Fluxon interaction in long Josephson junctions

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The interaction between nonlinear solitary magnetic-flux waves (fluxons) in a long Josephson junction is considered with account taken of dissipative processes and of the bias current. The system is described by the perturbed sine-Gordon equation. Perturbation theory calculations show that weak dissipation and a weak bias current can significantly affect the dynamics of a fluxon system and lead to the formation of soliton complexes moving as a unit with practically constant distances between the solitons (congealing of fluxons). The asymptotic results obtained by the perturbation-theory calculations are confirmed by numerical experiments.

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1. INTRODUCTION

The dynamics of nonlinear wave processes in Josephson junctions (see, e.g., Refs. 1 and 2) is of great fundamental and applied interest. A major role in these processes, which are described by the sine-Gordon (SG) equation, is played by solitons, which are in this case elementary excitations of the magnetic flux and are therefore called fluxons. Fluxons oscillating between the ends of a Josephson line (observed in Ref. 3) generate microwave radiation which, on the one hand, is one of the means for their diagnostics, and on the other hand can be effectively used in various applications (see, e.g., Ref. 4 and the literature cited there). Since the SG equation is integrable[5,6] the dynamics of unperturbed fluxons and of their systems is known. In Josephson lines, however, dissipative effects, even small ones, can play a substantial role. To compensate for them one uses external sources (bias currents). The influence of dissipation and bias currents on individual solitons has been investigated in sufficient detail by perturbation theory.[7] The action of these perturbations on soliton systems, however, have been far from adequately investigated. It is known, for example, that SG solitons of like (for example, kink–kink) polarity, are always repelled. However, as shown by experiments with mechanical models (coupled pendulums) and by numerical investigations,[1,8,9] in the presence of dissipation and of a bias current solitons of like polarity can form complexes that move as a unit. The attempts made so far to explain this phenomenon are, in our opinion, not convincing for reasons which will be discussed in Sec. 2.

The present paper is devoted to the interaction of fluxons with allowance for the dissipation and the bias current. To reveal the main physical factors that determine in this case the dynamics of the system, we consider first the simplest case, when the Josephson junction is infinitely long and the fluxons are always far enough from one another, so that the interaction between them is always weak.[1)] In addition, the external perturbations (dissipation and bias current) are assumed to be small. The dynamics of the system can then be relatively simply and sufficiently completely investigated analytically by perturbation theory (as is done for the analogous, in principle, case of Korteweg–de Vries solitons in Refs. 10 and 11). This is the subject of Sec. 2, one of the main results of which is a simple explanation of the congealing of fluxons, wherein complexes moving with practically the same velocity are produced.

In Sec. 3 it is shown by numerical methods that the asymptotic results obtained within the framework of perturbation theory remain valid also in more general cases, closer to experiment, when the distance between the solitons can also be small, and the Josephson junctions can be of finite length. In particular, from the results of the numerical experiments it is seen that the soliton complexes are preserved, despite reflections from the ends of the junction (at least if the losses at the ends are not large). These phenomena are perfectly observable: they should manifest themselves, for example, in a characteristic structure of the emission from the Josephson junctions (see the end of Sec. 3). It seems to us that the relevant experiments are feasible.

2. PERTURBATION THEORY FOR A SYSTEM OF TWO FLUXONS

A transmission line with a Josephson junction can be represented as two superconducting strips separated by a thin insulating layer (Fig. 1). We designate the layer thickness by d and its effective dielectric constant by ε_{eff}. The dissipation effect due to the current of the normal electrons across and along the junction can be taken into account by introducing the transverse conductivity σ and the longitudinal resistance r of the normal electrons per unit length of the junction. The propagation of the magnetic flux along such a transmission line, with allowance for the external current and dissipation effects, can be described in dimensionless variables by the equation[1]

$$v_{TT}-v_{XX}+\sin v=\varepsilon R[v], \tag{2.1}$$

$$\varepsilon R[v]=-\alpha v_T+\beta v_{XXT}-f, \quad \varepsilon=\max(\alpha,\ \beta,\ |f|), \tag{2.2}$$

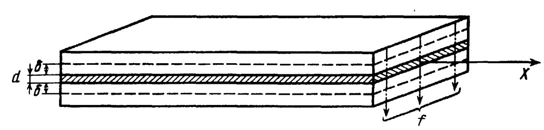

FIG. 1. Diagram of Josephson junction.

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where $v = 2\pi\Phi(X, T)/\Phi_0$ is the magnetic flux normalized to the value of the corresponding quantum $\Phi_0 = hc/2e$, and $f = j/j_m$ is the dimensionless density of the bias currents, normalized to the maximum density of the Josephson current j_m. We emphasize that in contrast to the Josephson current $j_m \sin v$ the bias current j is usually constant. We assume accordingly, that $f = \text{const}$. The dimensionless coordinate and time (X, T) are defined relative to the characteristic scales of length λ and time τ which describe the junction: $X = x/\lambda$, $T = t/\tau$.

$$\lambda=\delta[e\hbar n_s/2m(2\delta+d)j_m]^{1/2}, \quad \tau=\lambda/u, \tag{2.3}$$

where n_s is the density of the superconducting electrons, $u = c[d/\varepsilon_{\text{eff}}(2\delta + d)]^{1/2}$ is the propagation velocity of the electromagnetic wave along the junction, and $\delta = (mc^2/4\pi n_s e)^{1/2}$ is the London penetration depth. Finally, the transverse (α) and longitudinal (β) dissipation coefficients are expressed in terms of the parameters of the junction in the following manner:

$$\alpha=4\pi\left(\frac{2\delta+d}{d}\varepsilon_{eff}\right)^{1/2}\frac{\lambda}{c}\sigma, \quad \beta=\frac{4\pi}{r\lambda c}\left(\frac{d(2\delta+d)}{\varepsilon_{eff}}\right)^{1/2}. \tag{2.4}$$

If we neglect dissipation effects ($\alpha = \beta = 0$), then in the absence of the bias current ($f = 0$) the soliton solution of Eq. (2.1), which describes the fluxon in this case, will take the form

$$v_s(Z)=2\sigma \arcsin \operatorname{th} Z+\pi, \tag{2.5}$$

where $\sigma = \pm 1$ (kink and antikink),

$$Z=[X-X(T)](1-V^2)^{-1/2}, \tag{2.6}$$

and $dX/dT = V$, $V = \text{const}$. The influence of the small perturbation $\varepsilon R[v]$ (ε is a small parameter) on a single fluxon can be described by the expression $v = v_s(Z + \varepsilon\psi) + \delta v$, where v_s and Z are defined as before by expressions (2.5) and (2.6), but V and dX/dT now have a slow dependence on the time T in accordance with[2)] the equations[7,12,13]

$$dV/dT=-(1-V^2)^{1/2}\frac{\sigma}{4}\varepsilon\int_{-\infty}^{\infty}R[v_s]\operatorname{sech} Z dZ, \tag{2.7}$$

$$dX/dT=V+(1-V^2)\frac{\sigma}{4}\varepsilon\int_{-\infty}^{\infty}R[v_s]Z\operatorname{sech} Z dZ. \tag{2.8}$$

The quantity $\varepsilon\psi(Z, \varepsilon T)$ describes the change of the phase of the fluxon, and $\delta v(Z, \varepsilon T)$ describes its deformation, due to the perturbation $\delta R[v](\delta v \sim \varepsilon)$. These quantities were obtained and discussed in Refs. 12 and 13; we shall not consider them here.

Following Refs. 11 and 14, we can also use (2.7) and (2.8) to describe the interaction of two solitons when the interaction is small. To this end we seek an approximation solution of Eq. (2.1) in the form

$$v(X, T)=v_{1s}(Z_1)+v_{2s}(Z_2), \tag{2.9}$$

where v_{ns} ($n = 1, 2$) is obtained from (2.5) and (2.6) by the substitutions

$$\sigma\to\sigma_n, \quad Z\to Z_n, \quad X(T)\to X_n(T), \quad V\to V_n. \tag{2.10}$$

The approximation (2.9) can be called adiabatic, since we are neglecting here the increments $\varepsilon\psi_n$ and δv_n (allowance for them adds nothing substantial). The functions dV_n/dT and dZ_n/dT are determined from Eqs. (2.7) and (2.8), in which the substitutions (2.10) must be made, and $\varepsilon R[v_s]$ must be taken to mean a perturbing term that not only turns on the external perturbation (dissipation and bias current), but also the interaction between the solitons $\varepsilon_{12}R_{12}$, i.e.,

$$\varepsilon R[v_{ns}]=\varepsilon_{12}R_{12}-\alpha(v_{ns})_T+\beta(v_{ns})_{XXT}-f. \tag{2.11}$$

The term $\varepsilon_{12}R_{12}$ is easily obtained from the nonlinear term $\sin v = \sin(v_{1s} + v_{2s})$ of Eq. (2.1) and takes the form[14]

$$\varepsilon_{12}R_{12}=-(\sin v-\sin v_{1s}-\sin v_{2s})$$

or, after simple transformations

$$\varepsilon_{12}R_{12}=-4\sigma_1\frac{\operatorname{th} Z_1}{\operatorname{ch} Z_1 \operatorname{ch}^2 Z_2}-4\sigma_2\frac{\operatorname{th} Z_2}{\operatorname{ch} Z_2 \operatorname{ch}^2 Z_1}. \tag{2.12}$$

(It is assumed here that the fluxons are far enough from one another, so that their overlap may be regarded as small.) As a result, Eqs. (2.7) and (2.8) yield for each of the fluxons

$$\begin{aligned} dV_n/dT=&-(-1)^n 4\sigma_1\sigma_2(1-V^2)^{3/2}\exp[-r(1-V^2)^{-1/2}] \\ &-\alpha V_n(1-V_n^2)-{}^1/_3\beta V_n+\sigma_n(\pi/4)(1-V_n^2)^{3/2}f, \end{aligned} \tag{2.13}$$

$$dX_n/dT=V_n+2\sigma_1\sigma_2(1-V^2)\exp[-r(1-V^2)^{-1/2}], \tag{2.14}$$

where

$$V=(V_1+V_2)/2, \quad r=X_1(T)-X_2(T)>0$$

r is the distance between the solitons. In the derivation of Eqs. (2.13) and (2.14) it was assumed that

$$r(T)\gg(1-V^2)^{1/2}, \tag{2.15}$$

$$|(V+P/4)P|\ll 1-V^2, \tag{2.16}$$

$$|(V+P/4)P|r\ll(1-V^2)^{3/2}, \tag{2.17}$$

where $P = V_1 - V_2$. The inequalities (2.15) and (2.16) are necessary to be able to treat the interaction of the solitons by perturbation theory.[14] The last inequality (2.17) is not essential from the physical point of view, inasmuch as at those distances at which it can be violated the interaction of the solitons can be neglected (it was introduced only to simplify the calculations).

We now apply these equations to a system of fluxons of equal polarity, i.e., we assume that $\sigma_1 = \sigma_2 = \sigma$. Confining ourselves for simplicity to the case $\beta = 0$, we obtain from (2.13) and (2.14)

$$dP/dT=8(1-V^2)^{3/2}\exp[-r(1-V^2)^{-1/2}]-\alpha P-3V/(1-V^2)(dV/dT)P, \tag{2.18}$$

$$dV/dT=-\alpha V(1-V^2)+\sigma(\pi/4)f(1-V^2)^{3/2}, \tag{2.19}$$

$$dr/dT=P. \tag{2.20}$$

If $\alpha = f = 0$, then $dV/dT = 0$, and Eqs. (2.18) and (2.20) lead to

$$r(T)=2(1-V^2)^{1/2}\ln[4(1-V^2)\Pi^{-1}\operatorname{ch}(\Pi T(1-V^2)^{-1/2}/2)], \tag{2.21}$$

$$r(0)=2(1-V^2)^{1/2}\ln[4(1-V^2)\Pi^{-1}], \tag{2.22}$$

$$P(T)=\Pi \operatorname{th}[\Pi T(1-V^2)^{-1/2}/2], \tag{2.23}$$

where $\Pi \equiv P(\infty)$. From (2.21)–(2.23) it is seen, in particular, that the conditions (2.15) and (2.16) will be satisfied if $\Pi \ll 1 - V^2$. We indicate also that Eqs. (2.21)–(2.23) are in full agreement with the exact two-soliton solution which follows from the inverse-problem method, as shown in Ref. 14 for an equivalent problem.

Let now $\alpha \neq 0$ and $f \neq 0$. We note first of all that Eq.

(2.19) can be considered independently of Eqs. (2.18) and (2.20). Integrating (2.19), we obtain

$$V(T)=\frac{Ce^{-\alpha T}+\sigma\pi f/4\alpha}{[1+(Ce^{-\alpha T}+\sigma\pi f/4\alpha)^2]^{1/2}}, \tag{2.24}$$

where

$$C=V(0)(1-V^2(0))^{-1/2}-\sigma\pi f/4\alpha. \tag{2.25}$$

We see that as $T\to\infty$ the velocity $V(T)$ tends to the limiting value:

$$V_*=(\sigma\pi f/4\alpha)[1+(\pi f/4\alpha)^2]^{-1/2}. \tag{2.26}$$

Expression (2.26) coincides with the limiting velocity obtained for one soliton in Ref. 7. If $V(0)=V_*$, then it follows from (2.25) that $C=0$, and therefore $V(T)=V_*$. To simplify the analysis we shall assume $V(0)=V_*$. (It is seen from what follows that this does not make the final results less general.) We then obtain from (2.18) and (2.20)

$$dP/dT=8(1-V_*^2)^{1/2}\exp[-r(1-V_*^2)^{-1/2}]-\alpha P, \tag{2.27}$$

$$dr/dT=P. \tag{2.28}$$

Equations (2.27) and (2.28) have the appearance of the equations of motion of a particle acted upon by a potential repulsive force and by friction $-\alpha P$. Assume for the sake of argument that at $T=0$ the particle is at the turning point r_0. Assuming that the dissipative coefficient α is sufficiently small, we can neglect friction in the time interval $\theta< T\ll 1/\alpha$. During this initial stage the solution of the equations (2.27) and (2.28) can be represented in the form

$$r(T)=r_0+2(1-V_*^2)^{1/2}\ln \mathrm{ch}(T/T_0), \tag{2.29}$$

$$T_0={}^1/_2\exp[(r_0/2)(1-V_*^2)^{-1/2}](1-V^2)^{-1/2}. \tag{2.30}$$

At sufficiently large T (but $T\ll 1/\alpha$) the repulsion force can be neglected, and the friction assumes the principal role. In this case it follows from (2.27) and (2.28) that

$$dr/dT=-\alpha(r-r_*), \tag{2.31}$$

where r_* is an integration constant. Since α is small, Eqs. (2.29) and (2.31) can be matched together and r_* obtained. Assuming that $T_0\ll T\ll 1/\alpha$ and substituting (2.29), (2.30) in (2.31), we obtain[12]

$$r_*\approx r_0+4\alpha^{-1}(1-V_*^2)\exp[(-r_0/2)(1-V_*^2)^{-1/2}]. \tag{2.32}$$

The condition that α be small can in this case be written in the form $\alpha\ll T_0^{-1}$.

Next, integrating (2.31), we have

$$r=r_*+\mathrm{const}\cdot e^{-\alpha T}. \tag{2.33}$$

It is seen from that that at $T\gg 1/\alpha$ the distance $r(t)$ between the solitons tends to a constant limit r_*. It must be taken into account here, however, that at very large αT the friction force, which is proportional to dr/dT, decreases very rapidly and becomes comparable with the potential force, so that the latter can no longer be neglected. During this concluding stage we can use another approximation, namely discard the term dP/dT of Eq. (2.27), which is now small compared with each of the terms of the right-hand side. Determining next P from (2.27) and substituting in (2.28), we obtain

$$r(T)=(1-V^2)^{1/2}\ln[8(1-V^2)(T+T_1)/\alpha], \tag{2.34}$$

where T_1 is an integration constant. It is easy to verify that the condition that dP/dT be negligible is satisfied at $\alpha(T+T_1)\gg 1$.

Matching together (2.34) with (2.33) at $T_1\gg T\gg 1/\alpha$, we obtain[12]

$$T_1\approx{}^1/_8\alpha(1-V^2)^{-1}\exp[r_*(1-V^2)^{-1}]. \tag{2.35}$$

We see that T_1 is an exponentially large quantity at small α, and the logarithmic growth of $r(T)$, which can become noticeable at $T\gg T_1$, hardly appears if $T\lesssim T_1$.

It can thus be assumed that the distance $r(T)$ between the fluxons reaches a certain limit r_* within a time $T\sim 1/\alpha$ and then remains approximately constant, although in essence there is no bound state of fluxons here. It is almost obvious that this result does not depend on the assumption that $V(0)=V$. We shall not go into the details of the rigorous but rather tedious proof of this statement, since this fact follows directly from the numerical results of the next section.

This entire picture was obtained under the assumption that $\beta=0$ in (2.2). We indicate in this connection that in certain papers (for example Refs. 1 and 7) another theory is proposed, in which the formation of fluxon complexes is essentially connected with the terms with the third derivative in (2.2). These ideas, however, do not agree with the model and the numerical experiments in Refs. 8 and 9 where, as above, it was assumed that $\beta=0$ and nonetheless soliton complexes were observed. We assume therefore that our theory offers a more adequate explanation of the congealing effect observed in Refs. 8 and 9. We indicate also that it can be easily extended to include the case of n fluxons, i.e., quasistationary complexes with an arbitrary number of fluxons can be formed on the basis of the very same quasiequilibrium mechanism between the repulsion forces, the dissipative effect due to the transverse current of the normal electrons, and the external source.

3. NUMERICAL SIMULATION

We discuss now some results of numerical integration of Eq. (2.1) with the right-hand side (2.2) at $\beta=0$, and compare them with the asymptotic perturbation-theory analysis of the preceding section.

Equation (2.1) was solved numerically with two types of boundary conditions (BC) at the end points of the segment (a,b) of the real axis:

$$v_x(a,T)=v_x(b,T), \tag{3.1}$$

$$v_x(a,T)=0,\quad v_x(b,T)=0. \tag{3.2}$$

The initial conditions were assumed to be represented by the profile of the exact two-soliton solution of the unperturbed SG equation[3)]:

$$v(X,0)=4\,\mathrm{arctg}\left\{\frac{|\nu_1-\nu_2|}{2\nu}\frac{\mathrm{sh}[(Z_1+Z_2)/2]}{\mathrm{ch}[(Z_1-Z_2)/2]}\right\}, \tag{3.3}$$

$$Z_n=[X-X_n(0)](1-V_n^2)^{-1/2},\quad n=1,2, \tag{3.4}$$

$$\nu_n=[(1-V_n)/(1+V_n)]^{1/2}/2,\quad \nu=(\nu_1+\nu_2)/2.$$

The BC (3.1) define a periodic continuation of the solu-

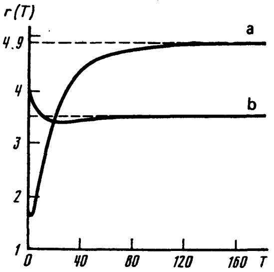

FIG. 2. Plot of $r(T)$ obtained from the numerical solution of Eqs. (2.1) and (2.2) with the BC (3.1), where $a = -6$ and $b = 8$, and with initial conditions (3.3) and (3.4) corresponding to a system of two fluxons with initial velocities: a) $V_1 = 0.77$; $V_2 = 0.93$ ($V = V_*$) at $\beta = 0$, $\alpha = 0.8$, and $f \approx 0.16$; b) $V_1 = 0.8$; $V_2 = 0.9$ ($V \neq V_*$) at $\beta = 0$, $\alpha = 0.1$, and $f = 0.3$.

tion over the entire real axis. It corresponds to propagation of the waves along a ring and, obviously, simulate an infinite line, if the fluxon dimensions and the distances between them are much smaller than the length of the segment $b-a$.

The BC (3.2) correspond to a Josephson line of finite length neglecting radiation (and other losses) at the ends. The influence of the losses at the end of the line on the evolution of one fluxon was recently investigated (numerically) in Ref. 16. It can be proposed on the basis of the results of that reference that if the losses at the end of the line are small enough, the results that follow from the BC (3.2) do not change substantially.

The actual values of the parameters in the initial conditions were chosen such as to clarify the meaning of the assumptions made in the derivation of the analytic relations of the preceding section. Figure 2 shows the distance $r(T)$ between the solitons for two cases. Curve a was obtained for the case when the solitons overlap strongly[4)] at $T = 0$, and their velocities under the initial conditions (3.4) are $V_1 = 0.77$ and $V_2 = 0.93$. In this case $V = (V_1 + V_2)/2 = 0.85$, which agrees with the value of V_* calculated from formula (2.26), ($\alpha = 0.08$, $f \approx 0.16$). Thus, the conditions under which Eq. (2.32) was obtained are well satisfied here. In the case considered, this equation yields $r_* \approx 4.1$, whereas the numerical value is $r(160) \approx 4.9$.

Curve b obtained for the case when the distance between the solitons at $T = 0$ is quite large (larger than the sum of their characteristic dimensions), and the initial velocities are such that[4)] $V = 0.85 \neq V_* = 0.92$. The numerical results, which are not presented here, show that with increasing T the average velocity of the system approaches the asymptotic value $V_* = 0.92$ in accordance with (2.24). As for the distance between the solitons, it is seen from the figure that it initially decreases and then increases slightly, approaching the stationary value $r_* \approx 3.5$. Although a quantitative comparison with the asymptotic formula (2.32) cannot in fact be carried out here, since the conditions under which this formula was derived are not satisfied, the qualitative result, namely that r tends to an asymptotic value of the same order as (2.32), remains in force.

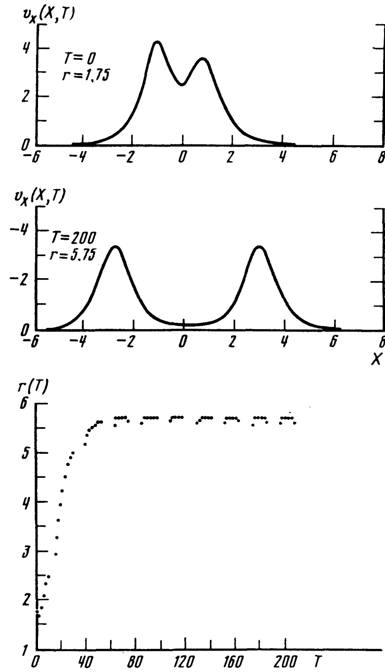

FIG. 3. Numerical solution of Eqs. (2.1) and (2.2) for two fluxons with BC (3.2) at $a = -8$, $b = 10$, $\beta = 0$, $\alpha = 0.08$, and $f \approx 0.16$ and initial velocities $V_1 = 0.77$ and $V_2 = 0.93$ ($V = V_*$).

Figures 3 and 4 show the numerical solution and the distance between the solitons for the BC (3.2), i.e., for a Josephson junction of finite length, under initial conditions that agree with those for Figs. 2(a) and 2(b), respectively. It is seen that there is good agreement between the two types of BC. Notice must be taken, however, of certain characteristic features of the BC (3.2). Upon reflection from the end, the polarity of the

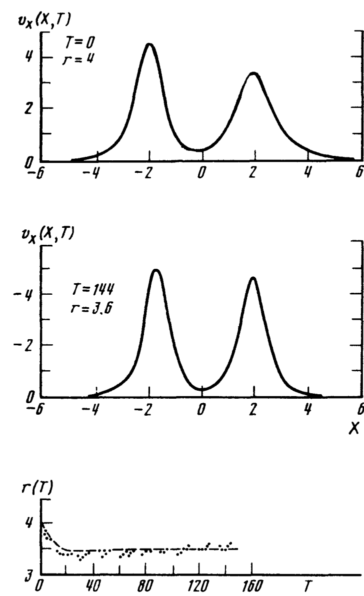

FIG. 4. Numerical solution of Eqs. (2.1) and (2.2) for two fluxons with BC (3.2) at $a = -6$, $b = 8$, $\beta = 0$, $\alpha = 0.1$, and $f = 0.3$ and initial velocities $V_1 = 0.8$ and $V_2 = 0.9$ ($V \neq V_*$). The dashed lines shows the plot of $r(T)$ for the BC (3.1) from Fig. 2b.

fluxon is reversed ($\sigma \to -\sigma$). Therefore when choosing the instants of time T for which the distance between the solitons was determined, only those T were considered at which the fluxons had the same polarity and were far enough from the end points. Since it is quite difficult to satisfy well the last requirement, a certain straggling $r(T)$ was obtained. It is seen also that for the case shown in Fig. 3 the limiting distance is somewhat larger than in Fig. 2(a). We note that with increasing length of the line $b-a$, the difference between the two cases decreased. As for the case shown in Fig. 4, $r(T)$ agrees there very well with Fig. 2(b) [the dashed line of Fig. 4 is a plot of $r(T)$ from Fig. 2(b)].

Thus, a finite length of the Josephson junction and strong interaction between the solitons do not change the main results obtained on the basis of perturbation theory in Sec. 2. The gist of the latter is that fluxons of equal polarity, in the presence of a bias current and of dissipation due to the finite conductivity of the junction, form despite their mutual repulsion complexes having the following basic properties: 1) the distance $r(T)$ between solitons in the complex tends to an asymptotic value r_* that is practically constant [in fact it increases slowly (logarithmically) at very large T]; 2) the velocity of the complex (i.e., the arithmetic mean of the soliton velocities) tends to an asymptotic value V_* that coincides with the limiting velocity of one soliton at the same bias current and dissipation.

An experimental check on these conclusions is realizable in junctions with length of the order of or larger than $(2-3)\lambda$, where λ is the Josephson length (2.3). As already indicated in the introduction, the fluxons oscillating between the ends of the line generate electromagnetic radiation whose spectrum is determined by the frequency of their oscillations in the junction and by their number. From the current-voltage characteristic it is always possible to determine how many fluxons are excited in the junction.[3] If n fluxons ($n > 1$) are excited, and the fundamental frequency in the spectrum is close enough to the frequency of one fluxon, then this obviously should be evidence that the fluxons form a single complex and move with the same velocity.

In conclusion the authors thank Professor K.K. Likhorev and Professor A. Scott for helpful discussion of the results and for valuable bibliographical hints.

Note (10 July 1981). In a recent article, B. Dueholm *et al.* [Phys. Rev. Lett., **46**, 1299 (1981)], report experimental observations, in long Josephson junctions, of bound fluxons with like polarity (in their terminology—"bunched fluxon configurations"), which agree well with the theory developed above.

[1)] As seen from Sec. 2, this calls for the parameters of the fluxons to be close enough to one another.

[2)] Equation (2.8) differs from that given in Ref. 7 and was taken from Refs. 12 and 13.

[3)] The use of the asymptotic solutions of the unperturbed SG equation under the BC (3.2) obtained in Ref. 15 is less convenient.

[4)] The initial profiles of $v_x = 2\alpha/2\alpha$ for the cases a and b are shown in Figs. 3 and 4.

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