Character of decay instability

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(Submitted 11 February 1981; resubmitted 24 June 1981)
Zh. Eksp. Teor. Fiz. **81**, 1707–1713 (November 1981)

If the initial wave is unstable in the upper half plane Im $\omega > 0$ and there are no branch points of the quasiwave number, or if waves traveling in the same direction coalesce at a branch point, the instability is convective. On the other hand, if a branch point $k(\omega)$ does exist in the upper half-plane Im $\omega > 0$, and not all the waves that merge at this point travel in the same direction, the instability is absolute. A Green's function that describes the evolution of the perturbations of the initial wave in space and in time is constructed. The growth rates of the decay instability of the harmonics are determined. The produced waves are richer in harmonics than the initial waves. It is shown that the decay instability of an Alfvén wave is absolute.

PACS numbers: 52.35.Py

1. INTRODUCTION

In investigations of decay instabilities[1,2] it is customarily assumed that all waves are monochromatic and have infinitesimal amplitudes of the same order of magnitude. (An exception is Ref. 3, dealing with the decay of an Alfvén wave of large amplitude with sawtooth magnetic force lines). Yet the interaction of higher harmonics plays an important role in plasma turbulence.[4] In addition, the amplitude of the initial wave is frequently considerably higher than that of the produced waves. Therefore the produced waves move in an inhomogeneous medium and this leads to a change in the dispersion equation. In the present article we obtain a general dispersion equation that connects the frequency and the quasiwave number of the perturbation with the monodromy (single-valuedness) matrix. No restrictions whatever are imposed on the amplitude and profile of the stationary initial wave.

An important question is that of the character of the instability (whether it is absolute or convective[5]). For the case when the amplitudes of the initial and produced waves are infinitesimals of the same order, so that all the waves move in a homogeneous medium, the character of the decay instability was investigated in Refs. 6–8. If, however, the amplitude of the initial wave greatly exceeds the amplitudes of the produced waves, the latter are reflected by the inhomogeneities due to the initial waves, and this can convert the absolute instability into convective and vice versa. In the case of a spatially periodic medium that is at rest, this question was investigated in Ref. 9. Since the character of the instability changes on going to a moving reference frame, the investigation of the character of decay instability calls for an additional analysis, and this is done in the present article. A Green's function is constructed, describing the evolution in space and in time of the perturbations of the initial stationary wave. It is shown that the instability is absolute if the upper half-plane of the complex frequency contains at least one branch point of the quasiwave number, in which waves that do not all travel in the same direction converge. The results pertain not only to decay instability, but also to instability of any moving spatially periodic system, for example at instability of a modulated electron beam. The general investigation is illustrated with the decay of an Alfvén wave of small amplitude but of arbitrary profile as the example. For a wave with sinusoidal profile, the results agree with Ref. 1. The growth increment of the harmonics is directly proportional to the number of the harmonic. It is shown that when an Alfvén wave decays into an Alfvén and slow magnetosonic wave, the instability is absolute.

2. DISPERSION EQUATION

The evolution of the perturbation $\varphi(z,t)$ of a stationary wave is described by a system of linear partial differential equations

$$\left[I\frac{\partial}{\partial t}+K(z-vt)\frac{\partial}{\partial z}+Q(z-vt)\right]\varphi(z,t)=0, \tag{2.1}$$

where $\varphi=(\varphi_1,\varphi_2,\ldots,\varphi_N)$ is the perturbation vector, K and Q are square matrix functions of order N, which depend on the coordinate z and on the time only via the combination $z'=z-vt$, v is a constant equal to the propagation velocity of the stationary wave, and I is a unit matrix. The matrices K and Q are periodic functions of z':

$$K(z'+L)=K(z'),\qquad Q(z'+L)=Q(z'),$$

where L is the period of the stationary wave.

In the reference frame (z',t) that moves together with the wave, the coefficients of the system (2.1) do not depend explicitly on the time, so that the solution can be sought in the form of a superposition of monochromatic waves:

$$\varphi(z',t)=e^{-i\omega t}\varphi(z',\omega). \tag{2.2}$$

The system of partial differential equations (2.1) is then converted into a system of ordinary differential equations with periodic coefficients:

$$\left\{[K(z')-vI]\frac{d}{dz'}+Q(z')-i\omega I\right\}\varphi(z',\omega)=0. \tag{2.3}$$

The fundamental system of the solutions $w^{(j)}(z',\omega)$ ($j=1,2,\ldots,N$) of this system is given by[10]

$$w^{(j)}(z',\omega)=e^{ik^{(j)}(\omega)z'}\pi^{(j)}(z',\omega), \tag{2.4}$$

where $\pi^{(j)}$ is a periodic vector:

$$\pi^{(j)}(z'+L,\omega)=\pi^{(j)}(z',\omega),$$

and $k^{(j)}(\omega)$ is the quasiwave number and is a solution of the equation

$$D(k,\ \omega)\equiv\det\{M(\omega)-Ie^{ikL}\}=0, \tag{2.5}$$

where $M(\omega)$ is the monodromy matrix.[10]

This relation is the general dispersion equation that is valid for the perturbation of a stationary wave of any amplitude and any profile. Its value lies in the fact that to obtain the monodromy matrix it is necessary to solve the system (2.1) over a finite interval $(0,L)$, so that approximate methods can be used. On the other hand, approximate methods cannot be used directly to assess the stability of the initial wave, since this calls for obtaining the asymptotic form of the solution as $t\to\infty$.

The dispersion equation (2.5) is a periodic function of the quasiwave number k, with a period $2\pi/L$. This means that the quasiwave number is defined accurate to a term $2\pi/L$, i.e., the quasiwave numbers $k^{(1)}$ and $k^{(2)}$, which differ by $2\pi/L$:

$$k^{(1)}=k^{(2)}+2\pi/L, \tag{2.6}$$

are equivalent. Returning to the laboratory frame (z,t), it is easy to show that Eq. (2.6) corresponds to the relation

$$\Omega^{(1)}=\Omega^{(2)}+\Omega, \tag{2.7}$$

where $\Omega^{(1)}$, $\Omega^{(2)}$, and Ω are the frequencies of the produced wave end of the initial wave.

Equations (2.6) and (2.7) mean that the decay conditions[1,2] correspond to second-order branch points $k(\omega)$. Coupled three-plasma processes[4] correspond to branch points of third or higher order.

3. THE GREEN'S FUNCTION

Actually, small perturbations do not take the form of individual monochromatic waves (2.2), but constitute wave packets, i.e., superpositions of monochromatic waves.

If the perturbation in a wave packet $u(z,t)$ remains bounded at constant z and as $t\to\infty$ (usually it tends to zero in this case), despite the presence of components with $\operatorname{Im}\omega>0$, the instability is convective. On the other hand if the wave packet $u(z,t)$ increases without limit at fixed z and as $t\to\infty$, the instability is absolute.

The simplest wave packet in terms of which all the remaining wave packets are expressed is the Green's function $G(z,\zeta,t)$ which is the solution of the simpler equation

$$\left[I\frac{\partial}{\partial t}+K(z-vt)\frac{\partial}{\partial z}+Q(z-vt)\right]G(z,\zeta,t)=\delta(z-\zeta)\delta(t) \tag{3.1}$$

and satisfies the initial condition

$$G(z,\zeta,t)=0 \text{ for } t<0 \tag{3.2}$$

with boundary conditions

$$G(\pm\infty,\zeta,t)=0. \tag{3.3}$$

To determine the Green's function $G(z,\zeta,t)$ we change over to the reference frame (z',t). Putting

$$G(z,\zeta,t)=g(z-vt,\zeta,t), \tag{3.4}$$

we obtain

$$\left\{[K(z')-vI]\frac{d}{dz'}+Q(z')-i\omega I\right\}g_\omega(z',\zeta)=I\delta(z-\zeta), \tag{3.5}$$

$$g_\omega(z',\zeta)=\int_0^\infty g(z',\zeta,t)e^{i\omega t}dt. \tag{3.6}$$

The function $g_\omega(z',\zeta)$ can be expressed in terms of the solutions $w^{(j)}(z',\omega)$ of the homogeneous equation

$$\left\{[K(z')-vI]\frac{d}{dz'}+Q(z')-i\omega I\right\}w^{(j)}(z',\omega)=0. \tag{3.7}$$

To this end, we subdivide all the solutions $w^{(j)}$ into waves traveling to the right and waves traveling to the left in the laboratory frame (z,t). Substituting $z'=z-vt$ in (2.4), we obtain

$$w^{(j)}(z',\ \omega)=\exp[ik^{(j)}z-i\Omega^{(j)}t]\pi^{(j)}(z-vt,\ \omega), \tag{3.8}$$

where $\Omega^{(j)}$ is the frequency in the laboratory frame:

$$\Omega^{(j)}=\omega+k^{(j)}v. \tag{3.9}$$

If the inequality

$$\operatorname{Im}k^{(j)}>0 \tag{3.10}$$

holds at $\operatorname{Im}\Omega^{(j)}\to+\infty$, the jth wave travels to the right,[5] but if

$$\operatorname{Im}k^{(j)}<0, \tag{3.11}$$

the wave travels to the left. We assume that r waves travel to the right and $N-r$ waves travel to the left.

At $z\neq\zeta$, the function $g_\omega(z',\zeta)$ satisfies the homogeneous equation (3.7). Taking the boundary conditions (3.3) into account, we see that the function $g_\omega(z',\zeta)$ is a linear superposition of waves traveling to the right at $z>\zeta$ and traveling to the left at $z<\zeta$. This means that

$$\begin{aligned} g_\omega(z',\zeta)&=W(z',\omega)P_RA, \quad z>\zeta,\\ g_\omega(z',\zeta)&=-W(z',\omega)P_LA, \quad z<\zeta, \end{aligned} \tag{3.12}$$

where $W(z',\omega)$ is the matrix of the solutions $w^{(j)}(z',\omega)$, whose first r columns constitute waves traveling to the right, and the remaining $N-r$ columns waves traveling to the left. P_R and P_L are the projection operators on the subspaces of the waves traveling to the right and left, respectively (they are multiplied by the solution matrix from the right). These are diagonal matrices, with the first r diagonal elements of P_R equal to unity, and the remainder equal to zero, with the converse for the matrix P_L; A is a matrix whose elements do not depend on z' and which we shall now determine.

At $z=\zeta$ the Green's function satisfies the matching condition that is obtained from (3.5) by integration with respect to z over an infinitesimal interval $\zeta-\varepsilon<z<\zeta+\varepsilon$:

$$g_\omega(\zeta+0,\ \zeta)-g_\omega(\zeta-0,\ \zeta)=[K(\zeta)-vI]^{-1}. \tag{3.13}$$

From (3.12) and (3.13), taking into account the relation $P_R+P_L=I$, it follows that

$$\begin{aligned} g_\omega(z',\zeta)&=W(z',\omega)P_RW^{-1}(\zeta,\omega)[K(\zeta)-vI]^{-1}, \quad z>\zeta,\\ g_\omega(z',\zeta)&=-W(z',\omega)P_LW^{-1}(\zeta,\omega)[K(\zeta)-vI]^{-1}, \quad z<\zeta. \end{aligned} \tag{3.14}$$

Thus, according to (3.4) and (3.6) the Green's function in the laboratory frame is equal to

$$G(z,\zeta,t)=\frac{1}{2\pi}\int_B g_\omega(z-vt,\zeta)\,e^{-i\omega t}d\omega, \tag{3.15}$$

where the contour B is located in the complex ω plane above all the singularities of the integrand (the Bromwich contour).

4. ASYMPTOTIC FORM OF WAVE PACKET

As already indicated, the Green's function is the simplest wave packet to which all other wave packets reduce.

To calculate the asymptotic form of the Green's function (3.15) we deform the Bromwich contour B by shifting it on the real axis $\operatorname{Im}\omega = 0$. With this, loops appear and circle all the singular points of the integrand. The integral over the real axis tends to zero as $t\to\infty$. The contribution from the singular point ω_0 is of the form $\exp(-i\omega_0 t)$ (apart from the pre-exponential factor). Thus, the character of the instability depends on whether the function $g_\omega(z-vt,\zeta)$ is analytic in the upper half-plane $\operatorname{Im}\omega>0$.

According to (3.14) and (2.4), the singular points of the integrand (3.15) regarded as a function of ω are the following: 1) the singular points $k^{(j)}(\omega)$; 2) the singular points $\pi^j(z',\omega)$; and 3) the zeros of $\det W(\zeta,\omega)$. It can be shown that these three types of singular points always coincide,[9] so that it sufficies to consider only the singular points $k^{(j)}(\omega)$.

It is known[13] that the solutions of a system of ordinary differential equations having coefficients analytic in the parameter ω are also analytic in ω. By virtue of (2.3), the coefficients of the dispersion equation (2.5) are analytic at all finite values of ω. As for the quasiwave numbers $k^{(j)}(\omega)$, they have branch points where the dispersion equation (2.5) has multipole roots.

However, not all the branch points of a quasiwave number are branch points of the function $g_\omega(z',\zeta)$. If several waves traveling in the same direction coalesce at $\omega=\omega_0$, this point is not a branch point of the function $g_\omega(z',\zeta)$.

In fact, the function $g_\omega(z',\zeta)$ is defined in a unique manner that does not depend on the choice of the fundamental solution matrix $W(z',\omega)$, i.e., it does not change on going to another basis. In particular, the function $g_\omega(z',\zeta)$ remains invariant when the waves traveling in the same direction are renumbered. In other words, it is a symmetrical function of the quasiwave numbers of waves traveling in the same direction. On the other hand, symmetrical functions of the roots of a polynomial are single-valued functions of its coefficients,[8,14] in the present case of the coefficients of the dispersion equation (2.5). Therefore the Green's function is an analytic function of ω at the point ω_0.

Since the contribution from the singular point ω_0 to the integral (3.15) is of the form $\exp(-i\omega_0 t)$, the instability is absolute if at least one branch point of the quasiwave number $k(\omega)$, at which at least two waves traveling in opposite directions coalesce, is located in the upper half-plane $\operatorname{Im}\omega>0$. The instability is convective in the opposite case.

5. DECAY OF AN ALFVÉN WAVE

We use now the theory developed to investigate the character of the decay of an Alfvén wave.

We shall use the equations of ideal magnetohydrodynamics. In the zeroth approximation, the density of the medium $\rho^{(0)}$, the pressure $p^{(0)}$, and the magnetic field $\mathbf{B}^{(0)}$ are constant, while the velocity $\mathbf{v}^{(0)}$ of the medium is zero. We choose the coordinate axes to make $B_y^{(0)}=0$.

In the principal approximation there moves in the direction of the z axis, to the right, an initial stationary Alfvén wave of small amplitude, in which

$$\begin{gathered}B_y^{(1)}(z,t)=\sum_{n\neq 0}B_n^{(1)}\exp\frac{2\pi in}{L}(z-v_A t),\\ v_y^{(1)}(z,t)=B_y^{(1)}(z,t)/(4\pi\rho^{(0)})^{1/2}\end{gathered} \tag{5.1}$$

(n are integers), with $v_A=|B_z^{(0)}|/(4\pi\rho^{(0)})^{1/2}$. The presence of an infinite number of harmonics $B_n^{(1)}$ makes it possible to take into account an arbitrary profile of the wave (sinusoidal, rectangular, sawtooth, and others). The initial wave profile is periodic with a period L.

In the second approximation, perturbations of the magnetohydrodynamic quantities in the Alfvén wave appear. The dependence of these perturbations on the time if of the form $\exp(-i\omega t)$. These perturbations are described by the vector

$$\varphi(z,\omega)=[\rho^{(2)},v_x^{(2)},v_y^{(2)},v_z^{(2)},B_x^{(2)},B_y^{(2)}]. \tag{5.2}$$

By a suitable change of the variables, the operator (2.1) can be made self-adjoint.[11] Putting $B_n^{(1)}\ll B^{(0)}$ and $[B^{(0)}]^2\gg p^{(0)}$ and using perturbation theory,[15] we obtain the increment γ corresponding to the decay of an Alfvén wave into an Alfvén wave (A) and a magnetosonic wave (S):

$$\gamma^2=2\eta_A\eta_S\frac{V_z^2}{Va}\left(\frac{\pi n}{L}\right)^2|V_n|^2, \tag{5.3}$$

where $V=B^{(0)}/(4\pi\rho^{(0)})^{1/2}$, a is the speed of sound, $V_n=B_n^{(1)}/(4\pi\rho^{(0)})^{1/2}$, $\eta_{A,S}=1$ for waves traveling in the opposite z directions, and $\eta_{A,S}=-1$ for waves traveling in the opposite direction. For the first harmonic ($n=1$), the instability growth rate γ coincides with the results of Refs. 1, 11, and 16. As for the higher harmonics ($|n|>1$), according to (5.3) the growth rate of the decay instability increases with increasing $|n|$. Therefore the produced waves are "enriched" with harmonics.

The rate of the decrease of the Fourier coefficients $B_n^{(1)}$ as $n\to\infty$ is determined by the smoothness of the profile of the wave $B_y^{(1)}(z,t)$. If this function is continuous, but its derivatives are discontinuous, then $B_n^{(1)}\sim 1/n^2$. A discontinuous profile of the $B_y^{(1)}(z,t)$ corresponds to $B_n^{(1)}\sim 1/n$. On the other hands if $\lim B_n^{(1)}=\text{const}$ as $n\to\infty$, the function $B_y^{(1)}(z,t)$ has δ-like singularities.

It is seen from (5.3) that decay instability makes the wave profile less smooth: decay of a continuous profile with discontinuous derivatives results in a discontinuous profile, and decay of a discontinuous profile results in a δ-like profile.

We determine now the character of the instability. Since the initial Alfvén wave according to (5.1) moves in the negative direction of the z axis, it follows that $\eta_A = 1$ and according to (5.3) instability sets in at $\eta_S = -1$. The decay conditions (2.6) and (2.7) correspond to the branch point $\omega + i\gamma$ of the quasiwave number k. Inasmuch as at $\gamma > 0$ this point is located in the upper complex ω plane, and two waves moving in opposite direction coalesce at this point, the instability is absolute.

The authors thank A.I. Akhiezer, G. Ya. Lyubarskiĭ, V.F. Aleksin, and V.D. Khodusov for valuable discussions.

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Translated by J. G. Adashko