

Stability of crystals in a random field created by vibrations of the continuum

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An analysis is made of the dynamics of a crystal in a random field. This is done for the case when the field creates vibrations in a medium with which the crystal is interacting and also in the case of a static field. It is shown that in a short-range field of vibrations of a medium a crystal does not become unstable because of the anharmonicity of the interaction allowed for in all orders of perturbation theory. In a long-range field of vibrations a crystal is found to be unstable. A static short-range random field destroys the long-range order even if allowance is made for the anharmonicity. The results are used to analyze the stability of certain specific systems.

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Imry and Ma¹ showed that a static random field destroys an ordered state with a continuum symmetry in systems of dimensionality $2 \leq d \leq 4$. The instability is due to the fact that in the range of small wave vectors k the susceptibility of the system obeys $\chi \propto k^{-2}$ (see, for example, Ref. 2) and if the correlation function $\langle |h_k|^2 \rangle$ of the random forces h_k does not decrease in the limit $k \rightarrow 0$, the mean square of the fluctuations of the order parameter diverges as k^{-4} . This instability was known before the appearance of the paper by Imry and Ma¹ (Larkin³ considered "melting" of a lattice of vortices in a type II superconductor in a random field). It applies directly to the pinning of charge density waves in the case of interaction with defects^{4,5} and to the destruction of the long-range order formed in a crystal by electrons localized at impurities.⁶

The problem of destruction of the long-wave order in a crystal in a random field is interesting also in connection with several other systems. One can mention here, for example, crystals formed by atoms adsorbed on a glass substrate and also Wigner crystals formed by electrons or holes in the bulk or on the surface of a semiconductor or on the surface of liquid helium (Wigner crystals of the latter type had recently been discovered experimentally⁷).

It is important to stress that in all these systems a random field creates sources external to the crystal. One of these sources is in the form of vibrations of a medium with which the crystal is interacting. For example, in the case of Wigner crystals in semiconductors the interaction is with the fields of phonons and defects in semiconductors. We shall use a very general model to analyze the interaction with crystals of static random fields and of fields created by phonons or other vibrations of a medium with which the crystal is interacting.

The results of Imry and Ma¹ can be applied to crystals directly only in the case of long-range random potentials $V(r)$ and it is essential that the correlation function $\langle |V_k|^2 \rangle$ should diverge as k^{-2} (otherwise the correlation function of the forces $\langle |kV_k|^2 \rangle$ tends to zero in the limit $k \rightarrow 0$). However, a crystal may be unstable also in the case of a short-range field $V(r)$. We must bear in mind that the Fourier component of the displacement of an

atom from a lattice site is governed by a random force \tilde{h}_k calculated allowing for the umklapp processes:

$$\tilde{h}_k = -i \sum_{G+k} (G+k) V_{G+k}$$

(G is the reciprocal lattice vector); clearly, $\langle |\tilde{h}_k|^2 \rangle$ is approximately constant in the range of low values of k even in the limit $kV_k \rightarrow 0$. It was this type of instability which was considered by Larkin³ in the case of a vortex lattice. He allowed only for linear (in respect of the displacements) terms in the Hamiltonian of the interaction with the field. However, if a crystal melts and the rms displacement diverges, it is necessary to allow for the nonlinearity of the interaction, i. e., for the dependence of the force acting on an atom on the displacement of the atom. In principle, this may alter the result in the case of a short-range field since the potential varies over distances which are short compared with the lattice constant.

An analysis of the stability in a random field will be made below allowing for the nonlinearity of the interaction. It will be shown that if the field creates vibrations in a medium, the anharmonicity makes a crystal stable if the interaction with the medium is of the short-range type. The frequency of these vibrations can be infinitesimally low and the temperature can be high. If the interaction with the medium is of the long-range type, an instability should appear. In the case of a static short-range field the anharmonicity of the interaction does not suppress the divergence of the rms displacement and the system has a short correlation radius r_c (in general, it is necessary to allow for the anharmonicity in calculating this radius).

At first sight it might appear that if $sr_c^{-1} > \omega_m$ (s is the velocity of sound in a crystal and ω_m is the characteristic frequency of the vibrations of a medium), there should be no difference between a static random field and a field created by the vibrations of the medium. However, the results of the action of these fields on a crystal differ basically. Physically, this is due to the possibility of some tuning of the vibrations of the medium to the vibrations of the crystal and particularly due to the fact that a "polaron jacket" with the lattice period appears in the medium and it stabilizes the crys-

tal if the interaction is unharmonic. Formally, the difference is due to the different procedures of averaging over a random external field and over a field created by the vibrations of the medium (see Sec. 3).

1. ABSENCE OF MELTING IN THE CASE OF A SHORT-RANGE POTENTIAL OF THE INTERACTION OF A CRYSTAL WITH VIBRATIONS OF A CONTINUUM

We shall consider the specific case when a medium whose vibrations are created by a random field is continuous and isotropic (in our case, the structure of the medium is unimportant). The general form of the Hamiltonian of the system is

$$\begin{aligned} H &= H_0 + H_1, \quad H_0 = \sum_{\mathbf{k}j} \omega_{\mathbf{k}j} a_{\mathbf{k}j}^+ a_{\mathbf{k}j} + \sum_{\mathbf{q}} \omega_{\mathbf{q}} b_{\mathbf{q}}^+ b_{\mathbf{q}}, \\ H_1 &= \sum_{\mathbf{q}} H_{\mathbf{q}}, \quad H_{\mathbf{q}} = \sum_{\mathbf{n}} V_{\mathbf{q}} \exp(i\mathbf{q}\mathbf{r}_n) c_{\mathbf{q}}, \\ \mathbf{r}_n &= \mathbf{R}_n + \sum_{\mathbf{k}j} x_{\mathbf{k}j} \exp(i\mathbf{k}\mathbf{R}_n) \mathbf{e}_{\mathbf{k}j}, \\ x_{\mathbf{k}j} &= A_{\mathbf{k}j} a_{\mathbf{k}j} + A_{-\mathbf{k}j}^+ a_{-\mathbf{k}j}^+, \quad c_{\mathbf{q}} = b_{\mathbf{q}} + b_{-\mathbf{q}}^+, \quad \hbar = 1. \end{aligned} \quad (1)$$

Here, $\mathbf{e}_{\mathbf{k}j}$ is the polarization vector of a crystal vibration branch j with a wave vector \mathbf{k} ; $a_{\mathbf{k}j}$ and $b_{\mathbf{q}}$ are the annihilation operators of the crystal and continuum of vibrations, respectively; \mathbf{R}_n is the coordinate of the n -th lattice site. It is interesting to consider the systems for which the wave vector \mathbf{q} may exceed the reciprocal lattice period a_0^{-1} and, consequently, the scattering of the crystal vibrations by the continuum of vibrations may be accompanied by the umklapp processes. Therefore, expansion of H_1 as a series in $\mathbf{q} \cdot (\mathbf{r}_n - \mathbf{R}_n)$ will not be made.

The crystal dynamics at $T \neq 0$ is governed by the Green function

$$G_{\mathbf{k}j}(\omega_n) = \langle x_{\mathbf{k}j} | x_{-\mathbf{k}j} \rangle_n, \quad (2)$$

$$\omega_n = 2\pi nT, \quad \langle \langle A | B \rangle \rangle_n = - \int_0^{\hbar} d\tau \langle A(-i\tau) B(0) \rangle \exp i\omega_n \tau, \quad \lambda = 1/T.$$

In the zeroth approximation with respect to $V_{\mathbf{q}}$, we have

$$G_{\mathbf{k}j}^{(0)}(\omega_n) = -2\omega_{\mathbf{k}j} |A_{\mathbf{k}j}|^2 / (\omega_n^2 + \omega_{\mathbf{k}j}^2). \quad (3)$$

The absence or occurrence of melting in the weak interaction case depends on the nature of the divergence of $G_{\mathbf{k}j}$ in the range of small values of k . Generally speaking, the divergence should be strongest for the transverse modes which have the highest susceptibility in the zeroth approximation²: $G_{\mathbf{k}j}^{(0)}(0) \propto k^{-2}$. Since we shall consider the relevant vibration branch, we shall omit the index j .

The equation for $G_{\mathbf{k}}$ can be obtained allowing for the interaction (see Appendix 1) by differentiating Eq. (2) on the left and right of τ . This equation is

$$\begin{aligned} [G_{\mathbf{k}}(\omega_n) - G_{\mathbf{k}}^{(0)}(\omega_n)] (\omega_n^2 + \omega_{\mathbf{k}}^2)^2 &= -4 |A_{\mathbf{k}}|^2 J_{\mathbf{k}}(\omega_n), \\ J_{\mathbf{k}}(\omega_n) &= \sum_{\mathbf{q}} (\mathbf{q}\mathbf{e}_{\mathbf{k}})^2 \langle H_{\mathbf{q}} \rangle + \sum_{\mathbf{q}\mathbf{q}'} (\mathbf{q}'\mathbf{e}_{\mathbf{k}}) \langle H_{\mathbf{q}'}(-\mathbf{k}) | H_{\mathbf{q}}(\mathbf{k}) \rangle_n, \end{aligned} \quad (4)$$

where

$$H_{\mathbf{k}}(\mathbf{k}) = \sum_{\mathbf{n}} V_{\mathbf{q}} c_{\mathbf{n}} \exp(i\mathbf{q}\mathbf{r}_n + i\mathbf{k}\mathbf{R}_n), \quad H_{\mathbf{q}} = H_{\mathbf{q}}(0). \quad (5)$$

It is clear from Eq. (4) that $|G_{\mathbf{k}}(\omega_n) - G_{\mathbf{k}}^{(0)}(\omega_n)| \propto \omega_n^{-4}$ in the limit $\omega_n \rightarrow \infty$, and the correction to $G_{\mathbf{k}}(\omega_n)$ which is the only one diverging as $k \rightarrow 0$ and $\omega_k \rightarrow 0$, may appear only for $n=0$. We shall show that in the case of a short-range interaction with the continuum vibrations this correction increases in the limit $k \rightarrow 0$ at least as $k^2 \omega_k^2 |A_{\mathbf{k}}|^4$ for all perturbation theory orders in H_1 . Therefore, since for acoustic phonons we have $\omega_k |A_{\mathbf{k}}|^2 = \text{const}$, it follows that $G_{\mathbf{k}}(0) \propto k^{-2}$, as in the absence of the interaction, and no melting takes place.

According to Eq. (4), the correction to $G_{\mathbf{k}}(0)$ is determined by the function $J_{\mathbf{k}}(0)$. In the calculation of $J_{\mathbf{k}}(0)$ it is convenient to separate the term with $\mathbf{q}' = \pm \mathbf{q}$:

$$\begin{aligned} J_{\mathbf{k}}(0) &= J_{\mathbf{k}}^{(1)} + J_{\mathbf{k}}^{(2)}, \\ J_{\mathbf{k}}^{(1)} &= \sum_{\mathbf{q}} (\mathbf{q}\mathbf{e}_{\mathbf{k}})^2 [\langle H_{\mathbf{q}} \rangle - \langle H_{-\mathbf{q}}(-\mathbf{k}) - H_{\mathbf{q}}(-\mathbf{k}) | H_{\mathbf{q}}(\mathbf{k}) \rangle_0], \\ J_{\mathbf{k}}^{(2)} &= \sum_{\mathbf{q}\mathbf{q}'} (\mathbf{q}\mathbf{e}_{\mathbf{k}}) (\mathbf{q}'\mathbf{e}_{\mathbf{k}}) \langle H_{\mathbf{q}'}(-\mathbf{k}) | H_{\mathbf{q}}(\mathbf{k}) \rangle_0, \end{aligned} \quad (6)$$

(a primed summation sign means that $\mathbf{q}' \neq \pm \mathbf{q}$). Bearing in mind the equality obtained in Appendix 2

$$\langle H_{\mathbf{q}} \rangle = \langle H_{-\mathbf{q}} - H_{\mathbf{q}} | H_{\mathbf{q}} \rangle_0, \quad (7)$$

we can write down the expression for $J_{\mathbf{k}}^{(1)}$ in the form

$$J_{\mathbf{k}}^{(1)} = \sum_{\mathbf{q}} (\mathbf{q}\mathbf{e}_{\mathbf{k}})^2 [\langle H_{-\mathbf{q}} - H_{\mathbf{q}} | H_{\mathbf{q}} \rangle_0 - \langle H_{-\mathbf{q}}(-\mathbf{k}) - H_{\mathbf{q}}(-\mathbf{k}) | H_{\mathbf{q}}(\mathbf{k}) \rangle_0]. \quad (6a)$$

Clearly, $J_0^{(1)} = 0$. For finite \mathbf{k} in the lowest order in $V_{\mathbf{q}}$, we have

$$\begin{aligned} J_{\mathbf{k}}^{(1)} &= - \sum_{\mathbf{q}} |V_{\mathbf{q}}|^2 (\mathbf{q}\mathbf{e}_{\mathbf{k}})^2 \sum_{\mathbf{m}} \exp(-i\mathbf{q}\mathbf{R}_{\mathbf{m}}) [1 - \exp(-i\mathbf{k}\mathbf{R}_{\mathbf{m}})] \int_0^{\hbar} d\tau \varphi(\omega_{\mathbf{q}}, \tau) \\ &\times \exp \left\{ - \sum_{\mathbf{k}'} (\mathbf{q}\mathbf{e}_{\mathbf{k}'})^2 |A_{\mathbf{k}'}|^2 [\varphi(\omega_{\mathbf{k}'}, 0) - \varphi(\omega_{\mathbf{k}'}, \tau) \cos(\mathbf{k}'\mathbf{R}_{\mathbf{m}})] \right\}, \end{aligned} \quad (8)$$

$$\begin{aligned} \varphi(\omega, \tau) &= \bar{n}(\omega) e^{i\omega\tau} + [\bar{n}(\omega) + 1] e^{-i\omega\tau}, \\ \bar{n}(\omega) &= [e^{\hbar\omega} - 1]^{-1}, \quad \mathbf{R}_{\mathbf{m}} = \mathbf{R}_m - \mathbf{R}_n. \end{aligned}$$

It is clear from Eqs. (6a) and (8) that of the function $|V_{\mathbf{q}}|^2 \omega_{\mathbf{q}}^{-1}$ does not diverge in the limit $q \rightarrow 0$, then for small values of k , we have

$$J_{\mathbf{k}}^{(1)} \approx N C k^2, \quad (9)$$

where N is the number of atoms in a crystal.

Cancellation of the terms in $J_{\mathbf{k}}^{(1)}$ not proportional to k^2 is due to the fact that in addition to a random vibration field a crystal is subjected by the medium to some regular field described by the term $\sim \langle H_{\mathbf{q}} \rangle$ in Eq. (6). This field is due to the reaction of the medium on the crystal. It is clear from Eqs. (6)–(9) that the term in question hardens the crystal and this effect compensates the “softening” influence of the fluctuation field due to the vibrations of the medium. The coefficient C in Eq. (9) is exponentially small in the case of a short-range interaction with the medium:

$$C \sim \exp(-1/2 q_m^2 a_0^2), \quad q_m \sim \langle u^2 \rangle^{-1/2},$$

where a_0 is the lattice constant and $\langle u^2 \rangle^{1/2}$ is the rms displacement of an atom in the crystal¹ [if the interac-

tion decreases steeply beginning from some value $q = \bar{q}_m$ such that $a_0^{-1} \ll \bar{q}_m \ll \langle \mu^2 \rangle^{-1/2}$, then C is small compared with the parameter $(\bar{q}_m a_0)^{-1} \ll 1$, but this smallness may be nonexponential; it should be pointed out that in the case of Wigner crystals in semiconductors or on the surface of helium there is no such reduction in the interaction on increase in q and the exponential estimate applies to C].

It is easy to calculate $J_k^{(1)}$ in the higher orders in H_1 and to show that the renormalization of C in the case of a weak interaction is small and the terms not proportional to k^2 do not appear. This result is evident in the case when the total number of the vibrational degrees of freedom of a crystal is much smaller than of the medium with which it interacts and the medium acts as the crystal thermostat (this is true, for example, in the case of Wigner crystals formed on the surface of helium). Then, it is sufficient to allow for the reaction of the medium only in the lowest perturbation theory order and to ignore in $J_k^{(1)}$ the terms $\sim N |V_q|^4$ compared with the terms $\sim |V_q|^2$. The terms with higher powers of H_1 then result mainly in "clothing" of the Green functions $\varphi(\omega_{\mathbf{k}j}, \tau)$ of phonons in the crystal, i. e., they cause a slight renormalization of q_m . A somewhat more cumbersome (but still elementary) analysis is required to allow for the reaction of the medium in the higher orders, because the Green functions $\langle\langle c_{\mathbf{q}} | c_{\pm\mathbf{q}} \rangle\rangle_0$ have singularities at $\mathbf{q} = \mathbf{G}$ (\mathbf{G} is the reciprocal lattice vector). These singularities correspond to a periodic (with the crystal period) relief which appears in the medium as a result of the interaction. It follows from physical considerations that such a relief should not result in melting of the crystal.

It follows from Eq. (6) that calculation of $J_k^{(2)}$ reduces to determination of a Green function of the type

$$\sum_{\mathbf{q}} (\mathbf{q}e_{\mathbf{k}}) \langle\langle L | H_{\mathbf{q}}(\mathbf{k}) \rangle\rangle_0,$$

where L is an operator that does not contain $c_{\pm\mathbf{q}}$; in general, it can be described by the following expression, which is accurate apart from a numerical coefficient,

$$L = \prod_{i,j} a_i^{(\lambda_i)} b_j^{(\nu_j)}, \quad \lambda_{i,j} = \pm 1, \quad a_{\mathbf{k}}^{(+)} = a_{\mathbf{k}}^+, \quad a_{\mathbf{k}}^{(-)} = a_{\mathbf{k}}. \quad (10)$$

In the equation of motion for this Green function

$$\begin{aligned} & \sum_{\mathbf{q}} (\mathbf{q}e_{\mathbf{k}}) \left\{ \langle\langle L | H_{\mathbf{q}}(\mathbf{k}) \rangle\rangle_0 \left(\sum_i \lambda_i \omega_{\mathbf{k}_i} + \sum_j \lambda_j \omega_{\mathbf{q}_j} \right) \right. \\ & \quad \left. - \langle\langle [L, H_+ - H_-] | H_{\mathbf{q}}(\mathbf{k}) \rangle\rangle_0 \right\} \\ & = \sum_{\mathbf{q}} (\mathbf{q}e_{\mathbf{k}}) \left\{ \langle\langle [L, H_+ + H_-] | H_{\mathbf{q}}(\mathbf{k}) \rangle\rangle_0 + \langle\langle [L, H_{\mathbf{q}}(\mathbf{k})] \rangle\rangle \right\} \end{aligned} \quad (11)$$

the second term on the right-hand side can be transformed as follows using Eq. (A.2.1):

$$\sum_{\mathbf{q}} (\mathbf{q}e_{\mathbf{k}}) \langle\langle [L, H_{\mathbf{q}}(\mathbf{k})] \rangle\rangle = - \sum_{\mathbf{q}} (\mathbf{q}e_{\mathbf{k}}) \langle\langle [L, H_{\mathbf{q}}(\mathbf{k}) + H_{-\mathbf{q}}(\mathbf{k})] | H_{\mathbf{q}} \rangle\rangle_0. \quad (12)$$

It follows from Eq. (12) that the right-hand side of Eq. (11) vanishes at $k=0$. The second term on the left-hand side of Eq. (11) represents a Green function

of the same type as the first term, and for this second term we can derive an equation analogous to Eq. (11). This gives an infinite chain of homogeneous equations for $k=0$ and this chain has a trivial solution for all orders in $V_{\mathbf{q}}$. Consequently, we have $J_k^{(2)} = 0$ when $k=0$.

If k is small, we can estimate $J_k^{(2)}$ in the same way as has been done for $J_k^{(1)}$; clearly, we have $J_k^{(2)} \propto k^2$, i. e., Eq. (9) describes not only $J_k^{(1)}$, but also the whole of $J_k(0)$. It follows that the correction to $G_k(0)$ is indeed proportional to k^{-2} and in the case of a short-range random potential $\sum_{\mathbf{q}} V_{\mathbf{q}} c_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{r}}$ it contains an exponentially small factor $\exp(-\frac{1}{2} q_m^2 a_0^2)$.

2. DYNAMICS OF A CRYSTAL IN THE CASE OF A HARMONIC INTERACTION WITH VIBRATIONS OF A CONTINUUM

If H_1 in (1) is expanded as a series in displacements of atoms in a crystal from their equilibrium positions and only the linear terms are retained in the expansion,

$$H_i^{(1)} = \sum_{\mathbf{n}\mathbf{q}} V_{\mathbf{q}} c_{\mathbf{q}} \left[1 + i \sum_{\mathbf{k}j} (\mathbf{q}e_{\mathbf{k}j}) x_{\mathbf{k}j} \exp i\mathbf{k}\cdot\mathbf{R}_{\mathbf{n}} \right] \exp i\mathbf{q}\cdot\mathbf{R}_{\mathbf{n}}, \quad (13)$$

then in the approximation of a single vibration branch ($j=1$) we readily find the Green function of Eq. (2):

$$\begin{aligned} G_k^{(1)}(\omega_n) &= -2\omega_{\mathbf{k}} |A_{\mathbf{k}}|^2 / [\omega_n^2 + \omega_{\mathbf{k}}^2 - N\omega_{\mathbf{k}} |A_{\mathbf{k}}|^2 \Pi_{\mathbf{k}}(\omega_n)]; \\ \Pi_{\mathbf{k}}(\omega_n) &= 4N \sum_{\mathbf{q}} \sum_{\mathbf{q}'} \delta_{\mathbf{q}, \mathbf{q}+\mathbf{k}} (\mathbf{q}e_{\mathbf{k}})^2 |V_{\mathbf{q}}|^2 \omega_{\mathbf{q}} / (\omega_n^2 + \omega_{\mathbf{q}}^2). \end{aligned} \quad (14)$$

According to Eq. (14), a medium softens the normal modes of a crystal and causes the acoustic mode frequency to vanish at $k=k_0$, where

$$k_0 = s^{-1} [N\omega_{k_0} |A_{k_0}|^2 \Pi_{k_0}(0)]^{1/2}. \quad (15)$$

If the interaction is of a short-range type and if the polarization operator $\Pi_{\mathbf{k}}(0)$ is dominated by the umklapp processes ($G \neq 0$), it follows that $\Pi_{\mathbf{k}}(0)$ in the limit $k \rightarrow 0$ is independent of k and does not vanish, so that Eq. (15) has a nontrivial solution. This result is in conflict with the result of the preceding section and is incorrect. It is obtained because the higher terms of the series in $x_{\mathbf{k}j}$ are dropped unjustifiably from Eq. (13). In fact, if an rms displacement from a lattice site becomes comparable with the lattice constant, then the whole series should be included in the expansion of H_i in terms of $\mathbf{G}(\mathbf{r}_{\mathbf{n}} - \mathbf{R}_{\mathbf{n}})$ when $G \neq 0$. In particular, the terms in the polarization operator (14) not proportional to k^2 cancel out if in addition to Eq. (13) we include in H_i the terms $\sim x_{\mathbf{k}} x_{-\mathbf{k}}$. The Green function can again be calculated exactly. It is described by Eq. (14) where $\Pi_{\mathbf{k}}(\omega_n)$ is replaced with $\bar{\Pi}_{\mathbf{k}}(\omega_n)$,

$$\bar{\Pi}_{\mathbf{k}}(\omega_n) = \Pi_{\mathbf{k}}(\omega_n) - \Pi_0(0), \quad (16)$$

and it does not have divergences in the case of a short-range interaction.²⁾ We have pointed out above that the term $\langle H_{\mathbf{q}} \rangle$ in Eq. (4), which results in cancelling in $J_k(0)$ of the terms not proportional to k^2 , is precisely due to the terms $\sim x_{\mathbf{k}} x_{-\mathbf{k}}$ in H_i .

We shall now consider the case when the coupling to the continuum is of the long-range type, i. e., when $|V_{\mathbf{q}}|^2 \omega_{\mathbf{q}}^{-1}$ diverges in the limit $q \rightarrow 0$. It is then that Eq.

(13) can be reduced to just the term

$$H_i^{(h)} = iN \sum_{\mathbf{k}} V_{\mathbf{k}}(\mathbf{k}e_{-\mathbf{k}i}) x_{-\mathbf{k}i} c_{\mathbf{k}}, \quad (13a)$$

which describes the harmonic interaction of the long-wavelength longitudinal vibrations of a crystal with the vibrations of a continuum [the first term in Eq. (13) is unimportant in the long-wavelength limit; this becomes clear if the summation with respect to \mathbf{R}_n is replaced with integration and if allowance is made for the fact that we can assume that $V_{\mathbf{q}} = 0$ when $q = 0$ if the total number of particles is conserved]. It should be noted that Eq. (13a) is valid also in the absence of a long-range order.

In the case of a long-range interaction in a system of coupled modes we can expect an instability described by Eqs. (14) and (15). This instability does not disappear when the unharmonicity of the interaction is allowed for: going over to integration with respect to \mathbf{R}_n in Eq. (8), we obtain

$$J_{\mathbf{k}}^{(h)} \sim N^2 k^2 (|V_{\mathbf{q}}|^2 \omega_{\mathbf{q}}^{-1})_{\mathbf{q}=\mathbf{k}};$$

hence, it follows that if $|V_{\mathbf{q}}|^2 \omega_{\mathbf{q}}^{-1}$ diverge in the limit $q \rightarrow 0$, then $J_{\mathbf{k}}$ decreases in the limit $k \rightarrow 0$ more slowly than k^2 , and longitudinal acoustic phonons are unstable.

The nature of the instability in Eqs. (14) and (15) in the case of a long-range interaction with the vibrations of the medium differs from the nature of the instability in a long-range static field.¹ In the case considered here a new periodic structure is clearly established in the system. The internal anharmonicity of a crystal and of the continuum vibrations is important for the formation of this structure.

3. INSTABILITY OF A CRYSTAL IN A STATIC SHORT-RANGE FIELD

The dynamics of a crystal in an external field is described by the Hamiltonian (1) where the second term should be dropped from H_0 and in H_1 the operators $c_{\mathbf{q}}$ should be replaced with random c numbers (they can be included in $V_{\mathbf{q}}$ and the parameters $V_{\mathbf{q}}$ may be regarded as random). Averaging over realizations of a random field will be denoted by the symbol $(\dots)_{r.m.}$:

$$\langle A \rangle = (\text{Sp} [A e^{-\lambda H}] / \text{Sp} [e^{-\lambda H}])_{r.m.} \quad (17)$$

Since averaging is applied to the ratio of traces of two operators (in the case when the random field creates vibrations of the continuum, calculation of a trace includes effectively averaging over the field realizations), it follows that the relationships given by Eqs. (7), (12), and (A.2.1) are not satisfied even in a Gaussian static field. Therefore, in $J_{\mathbf{k}}(0)$ described by Eq. (4) the terms not proportional to k^2 do not cancel and we have $G_{\mathbf{k}}(0) \propto k^{-4}$ in the limit $k \rightarrow 0$, which means that the crystal is unstable and the long-range order is destroyed (by melting) due to the divergence of the rms displacement.

We shall estimate $J_{\mathbf{k}}(0)$ in the simplest case when the random field is Gaussian (with the zero average value) and all the averages $(H_{\mathbf{q}1}, \dots, H_{\mathbf{q}n})_{r.m.}$ in which the same vector $\pm \mathbf{q}$ is encountered more than twice can be

ignored. It then follows from Eqs. (6) and (17) that

$$J_{\mathbf{k}}^{(1)} = \lambda N J + \sum_{\mathbf{q}} (q e_{\mathbf{k}})^2 [\langle H_{-\mathbf{q}} | H_{\mathbf{q}} \rangle_0 - \langle H_{-\mathbf{q}}(-\mathbf{k}) | H_{\mathbf{q}}(\mathbf{k}) \rangle_0], \quad (18)$$

$$J = \frac{1}{2N} \sum_{\mathbf{q}} q^2 (|\text{Sp} [H_{\mathbf{q}} e^{-\lambda H}]|^2 / \{\text{Sp} [e^{-\lambda H}]\}^{-2})_{r.m.}$$

In a short-range field with a finite correlation radius we have $J_{\mathbf{k}}^{(1)} \rightarrow \lambda N J$ in the limit $k \rightarrow 0$, and we also find that $J > 0$. Since $J_{\mathbf{k}}^{(2)}$ contains terms only of the fourth or higher orders of $V_{\mathbf{q}}$, it follows that in the case of a sufficiently weak interaction the value in the long-wavelength limit is $J_{\mathbf{k}}(0) \approx \lambda N J$.

The quantity J describes the characteristic distance r_c in which the system retains a short-range order. The parameter r_c can be estimated by equating³ the rms difference between the atomic displacements at a distance r_c to the lattice constant a_0 :

$$\sum_{\mathbf{k}} (1 - \cos k r_c) \langle |x_{\mathbf{k}}|^2 \rangle \approx a_0^2,$$

which gives, subject to Eqs. (4) and (18),

$$r_c \sim a_0 (m^2 s^4 / a_0^2 J)^{1/(4-d)}, \quad r_c \gg a_0 \quad (19)$$

(m is the mass of an atom in the crystal).

It follows from Eq. (19) that the correlation radius at r_c increases on increase in the rigidity of the system (on increase in the velocity of sound s) and on reduction in the fluctuation intensity J for a short-range random field (a similar result was obtained by Larkin³ for a vortex lattice). The dependence of r_c on s^2 and J is a power law with the exponent governed by the dimensionality of the system d ($d = 4$ is the critical dimensionality^{1,6}). It is clear from Eqs. (18) and (19) that the anharmonicity of the interaction with a random field does not stabilize a crystal if the field has no dynamic degrees of freedom. However, allowance for the anharmonicity is very important in the case of random potentials with a small correlation radius $\rho_0 \ll \Delta$ (Δ is the rms displacement of an atom relative to its equilibrium position in a glassy system formed as a result of pinning in a crystal). In this case we can calculate J in Eq. (18) by assuming that $|V_{\mathbf{q}}|^2 = \text{const}$ and by removing a short-wavelength divergence of the sum with respect to \mathbf{q} by a factor of the $\exp(-\frac{1}{2} q^2 \Delta^2)$ type. Without allowance for the anharmonicity the expression for J would have been approximately $(\Delta/\rho_0)^{2+d}$ times greater and the transition to the limit of a δ -correlated potential ($\rho_0 \rightarrow 0$) would not have been possible.

An estimate of r_c described by Eq. (19) can be obtained on the basis of considerations similar to those used in Ref. 6. The deformation of a crystal for which the rms difference between displacements of atoms at a distance r_c is equal to a_0 is related to the loss of the elastic energy. In a volume of r_c^d this loss is

$$\sim r_c^d m a_0^{-d} s^2 (a_0/r_c)^2 = m s^2 (r_c/a_0)^{d-2}$$

($m a_0^{-d}$ is the density of the crystal and a_0/r_c is the deformation per unit length). The rms gain in the energy in the same volume due to the alignment with the short-wavelength random field is $\sim [J a_0^2 (r_c/a_0)^d]^{1/2} [J a_0^2]$ is a typical value of the square of the work done by a

random force in displacing an atom by a_0 and $(r_c/a_0)^d$ is the number of atoms in the volume r_c^d . We obtain Eq. (19) by equating the energy loss to the energy gain.

4. CONCLUSIONS

The instabilities in a random field considered above do not have a field threshold. The interaction determines only the correlation radius in a static external field and the characteristics of a superstructure in the case of a long-range interaction with vibrations of a medium in which a crystal is interacting. The internal anharmonicity of the crystal does not influence the presence or absence of an instability (in contrast to the anharmonicity of the interaction with the field) because the instability relates to long-wavelength modes and their unharmonicity is weak.

As pointed out earlier, interesting examples of crystals interacting with a random field are Wigner crystals or charged density waves. It is clear from the above results that since both Wigner crystals and charged density waves interact with defects in the main crystal, they always melt (or, more exactly, become pinned). Similarly, a random field created by a glassy substrate results in "melting" of an adsorbed monolayer. The resultant correlation radius depends, in accordance with Eq. (19), on the concentration of defects and on the interaction parameters. In the case of a short-range random field the correlation radius r_c for Wigner crystals in a semiconductor may be related to the momentum relaxation time $\tau(p)$ of a free electron scattered by fluctuations of the random field. The order of magnitude of the correlation radius is given by

$$r_c \sim a_0 (m^* s^2 \tau / \hbar)^{1/(4-d)},$$

where m^* is the effective mass of an electron and $p_0 \sim \hbar/\Delta$ [Δ is the rms displacement of an electron relative to its equilibrium position in a glassy system formed as a result of pinning of Wigner crystals; at low temperatures we have $\Delta \sim (\hbar a_0 / m^* s)^{1/2}$]. It follows from the above estimate that, in particular, in the case of metal-insulator-semiconductor systems which are of great current interest the correlation radius may be several times greater than the average distance between electrons in an inversion layer of a semiconductor. It should be noted that pinning of Wigner crystals (or charged density waves) appears because defects in the main crystal act as an external field in relation to Wigner crystals: these defects are not matched to Wigner crystals (in contrast to the usual situation of a crystal with impurities where self-matching occurs and, therefore, the long-range order is retained—see Ref. 8).

In the case of electrons on the surface of liquid helium a random field creates capillary waves (ripples). Under the experimental conditions of Ref. 7 the random field is of the short-wavelength type and the occupation numbers of ripples are large. The energy spectrum of a system of bound electrons and ripples differs considerably from the spectrum of noninteracting subsystems. However, the relative change in the static correlation function of the displacements in an

electron crystal is very small for low values of k ; it is proportional to the constant C in Eq. (9) and amounts to less than 10^{-5} . The experimentally observed⁷ crystallization of electrons does in fact confirm the conclusion of our investigation that a short-range interaction with the vibrations of a continuum does not melt a crystal.

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APPENDIX 1

The equation of motion for the k -th component of the displacement of an atom in a crystal from its lattice site is

$$\frac{d^2 x_{kj}(\tau)}{d\tau^2} + \omega_{kj}^2 x_{kj}(\tau) = -2i\omega_{kj} |A_{kj}|^2 \sum_q (q e_{kj}) H_q(-k, \tau) \quad (\text{A. 1. 1})$$

[the operator $H_q(-k)$ is defined by Eq. (5)]. Going over in Eq. (A. 1. 1) to imaginary time $-i\tau$, multiplying the right-hand side by $x_{-kj}(0)$, averaging, and integrating with respect to τ with a weight $\exp(i\omega_n \tau)$, we obtain the following expression for the Green function $G_{kj}(\omega_n)$:

$$[G_{kj}(\omega_n) - G_{kj}^{(0)}(\omega_n)] (\omega_n^2 + \omega_{kj}^2) = -2i\omega_{kj} |A_{kj}|^2 \sum_q (q e_{kj}) \langle H_q(-k) | x_{-kj} \rangle_n. \quad (\text{A. 1. 2})$$

Writing down the Green function on the right-hand side of Eq. (A. 1. 2) in the form

$$\langle H_q(-k) | x_{-kj} \rangle_n = - \int_0^{\lambda} d\tau \exp(i\omega_n \tau) \langle H_q(-k, 0) x_{-kj}(i\tau) \rangle \quad (\text{A. 1. 3})$$

and using once again Eq. (A. 1. 1), we obtain

$$\langle H_q(-k) | x_{-kj} \rangle_n (\omega_n^2 + \omega_{kj}^2) = -2i\omega_{kj} |A_{kj}|^2 \{ (q e_{kj}) \langle H_q \rangle + \sum_{q'} (q' e_{kj}) \langle H_q(-k) | H_{q'}(k) \rangle_n \}. \quad (\text{A. 1. 4})$$

Equation (4) follows directly from Eqs. (A. 1. 2) and (A. 1. 4).

The first term in Eq. (A. 1. 4) is proportional to the derivative $\partial H_q(-k) / \partial x_k$. Since $H_q(-k)$ is, in its turn, proportional to $\partial H_q / \partial x_{-k}$, it is clear that $\langle H_q \rangle$ is due to terms of the $x_k x_{-k} c_q$ type in H_i . The presence of such "non-Wick" (but very important) terms makes it impossible to use the standard technique in the calculation of $G_k(\omega_n)$.

APPENDIX 2

We shall show that for an arbitrary operator L not containing operators c_q and c_{-q} , we have the equality

$$\langle LH_q \rangle = \langle H_{-q} - H_q | LH_q \rangle_0. \quad (\text{A. 2. 1})$$

We shall do this by introducing an auxiliary Hamiltonian

$$H' = H - \sum_{\alpha=\pm 1} H_{\alpha q} \quad (\text{A. 2. 2})$$

and writing down the left-hand side of Eq. (A. 2. 1) in the form

$$\langle LH_q \rangle = Z^{-1} \text{Tr} \left\{ \exp(-\lambda H') T, \exp \left[- \int_0^{\lambda} d\tau \sum_{\alpha} H_{\alpha q}(-i\tau) \right] LH_q \right\},$$

$$\begin{aligned} H_{\alpha\mathbf{q}}(-i\tau) &= \exp(H'\tau) H_{\alpha\mathbf{q}} \exp(-H'\tau), \\ Z &= \text{Tr}[\exp(-\lambda H)], \end{aligned} \quad (\text{A. 2. 3})$$

where T_τ is the operator of a chronological ordering along the imaginary axis.

Since the operator $H_{\mathbf{q}}$ is proportional to $c_{\mathbf{q}}$, and H' contains $c_{\pm\mathbf{q}}$ only in the form of the combination $b_{\mathbf{q}}^+ b_{\mathbf{q}}$, it follows that Eq. (A. 2. 3) contains contributions of the odd terms of the expansion as a series of the exponential function associated with $\tilde{H}_{\alpha\mathbf{q}}$, and the general form of the $(2+1)$ -th term of the series is

$$\begin{aligned} & - \int_0^\lambda d\tau_1 \dots \int_0^{\tau_{2n}} d\tau_{2n+1} H_{\alpha_1\mathbf{q}}(-i\tau_1) \dots H_{\alpha_{2n+1}\mathbf{q}}(-i\tau_{2n+1}), \\ & \sum_{\alpha_i} \alpha_i = -1, \quad \alpha_1, \dots, \alpha_{2n+1} = \pm 1. \end{aligned} \quad (\text{A. 2. 4})$$

The right-hand side of Eq. (A. 2. 1) can be transformed similarly:

$$\begin{aligned} & \langle H_{-\mathbf{q}} - H_{\mathbf{q}} | L H_{\mathbf{q}} \rangle_0 = -Z^{-1} \int_0^\lambda d\tau \text{Sp} \left\{ \exp(-\lambda H') \right. \\ & \times \left(T_\tau \exp \left[- \int_\tau^\lambda d\tau_1 \sum_{\alpha} H_{\alpha\mathbf{q}}(-i\tau_1) \right] [H_{-\mathbf{q}}(-i\tau) - H_{\mathbf{q}}(-i\tau)] \right. \\ & \left. \left. \times \left(T_\tau \exp \left[- \int_0^\tau d\tau_2 \sum_{\alpha} H_{\alpha\mathbf{q}}(-i\tau_2) \right] \right) L H_{\mathbf{q}} \right\}. \end{aligned} \quad (\text{A. 2. 5})$$

In Eq. (A. 2. 5) the general form of the $(2+1)$ -th term of the series in $\tilde{H}_{\alpha\mathbf{q}}$ is

$$\begin{aligned} & - \sum_{m=0}^{2n} \int_0^\lambda d\tau \int_\tau^\lambda d\tau_1 \dots \int_\tau^{\tau_{m-1}} d\tau_m H_{\alpha_m\mathbf{q}}(-i\tau_1) \dots H_{\alpha_m\mathbf{q}}(-i\tau_m) \\ & \times [H_{-\mathbf{q}}(-i\tau) - H_{\mathbf{q}}(-i\tau)] \int_0^\tau d\tau_{m+1} \dots \int_0^{\tau_{2n-1}} d\tau_{2n} \\ & \times H_{\alpha_{m+1}\mathbf{q}}(-i\tau_{m+1}) \dots H_{\alpha_{2n}\mathbf{q}}(-i\tau_{2n}), \end{aligned} \quad (\text{A. 2. 6})$$

and if in each bracket we select the term $H_{-\mathbf{q}}(-i\tau)$ then $\sum_i \alpha_i = 0$, whereas if we select $\tilde{H}_{\mathbf{q}}(-i\tau)$, we have $\sum_i \alpha_i = -2$.

We shall assume that a certain sequence of numbers $\{a_i\}$ ($i = 1, \dots, 2n+1$) is selected in Eq. (A. 2. 4), i. e., a sequence of signs of the vector \mathbf{q} in the product of the operators $\tilde{H}_{\pm\mathbf{q}}(-i\tau)$; we encounter the "+" sign n times and "-" sign $n+1$ times. Clearly, we can select in Eq. (A. 2. 6) the same sequence of signs of \mathbf{q} for each value of m . In the summation with respect to m in Eq. (A. 2. 6) (for a fixed sequence of signs) the operator

$\tilde{H}_{\mathbf{q}}(-i\tau)$ in the brackets is used $n+1$ times, whereas the operator $\tilde{H}_{\mathbf{q}}(-i\tau)$ is used n times. Since these operators occur with different signs, the expressions (A. 2. 4) and (A. 2. 6) are equal and this proves the relationship (A. 2. 1).

It should be noted that although $V_{\mathbf{q}} \propto \Omega^{-1/2}$ (Ω is the volume of the system), the terms $\sim |V_{\mathbf{q}}|^{2n}$ with $n > 1$ in Eq. (A.2.1) can be of the same order as the term $\sim |V_{\mathbf{q}}|^2$ in the limit $\Omega \rightarrow \infty$ because $H_{\mathbf{q}}$ contains a sum of the atoms in a crystal, which results in cancellation of the factor Ω^{-n} in $|V_{\mathbf{q}}|^{2n}$.

¹According to Eq. (8), the value of $\langle u^2 \rangle$ should be calculated allowing only for short-wavelength fluctuations. This is particularly important in the case of two-dimensional systems when the temperature correction to the rms displacement due to long-wavelength fluctuations diverges. The exponential fall of the contribution to $J_k^{(1)}$ due to large values of R_{nm} justifies the use of the Hamiltonian (1) in describing the influence of the interaction with the continuum of the vibrations of a 2D crystal.

²This important example can be used to demonstrate the validity of the general treatment given in Sec. 1 and, in particular, the correctness of Eq. (9) for $J_k(0)$ [if $\gamma_{\mathbf{r}_n} \ll \langle (\mathbf{r}_n - \mathbf{R}_n)^2 \rangle^{1/2}$]. We can also see that stabilization of a crystal is due to the fact that $\langle c_{\mathbf{q}} \rangle \neq 0$ for $\mathbf{q} = \mathbf{G}$, i. e., a periodic relief with the same period as the crystal appears in the medium.

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