Quasi-one-dimensional weak turbulence spectra

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We analyze the kinetic equation describing the four-wave nonlinear interaction for weak-turbulence spectra for the case where the spectrum is concentrated along a selected direction in k-space. The kinetic equation is strongly simplified in the quasi-one-dimensional limit and this enables us to obtain simple formulae for the angular dependence of the spectrum and for the nonlinear energy transfer along the spectrum. We apply the results obtained to the problem of Langmuir turbulence of an isothermal plasma which occurs when the plasma is parametrically heated by an oscillating electric field and to the problem of wind waves on a liquid surface. We suggest that there exist two scales in the angular spectrum of wind waves.

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INTRODUCTION

In weak-turbulence theory one often meets with a situation when the dispersion law $\omega(\mathbf{k})$ forbids three-wave processes with a resonance condition:

$$\omega(\mathbf{k}_1+\mathbf{k}_2)=\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2). \tag{1}$$

Such a dispersion law is usually called a non-decay law; one can also call the corresponding weak turbulence non-decay turbulence. Langmuir turbulence in an isothermal plasma, and also turbulence of the waves on a liquid surface, are examples of non-decay weak turbulence.

It is well known from the theory of Langmuir turbulence that its spectra as a rule turns out to be singular ("jet-like"), i.e., they are located in k-space on special surfaces, or lines—"jets."[1] In a number of cases, in particular, when the plasma is parametrically heated by an oscillating electric field, the jets are straight lines passing through the origin, i.e., the spectrum is quasi-one-dimensional. In Langmuir turbulence the basic mechanism for the interaction of the waves is the non-conservative mechanism of induced scattering of waves by ions.

In an other important physical problem of the turbulence of waves on the surface of a liquid, the main wave interaction mechanism is their scattering by one another which satisfies the resonance conditions

$$\omega(\mathbf{k})+\omega(\mathbf{k}_1)=\omega(\mathbf{k}_2)+\omega(\mathbf{k}_3), \quad \mathbf{k}+\mathbf{k}_1=\mathbf{k}_2+\mathbf{k}_3. \tag{2}$$

However, also in this purely conservative case the spectra are often quasi-one-dimensional. Measurements in nature show (see, e.g., Ref. 2) that the typical angular width of developed sea swell is of the order of 20–30°. An attempt to explain such a small width of the spectra stimulated the present paper. We show in it that conditions when the turbulence spectrum is quasi-one-dimensional can easily be realized in the conservative case (2). The kinetic equation describing the turbulence simplifies considerably for the quasi-one-dimensional case and this enables us to calculate explicitly the form of the stationary spectrum. This result is relevant not only for the conservative situation. In the problem of the Langmuir turbulence of an isothermal plasma allowance for only the single mechanism of induced scattering enables us to determine only the gross characteristics of the spectrum—the position and intensity of the jets. The fine structure of the jets is, however, determined as before by processes of the kind (2). One can therefore say that we solve in the present paper the problem of the structure of one-dimensional jet spectra.

The basic technique used in the present paper is the expansion of the kinetic equation for the waves in powers of the quantity $\langle\theta^2\rangle$—the mean square angular width of the spectrum.

§1. ZEROTH APPROXIMATION

A conservative medium with a non-decay dispersion law is described by the effective Hamiltonian

$$H=\int\omega(\mathbf{k})a_{\mathbf{k}}a_{\mathbf{k}}^*\,d\mathbf{k}+\frac{1}{2}\int T_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}a_{\mathbf{k}}a_{\mathbf{k}_1}\times a_{\mathbf{k}_2}^*a_{\mathbf{k}_3}^*\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\,d\mathbf{k}\,d\mathbf{k}_1\,d\mathbf{k}_2\,d\mathbf{k}_3. \tag{3}$$

The statistical behavior of the medium in the framework of weak turbulence is described by the quantity $n_{\mathbf{k}}(t)$ given by the formula

$$n_{\mathbf{k}}\delta(\mathbf{k}-\mathbf{k}')=\langle a_{\mathbf{k}}a_{\mathbf{k}'}^*\rangle.$$

This quantity is commonly called the quasi-particle number density in phase space. It satisfies the kinetic equation

$$\partial n_{\mathbf{k}}/\partial t+\gamma_{\mathbf{k}}n_{\mathbf{k}}=I_{\mathbf{k}}. \tag{4}$$

Here $\gamma_{\mathbf{k}}$ is the phenomenologically introduced damping of the wave and $I_{\mathbf{k}}$ the collision term describing four-wave processes:

$$I_{\mathbf{k}}=2\pi\int|T_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}|^2\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\delta[\omega(\mathbf{k})+\omega(\mathbf{k}_1)-\omega(\mathbf{k}_2)-\omega(\mathbf{k}_3)][n_{\mathbf{k}_2}n_{\mathbf{k}_3}(n_{\mathbf{k}}+n_{\mathbf{k}_1})-n_{\mathbf{k}}n_{\mathbf{k}_1}(n_{\mathbf{k}_2}+n_{\mathbf{k}_3})]\,d\mathbf{k}_1\,d\mathbf{k}_2\,d\mathbf{k}_3. \tag{5}$$

If $\gamma_{\mathbf{k}}$ is negative in some region of k-space, it corresponds to the instability of the medium for the generation of waves. Non-conservative mechanisms for the interaction of the waves can be taken into account by renormalizing $\gamma_{\mathbf{k}}$:

$$\gamma_{\mathbf{k}}\to\gamma_{\mathbf{k}}+\int V_{\mathbf{k}\mathbf{k}_1}n_{\mathbf{k}_1}\,d\mathbf{k}_1. \tag{6}$$

In particular, for the induced scattering of Langmuir waves by ions, the function $V_{\mathbf{k}\mathbf{k}_1}$ is antisymmetric: $V_{\mathbf{k}\mathbf{k}_1}=-V_{\mathbf{k}_1\mathbf{k}}$.

If $\gamma_{\mathbf{k}}=0$, Eq. (4) has the obvious conservation laws

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$$N=\int n_{\mathbf{k}}d\mathbf{k},\quad P=\int \mathbf{k}n_{\mathbf{k}}d\mathbf{k},$$

$$\mathscr{E}=\int \omega(\mathbf{k})n_{\mathbf{k}}d\mathbf{k}, \tag{7}$$

which have, respectively, the meaning of the conservation of total number of quasi-particles, momentum, and energy of the waves. We shall in what follows consider primarily the two-dimensional case—the generalization to the three-dimensional case is obvious. We denote by k the coordinate in $\mathbf{k}$-space along the one-dimensional jet and by ξ the coordinate in the transverse direction. We now have for the dispersion law

$$\omega(\mathbf{k})=\omega(k)+s(k)\xi^2+\ldots,\quad s(k)=\frac{1}{2k}\frac{d\omega(k)}{dk}. \tag{8}$$

We expand the kernel of the kinetic equation in the parameter ξ^2/k^2 and consider the zeroth and first terms of this expansion. We have

$$\delta[\omega(\mathbf{k})+\omega(\mathbf{k}_1)-\omega(\mathbf{k}_2)-\omega(\mathbf{k}_3)]=\delta[\omega(k)+\omega(k_1)-\omega(k_2)-\omega(k_3)]$$
$$+[s(k)\xi^2+s(k_1)\xi_1^2-s(k_2)\xi_2^2-s(k_3)\xi_3^2]\delta'[\omega(k)$$
$$+\omega(k_1)-\omega(k_2)-\omega(k_3)]+\ldots. \tag{9}$$

We shall neglect the dependence on the transverse coordinates in the matrix element $T_{\mathbf{kk}_1,\mathbf{k}_2\mathbf{k}_3}$ and put

$$T_{\mathbf{kk}_1,\mathbf{k}_2\mathbf{k}_3}=T_{kk_1,k_2k_3}.$$

The zeroth approximation in ξ^2/k^2 thus corresponds to taking into account in (9) only the first term in the expansion. In that case

$$I_{k\xi}^{(0)}=2\pi\int|T_{kk_1,k_2k_3}|^2\delta[\omega(k)+\omega(k_1)-\omega(k_2)$$
$$-\omega(k_3)]\delta(k+k_1-k_2-k_3)\delta(\xi+\xi_1-\xi_2-\xi_3)[n_{k_2\xi_2}n_{k_3\xi_3}(n_{k\xi}+n_{k_1\xi_1})$$
$$-n_{k\xi}n_{k_1\xi_1}(n_{k_2\xi_2}+n_{k_3\xi_3})]dk_1\,dk_2\,dk_3\,d\xi_1\,d\xi_2\,d\xi_3. \tag{10}$$

We can immediately integrate over dk_2dk_3. We note that the set of equations

$$\omega(k)+\omega(k_1)=\omega(k_2)+\omega(k_3),\quad k+k_1=k_2+k_3 \tag{11}$$

have just two solutions $k_2=k, k_3=k_1$ and $k_2=k_1, k_3=k$. Using the symmetry properties of the kernel T we have

$$I_{k\xi}^{(0)}=4\pi\int dk_1\frac{|T_{kk_1,kk_1}|^2}{|\omega_k'-\omega_{k_1}'|}\delta(\xi+\xi_1-\xi_2-\xi_3)$$
$$\times[n_{k\xi_2}n_{k_1\xi_3}(n_{k\xi}+n_{k_1\xi_1})-n_{k\xi}n_{k_1\xi_1}(n_{k\xi_2}$$
$$+n_{k_1\xi_3})]d\xi_1d\xi_2d\xi_3. \tag{12}$$

There is a logarithmic divergence in the integral (12) and must be cut off at $|k-k_1|\sim\xi$. [Our approximation does not hold in that region and it is impossible to neglect the ξ-dependence of $\omega(\mathbf{k})$ and $T_{\mathbf{kk}_1\mathbf{k}_2\mathbf{k}_3}$.]

We restrict ourselves in what follows to the simplest case $n(k,\xi)=n(k,-\xi)$ corresponding to spectra which are symmetric with respect to the k-axis. Fourier transforming with respect to ξ we then get after a few calculations the simple result:

$$\tilde{I}_{ky}^{(0)}=\int I_{k\xi}^{(0)}e^{iy\xi}d\xi=\tilde{n}_{ky}\int V_{kk_1}(\tilde{n}_{k_1y}^2-\tilde{n}_{k_10}^2)dk_1. \tag{13}$$

Here

$$V_{kk_1}=\frac{4\pi|T_{kk_1,kk_1}|^2}{|\omega_k'-\omega_{k_1}'|},\quad \tilde{n}_{ky}=\int n_{k\xi}e^{iy\xi}d\xi.$$

The quantity

$$\tilde{n}_{k0}=\int n_{k\xi}d\xi=N_k$$

is the quasi-particle number density integrated over the transverse variable ξ. From (13) it follows that $I_{k0}^{(0)}=0$. The collision term in the zeroth approximation (12), (13) therefore does not transfer the quasi-particle number along the jet and accomplishes only a redistribution of wave energy in the transverse direction. To establish the nature of this distribution we expand $\tilde{n}_{ky}$ in powers of y:

$$\tilde{n}_{ky}=\int n_{k\xi}e^{iy\xi}d\xi=\int n_{k\xi}d\xi-\frac{y^2}{2}\int\xi^2n_{k\xi}d\xi$$
$$+\frac{y^4}{24}\int\xi^4n_{k\xi}d\xi+\ldots=N_k\left[1-\frac{y^2}{2}\langle\xi^2\rangle_k+\frac{y^4}{24}\langle\xi^4\rangle_k+\ldots\right].$$

Here $\langle\xi^2\rangle_k\equiv\delta_k^2$ and $\langle\xi^4\rangle_k\equiv\Lambda_k$ are the second and fourth moments of the distribution $n_{k\xi}$ with respect to ξ. Equating terms $\propto y^2$ in the Fourier transform of Eq. (4) (for the case where $\gamma_{\mathbf{k}}=0$) we get

$$\frac{\partial}{\partial t}(\delta_k^2)=2\int V_{kk_1}N_{k_1}^2\delta_{k_1}^2dk_1. \tag{14}$$

Equation (14) shows (since $V_{kk_1}>0$) that the collision term (12) leads to a broadening in angle of the wave spectrum. In this connection the problem arises of how, in general, spectra which are narrow in angle are possible. One can construct the simplest model of a narrow spectrum by assuming that the damping increases fast with increasing ξ. Let, as $\xi\to0$,

$$\gamma_{k\xi}=-f_k+g_k\xi^2+\ldots. \tag{15}$$

For sufficiently small ξ the damping is negative, corresponding to an instability. A stationary spectrum arises because the collision term "disperses" the quasi-particles originating in the instability region $\xi^2<f_k/g_k$ to the region of damping $\xi^2>f_k/g_k$.

One can calculate the spectrum explicitly by expanding in the parameter $(\ln k^2/\delta_k^2)^{-1}$ (for waves on water this parameter is of the order of 1/3 to 1/5). With logarithmic accuracy we have

$$I_{ky}^{(0)}=V_k\tilde{n}_{ky}(\tilde{n}_{ky}^2-N_k^2),$$

$$V_k=\frac{2\pi|T_{kk,kk}|^2}{|\omega_k''|}\ln\frac{k^2}{\delta_k^2}. \tag{16}$$

After Fourier transforming with respect to ξ, taking into account the first two terms of (15) and (16), Eq. (4) takes the form

$$\frac{\partial\tilde{n}_{ky}}{\partial t}=f_k\tilde{n}_{ky}+g_k\frac{\partial^2}{\partial y^2}\tilde{n}_{ky}+\tilde{n}_{ky}(\tilde{n}_{ky}^2-N_k^2)V_k. \tag{17}$$

In the stationary case we have

$$\frac{d^2\tilde{n}_{ky}}{dy^2}+\frac{f_k-V_kN_k^2}{g_k}\tilde{n}_{ky}+\frac{V_k}{g_k}\tilde{n}_{ky}^3=0. \tag{18}$$

Equation (18) has an exact solution

$$\tilde{n}_{ky}=\left[\frac{2(N_k^2V_k-f_k)}{V_k}\right]^{1/2}\mathrm{ch}^{-1}\left[y\left(\frac{V_kN_k^2-f_k}{g_k}\right)^{1/2}\right]. \tag{19}$$

The undetermined quantity f_k must be found from the boundary condition $\tilde{n}_{k0}=N_k$. We have

$$f_k={}^1/_2V_kN_k^2. \tag{20}$$

(The obvious physical requirement $f_k>0$ for the existence of a stationary solution in the model considered is once again clear from this.) Finally we have

$$\tilde{n}_{ky}=N_k/\mathrm{ch}\,[yN_k(V_k/2g_k)^{1/2}]. \tag{21}$$

Performing the inverse Fourier transformation we find

$$n_{k\xi}=\left(\frac{2g_k}{V_k}\right)^{1/2}\mathrm{ch}^{-1}\left[\frac{\pi\xi}{N_k}\left(\frac{g_k}{2V_k}\right)^{1/2}\right]. \tag{22}$$

We now have for the width of the spectrum $\langle\theta^2\rangle_k$

$$\langle\theta^2\rangle_k=\frac{\delta_k^2}{k^2}=\frac{N_k^2V_k}{2g_kk^2}=-\frac{\pi|T_{kk,kk}|^2N_k^2}{g_kk^2|\omega_k''|}\ln\langle\theta^2\rangle k. \tag{23}$$

The estimate (23) remains valid in the three-dimensional case.

In deriving Eqs. (21) to (23) we used the assumption that the ξ-dependence of $\gamma_{k\xi}$ is much smoother than that of $n_{k\xi}$. Indeed, in the opposite case it is illegitimate to neglect high-order terms in ξ in the expansion (15) for $\gamma_{k\xi}$, additional terms appear in Eq. (17), and expression (21) will no longer be a solution. In that case the angular width of the spectrum will be determined by the characteristic angular scale of the quantity $\gamma_{k\xi}$. The solution obtained has in **k**-space a unique characteristic transverse scale (23). In the more general case there may occur several characteristic scales. We introduce in Eq. (4) an external force $F_{\mathbf{k}}\equiv F_{k\xi}$ so that it takes the form

$$\partial n_k/\partial t+\gamma_kn_k=I_k+F_k.$$

In the problem of wind waves the external force $F_{\mathbf{k}}$ corresponds to that part of the kinetic integral $I_{\mathbf{k}}$ which was not taken into account by us earlier and which describes the effect of the isotropic high-frequency components.

If the external force decreases sufficiently weakly as $\xi\to\infty$, we have for large ξ

$$n_{k\xi}=F_{k\xi}/\gamma_{k\xi}. \tag{24}$$

The quantity $n_{k\xi}$ can decrease relatively slowly as $\xi\to\infty$. We may assume the spectrum to be narrow in angle, if the integral $\int n_{k\xi}d\xi=N_k$ converges and is "concentrated" on a scale $\delta_k\ll k$. However, the integral $\int\xi^2n_{k\xi}d\xi$ may diverge or be "concentrated" on a scale $\bar\delta_k\gg\delta_k$. This happens just in the case when $\gamma_{k\xi}$ is given by Eq. (15)—the scale $\bar\delta_k$ then is the same as the characteristic scale on which the function $F_{k\xi}$ decreases. In the case when $F_{k\xi}=F(k)/2\pi$ the spectrum can be found explicitly. The stationary Eq. (18) now has the form

$$\frac{d^2\tilde n_{ky}}{dy^2}+\frac{f_k-V_kN_k^2}{g_k}\tilde n_{ky}+\frac{V_k}{g_k}\tilde n_{ky}^3+\frac{F_k}{g_k}\delta(y)=0. \tag{25}$$

The solution of Eq. (25) is a continuous symmetric function of y which has at $y=0$ a discontinuity in the derivative such that

$$|\lim_{y\to0}d\tilde n_{ky}/dy|=F_k/2g_k.$$

When $y>0$

$$\tilde n_{ky}=\left[\frac{2(N_k^2V_k-f_k)}{V_k}\right]^{1/2}\mathrm{ch}^{-1}\left[(y-a_k)\left(\frac{N_k^2V_k-f_k}{g_k}\right)^{1/2}\right], \tag{26}$$

and from the condition on the discontinuity of the derivative and the normalization condition $\tilde n_{k0}=N_k$ we find

$$N_k^2=f_k/V_k+(f_k^2/V_k^2+F_k^2/2g_kV_k)^{1/2};$$

when $F_k>0$ a stationary solution exists, even if $f_k\le0$. For the displacement a_k we have

$$a_k=[g_k/(N_k^2V_k-f_k)]^{1/2}\,\mathrm{arsh}\,[F_k/N_k^2(2V_kg_k)^{1/2}].$$

The discontinuity of the derivative at zero determines the asymptotic behavior of the spectrum $n_{k\xi}\to F_k/g_k\xi^2$ as $\xi\to\infty$. The quantity

$$\delta_k^2=\int\xi^2n_{k\xi}d\xi$$

is infinite in this case, or, strictly speaking, is determined by the cutoff boundary of the function $F_{k\xi}$.

Of course, the entire analysis in the logarithmic approximation has a meaning only if $T_{kk,kk}\neq0$.

§2. FIRST APPROXIMATION

When calculating the next approximation it is, in general, impossible to neglect the dependence of $T_{\mathbf{kk}_1,\mathbf{k}_2\mathbf{k}_3}$ on the transverse coordinates. However, only allowance for the next term in the expansion (9) of the frequency δ-function in powers of ξ leads to a principally new effect. Therefore we shall consider only those terms. We have

$$\begin{aligned}I_k^{(1)}&=2\pi\int|T_{kk_1,k_2k_3}|^2\delta'[\omega(k)+\omega(k_1)-\omega(k_2)\\&-\omega(k_3)][s(k)\xi^2+s(k_1)\xi_1^2-s(k_2)\xi_2^2-s(k_3)\xi_3^2]\\&\times\delta(\xi+\xi_1-\xi_2-\xi_3)\delta(k+k_1-k_2-k_3)[n_{k_2\xi_2}n_{k_3\xi_3}\\&\times(n_{k\xi}+n_{k_1\xi_1})-n_{k\xi}n_{k_1\xi_1}(n_{k_2\xi_2}+n_{k_3\xi_3})]dk_1dk_2dk_3d\xi_1d\xi_2d\xi_3.\end{aligned} \tag{27}$$

The integration over two of the longitudinal variables, for instance, over k_1 and k_2, can be performed in the same way as was done in §1. However, the expression obtained is very complicated and we restrict ourselves to a simplified variant of the study. In fact, we evaluate the quantity $\partial N_k/\partial t$ which is zero in the zeroth approximation in ξ^2/k^2. Integrating over $d\xi$, $d\xi_1$, $d\xi_2$, $d\xi_3$, and after that over dk_1, dk_2, we get

$$\frac{\partial N_k}{\partial t}=\int_0^\infty\frac{dk_3}{(\omega_k'-\omega_{k_3}')|\omega_k'-\omega_{k_3}'|}\left[\frac{\omega_k''-\omega_{k_3}''}{\omega_k'-\omega_{k_3}'}F_{kk_3}-\frac{1}{2}\left(\frac{\partial}{\partial k}+\frac{\partial}{\partial k_3}\right)F_{kk_3}\right], \tag{28}$$

where

$$F_{kk3}=8\pi|T_{kk_3,kk_3}|^2N_kN_{k_3}\left[\frac{\omega_k'}{k}N_{k_3}\delta_{k_3}^2+\frac{\omega_{k_3}'}{k_3}N_k\delta_k^2\right]. \tag{29}$$

As before, the right-hand side of Eq. (28) contains a logarithmic divergence for $k=k_3$. This divergence must be cutoff at $(k-k_3)^2\sim\delta_k^2$. With logarithmic accuracy we can write

$$\frac{\partial N_k}{\partial t}=-\frac{\partial^2}{\partial k^2}\left[\frac{\omega_k'}{k|\omega_k''|}V_k\delta_k^2N_k^3\right]+R_k. \tag{30}$$

Here

$$R_k=\int d\xi[-\gamma_{k\xi}n_{k\xi}+F_{k\xi}]d\xi$$

is the term caused by the pumping, the damping, and the external force. The quantity δ_k^2 must be determined from the zeroth approximation equation. After multiplication by ξ^2 and integration over ξ in agreement with Eq. (14), this approximation takes in the logarithmic approximation the form

$$\begin{aligned}&\frac{\partial}{\partial t}(N_k\delta_k^2)=2V_k\delta_k^2N_k^3+S_k.\\&S_k=\int\xi^2(-\gamma_{k\xi}n_{k\xi}+F_{k\xi})d\xi.\end{aligned} \tag{31}$$

In the conservative case when $R_k=0$, $S_k=0$, Eqs. (30)

and (31) conserve the integrals of motion

$$N=\int N_k dk, \quad P=\int kN_k dk,$$
$$\mathscr{E}=\int\left(\omega_k+\frac{\omega_k'}{2k}\delta_k^2\right)N_k dk. \tag{32}$$

The integrals N and P in (32) correspond to the conservation of the number of quasi-particles and of the momentum of the waves, while the integral $\mathscr{E}$ is the energy of the waves, taking into account the ξ-dependence of $\omega_{\mathbf{k}}$.

Kolmogorov-type solutions play a very important role in weak turbulence theory; for them the flux of one of the integrals of motion is constant in the system considered. We shall discuss for what circumstances Kolmogorov solutions can exist for quasi-one-dimensional spectra. Let the solution be a "single-scale" one with respect to the transverse wave number [such as, e.g., solution (21) in §1]. In that case the integral for S_k converges at $\xi\sim\delta_k$ so that $S_k\approx\delta_k^2 R_k$. One must understand this relation in the rough sense that R_k stands for different terms occurring in it. Indeed, it now follows from Eq. (31) that in the stationary case $V_k N_k^3\approx R_k$.

On the other hand, the flux term in (30) is of the order of $(\delta_k^2/k^2)V_k N_k^3$ and hence is negligibly small compared to the individual terms in R_k. In the stationary case there follows now therefore only a balance equation for the number of quasi-particles $R_k=0$. The flux of the number of quasi-particles along the spectrum is then small.

Let, however, the spectrum have two scales so that $S_k\approx L\delta_k^2 R_k$ where $L\gg 1$ is a large parameter. In that case, if $Lk^2/\delta_k^2\gg 1$ we can neglect the term R_k in Eq. (30). The stationary Eq. (30) has now the solution

$$N_k=\left(\frac{|\omega_k''|}{k\omega_k' V_k\langle\theta^2\rangle_k}\right)^{1/3}(Qk+T)^{1/3}. \tag{33}$$

Here $\langle\theta^2\rangle_k=\delta_k^2/k$ is the square of the angular width of the spectrum.

In the solution (33) Q has the meaning of the flux of the number of quasi-particles in the region of small wavenumbers as $k\to 0$ and T the meaning of the momentum flux in the region of large wavenumbers as $k\to\infty$. The solution corresponding to a constant energy flux is not present as in the transfer process along the spectrum the energy is dissipated in the region of large ξ^2. As before, it is necessary for the determination of $\langle\theta^2\rangle_k$ to use the zeroth approximation Eq. (17). An example of this solution for which a Kolmogorov situation can be realized is given by Eq. (26).

§3. APPLICATIONS

We consider physical applications of the results. They are the simplest for the Langmuir turbulence of a plasma. In that case a "single-scale" situation is realized as the N_k distribution is completely determined by a process of lower order in comparison with the four-wave interaction—the induced scattering of Langmuir waves by ions. The four-wave interaction only determines the shape of the jet. We have for the matrix element of the interaction (see Ref. 3)

$$T_{\mathbf{k}\mathbf{k}_1,\mathbf{k}_2\mathbf{k}_3}=\frac{\omega_p^2}{4\pi nT}\left[\frac{(\mathbf{k}\mathbf{k}_2)(\mathbf{k}_1\mathbf{k}_3)}{kk_1k_2k_3}G(\omega_{\mathbf{k}_1}-\omega_{\mathbf{k}_3},\mathbf{k}_1-\mathbf{k}_3)+(\mathbf{k}_2\leftrightarrow\mathbf{k}_3)\right]. \tag{34}$$

Here ω_p is the plasma frequency, n and T the electron density and temperature, $\omega_{\mathbf{k}}^2=\omega_p^2\,1+3(k\lambda_D)^2$ the Langmuir wave dispersion law, and $G(\omega,\mathbf{k})$ the low-frequency plasma Green function which is the dimensionless response of the ion density to the slow external force. In the hydrodynamic approximation

$$G(\omega,\mathbf{k})=k^2c_s^2/(\omega^2-k^2c_s^2+2i\gamma_s c_s k).$$

Here c_s and γ_s are the ion sound speed and damping. The quantity $T_{kk,kk}$ depends on the relation between the quantity $(k\lambda_D)^2$ and the ratio of the electron to the ion masses m_e/m_i. In the case of most interest, $(k\lambda_D)^2\gg m_e/m_i$, we can in the denominator of the Green function neglect the terms ω^2 and $\gamma_s c_s k$ as $k\to 0$, $\omega\to 0$ so that $G(\omega,\mathbf{k})=-1$. In that case

$$T_{kk,kk}=(\omega_p^2/4\pi nT)^2. \tag{35}$$

The induced scattering changes the damping rate which now has the form

$$\gamma_{k\xi}=\int T_{\mathbf{k}\mathbf{k}_1}n_{\mathbf{k}_1}d\mathbf{k}_1.$$

Here

$$T_{\mathbf{k}\mathbf{k}_1}=\frac{\omega_p^2}{2\pi nT}\frac{(\mathbf{k}\mathbf{k}_1)^2}{k^2k_1^2}\operatorname{Im}G(\omega_{\mathbf{k}}-\omega_{\mathbf{k}_1},\mathbf{k}-\mathbf{k}_1).$$

The factor $(\mathbf{k}\cdot\mathbf{k}_1)^2/k^2k_1^2$ determines the main angular dependence of the growth rate. When $(k\lambda_D)^2\gg m_e/m_i$ the "differential approximation" is valid in which

$$g_k k^2\sim\gamma_k\approx\frac{\omega_p N_k k}{4\pi nT}\left(\frac{m_e}{m_i}\right)^{1/2}\frac{1}{k\lambda_D}.$$

We now have from Eq. (23) for the angular width a transcendental equation

$$\langle\theta^2\rangle_k=-\frac{\omega_p N_k k}{8\pi nT}\left(\frac{m_i}{m_e}\right)^{1/2}\frac{1}{k\lambda_D}\ln\langle\theta^2\rangle_k, \tag{36}$$

the solution of which gives the answer when $\langle\theta^2\rangle_k\ll 1$.

As regards wind waves on a liquid surface, the situation is more complicated for them since effective growth rate for the instability and the damping rate of the waves, determined by the interaction with the wind, are not known with sufficient accuracy.

The magnitude of the instability growth rate, given by Miles' well known theory,[4] leads, if one uses the concept of a "one-scale" spectrum, to a value of the wave energy in the energy-carrying region which differs from the experimental data by one and a half orders of magnitude. This fact, and also a number of other considerations (shape of the frequency spectrum, nature of the dependence of the energy and wavelength on the temporal dependence of the wind action) leads to the hypothesis that the narrow energy-carrying section in the spectrum of wind waves is "two-scale" in character and that in them there occurs a Kolmogorov flux of the number of quasi-particles to the region of small wavenumbers.

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