTATION, IN A NON-SINGLE-CONSTANT APPROXIMATION

We shall write here an explicit expression for an eigenfunction of the operator $\hat{\mathscr{L}}$. For this purpose, we express all the quantities A_j and B_j in terms of B_4 by means of equations (11) and (12) and substitute them in the equation (8) for the eigenfunction. Eliminating from consideration the third equation in (12) (in order to obtain a linearly independent system of equations), we get

$$\delta\mathbf{n}(\mathbf{r})=\frac{2B_4e^{iq\rho}}{i\mu_1\mu_2(\cos\mu_1L-\cos\mu_2L)+\mu_1\mu_2\sin\mu_2L+q_v^2\sin\mu_1L}$$
$$\times\left\{\frac{i\mathbf{e}_x}{\mu_2q_v}[q_v^2\mu_1\mu_2(\cos\mu_1L-\cos\mu_2L)(\cos\mu_2x-\cos\mu_1x)\right.$$
$$-(q_v^2\sin\mu_1L+\mu_1\mu_2\sin\mu_2L)(\mu_1\mu_2\sin\mu_1x+q_v^2\sin\mu_2x)]$$
$$+\mathbf{e}_v[(\cos\mu_1L-\cos\mu_2L)(q_v^2\sin\mu_1x+\mu_1\mu_2\sin\mu_2x)$$
$$\left.-(\cos\mu_1x-\cos\mu_2x)(\mu_1\mu_2\sin\mu_2L+q_v^2\sin\mu_1L)]\right\}. \qquad (A.1)$$

The normalization condition (5) determines the value of B_4. By use of (13) it is easy to verify that the expression (A.1) satisfies the boundary conditions (2). In the limit $K_{11}-K_{22}\to 0$, the expression (A.1) reduces to the eigenvector of the operator $\hat{\mathscr{L}}$ in the single-constant approximation:

$$\delta\mathbf{n}(\mathbf{r})=\{\delta n_x,0\},\quad \delta n_x\propto\sin mpx.$$

[1)]Optics Faculty, Erevan State University.

[1]B. Ya. Zel'dovich, N. F. Pilipetskiĭ, A. V. Sukhov, and N. V. Tabiryan, Pis'ma Zh. Eksp. Teor. Fiz. 31, 287 (1980) [JETP Lett. 31, 263 (1980)].
[2]S. A. Akopyan, S. M. Arakelyan, L. E. Arushanyan, and Yu. S. Chilingarvan, Kvant. Elektron. (Moscow) 4, 1387 (1977) [Sov. J. Quantum Electron. 7, 790 (1977)].
[3]P. G. de Gennes, Physics of Liquid Crystals, Oxford Univ. Press, 1974 (Russ. transl., Mir, 1977).
[4]S. V. Kir'yanov, Vestn. Mosk. Univ. Fiz. Astron. 20, 82 (1979).
[5]D. Langevin and M. A. Bouchiat, J. Phys. (Paris) 36, C1-197 (1975).
[6]B. Ya. Zel'dovich and N. V. Tabiryan, Preprint FIAN No. 201, 1978.
[7]N. B. Baranova and B. Ya. Zel'dovich, Pis'ma Zh. Eksp. Teor. Fiz. 32, 636 (1980) [JETP Lett. 32, 622 (1980)].
[8]S. M. Arakelyan, L. E. Arushanyan, and Yu. S. Chilingaryan, Zh. Eksp. Teor. Fiz. 80, 1186 (1981) [Sov. Phys. JETP 53, 607 (1981)].

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