Massless ghost pole in chromodynamics and the solution of the U_1 problem

D. I. D'yakonov and M. I. Eides

Leningrad Institute of Nuclear Physics, USSR Academy of Sciences
(Submitted 4 April 1981)
Zh. Eksp. Teor. Fiz. **81**, 434–451 (August 1981)

An analysis of the U_1 problem (which consists, in particular, in the absence of a ninth light pseudoscalar meson) from the point of view of quantum chromodynamics (QCD) leads to the conclusion that the theory should contain a massless pole in some of the gauge-invariant quantities. The nature of this pole is new to particle physics: it is connected with the periodicity of the OCD potential energy with respect to a certain "generalized" coordinate and to the possibility of "free motion" of the system with respect to a certain coordinate. Owing to the axial anomaly, mixing takes the place of the ghost with light (pseudo-Goldstone) quark-antiquark states; diagonalization gives rise to physical η and gh' mesons, and the large mass of the latter is determined mainly by the mixing amplitude. The masses of the η and η' mesons are calculated, and certain amplitudes of the processes in which they take part agree well with experiment.

PACS numbers: 12.40.Cc, 12.70. + q, 14.40.Ka

1. INTRODUCTION

Application of quantum chromodynamics (QCD)[1] to hard hadron processes at high energy has shown that QCD perturbation theory describes the experiment adequately (see, e.g., the reviews[2]). Perturbation theory, however, still apparently does not fully specify the theory, and we are at present at the stage of accumulation of information concerning a "genuine" exact theory, a "genuine" vacuum, etc., formulated in terms of QCD itself. Much is already known at present from the phenomenology. In particular, it is known that the QCD vacuum should contain a quark-antiquark condensate $\langle\bar{q}q\rangle$,[3,4] as well as a gluon condensate $\langle G^2\rangle$, etc.[5-7] ($G^a_{\mu\nu}$ is the intensity of the gluon field), and these averages over the vacuum do not include the usual zero-point oscillations of the field.

Perhaps the most valuable information on chromodynamics outside the framework of perturbation theory is provided by the nonet of pseudoscalar mesons (π, K, η, and η'). We note that even before the appearance of QCD, the low-energy dynamics of the π, K, and η mesons was understood within the framework of current algebra and chiral Lagrangians.[8,9] In this approach, the eight pseudoscalar mesons constituted Goldstone bosons connected with spontaneous violation of the chiral invariance of the theory. Nonzero but small masses of the octet (we recall that π, K, and η are much lighter than all other hadrons) are the result of a patent small violation of chiral symmetry in the Lagrangian.

The advent of chromodynamics as a strong-interaction theory did not change this logic; moreover, the only source of the patent violation of chiral symmetry became obvious, namely the mass term in the QCD Lagrangian.[10] In the limit of zero masses of the u, d, and s quarks, the eight pseudoscalar mesons are, by virtue of the spontaneous violation of the chiral symmetry (which is due to formation of the $\langle\bar{q}q\rangle$ condensate) are massless Goldstone bosons. In this case, however, the ninth meson η' (singlet in SU_3) should apparently also be massless, inasmuch as in QCD the spontaneous violation of the chiral SU_3 symmetry brings about also violation of chiral U_1 symmetry.[11] Yet the η' mass (958) is large and cannot be attributed to nonzero but small masses of the quarks. This difficulty of QCD has been named the "U_1 problem."[11,12] It turned out to be so serious, that for many years it served as a stimulus for the development of a theory. The formulation of the U_1 problem, its discussion, and solution are the subject of the present paper.

Let us review briefly the history of the problem. In 1975, Kogut and Susskind,[13] starting from the analogy with two-dimensional electrodynamics (the Schwinger model), noted that the necessary element for the solution of the U_1 problem should be the existence of a pole at $q^2=0$ in the matrix elements of certain gauge-invariant operators. In 1976, 't Hooft[14] noted that, as a result of the Adler-Bell-Jackiw axial anomaly,[15] instantons[16] can lead to the desired solution of the problem. In 1977, however, Crewther[17] has shown that the instantons, when taken literally, lead to an incorrect dependence of various quantities of the theory on such "parameters" as the number N_c of the colors, on the number of the flavors, or on the quark masses. Next, in 1979, Witten[18] explained how the U_1 problem should be solved from the point of view of chromodynamics as $N_c\to\infty$. Finally, Veneziano[19] made Witten's idea more concrete by introducing into the theory a ghost state (of the type proposed by Kogut and Susskind[13]), demonstrated its self-consistency, and pointed to a practical possibility of calculating the η'-meson mass.

We regard the Veneziano approach as correct and constructive. However, the introduction of the pole $1/q^2$ in the theory is so serious a step, that we deem it important to do the following: 1) ascertain whether this pole is indeed a necessity from the phenomenological point of view; 2) determine its physical nature in QCD; 3) demonstrate that it leads to observable consequences and compare them with experiment.

In Sec. 2 we recall briefly the principal ideas on which the discussions of pseudoscalar mesons are based: the small quark masses, the approximate chiral symmetry of QCD, and its spontaneous violation. We also formulate the U_1 problem on a qualitative level. In Secs. 3 and 4 we do this more precisely and demonstrate that in order to ensure an η'-meson mass the theory should indeed have a ghost pole at $q^2=0$ in some of the correlators of the gluon currents, and that this pole should incidentally be separated from physical (observable) quantities.

Such a pole means that the theory contains a gapless excitation, and it is of interest to understand the physical cause of its appearance. Gapless excitations are always a reflection of some profound properties of the theory; for example, massless Goldstone particles appear as a result of spontaneous breaking of continuous symmetry; the zero photon mass is ensured by gauge invariance, etc. In our case, as we shall show in Sec. 5, gapless excitation is a consequence of the periodicity of the potential energy of chromodynamics with respect to a certain generalized coordinate. For the system to move along this coordinate, an arbitrary low energy is sufficient, and in this sense the corresponding excitation is gapless.

Without allowance for this phenomenon, the singlet pseudoscalar meson (the prototype of the η' meson) is a massless Goldstone particle (in the limit of massless quarks). Owing to the axial anomaly,[15] however, a mixing of this Goldstone particle with the aforementioned gapless excitation (ghost pole) takes place. It is their diagonalization which results in the massive η' meson, as will be demonstrated in Sec. 6. A simple formalism then makes it possible to calculate in fair agreement with experiment not only the masses of the η and η' mesons, but also the amplitudes of various processes in which they take part, (Sec. 7). Some conclusions are summarized in Sec. 8.

2. MASSLESS OR PSEUDOSCALAR MESONS

The characteristic mass scale of hadrons is approximately several hundred MeV. At the same time, there are substantial grounds for assuming (see below) that the masses of the three light quarks are small in this scale: $m_u \approx 4$ MeV, $m_d \approx 7$ MeV, and $m_s \approx 150$ MeV. It follows therefore that the hadron masses are determined not at all by the quark masses, but by some scale M_{str} that is peculiar to strong interactions, which can reasonably be assumed to be the mass of a typical hadron, $m_\rho \approx 770$ MeV. A good approximation is therefore the so-called chiral limit: $m_u = m_d = m_s = 0$. In this limit, the QCD Lagrangian has high symmetry, namely with respect to independent mutual transformations of the left-hand and right-hand components (helicities) of the quarks u, d, and s separately into one another. Since by the same token there exist invariants with respect to the transformations that mix states with different parity, hadrons with identical quantum numbers must be parity-degenerate: e.g., the vector meson 1^- should have the same mass as the axial meson 1^+, etc. The real splitting ($m_{A_1} - m_\rho \approx 400$ MeV) is too large to be able to associate it with the nonzero masses of the quarks. This means that the almost exact initial chiral symmetry of the theory is violated spontaneously in thermodynamics on account of interactions, e.g., on account of formation of a quark-antiquark condensate $\langle \bar{u}u + \bar{d}d + \bar{s}s\rangle$, and this condensate should appear also for really massless quarks. But if some symmetry is spontaneously broken, then according to the Goldstone theorem there should exist massless bosons. In fact their masses are not zero, but small: they are proportional to the parameters of the chiral-symmetry breaking in the Lagrangian, i.e., in this case to the masses of the quarks. The role of these so-called pseudo-Goldstone bosons, is played by the octet of pseudoscalar mesons (π, K, η); their masses are indeed much lower than the masses of the remaining hadrons.

Since the quark masses serve as a small parameter, it is possible to calculate the masses of the pseudoscalar mesons in first order in this parameter as the matrix element of the perturbation, in this case of the mass mass term in the QCD Hamiltonian, which violates explicitly the chiral invariance[20]:

$$\mathscr{H} = m_u \bar{u}u + m_d \bar{d}d + m_s \bar{s}s + \ldots .$$

Using the soft-pion theorem (and its generalization to mesons that include s-quarks), we obtain

$$\begin{aligned}
m_{\pi^\pm}^2 &= \langle \pi | m_u \bar{u}u + m_d \bar{d}d + m_s \bar{s}s + \ldots | \pi \rangle = -\frac{1}{f_\pi^2}(m_u + m_d)\langle \bar{u}u + \bar{d}d\rangle + O(m_{u,d}^2),\\
m_{\pi^0}^2 &= -\frac{1}{f_\pi^2}(2m_u\langle \bar{u}u\rangle + 2m_d\langle \bar{d}d\rangle) + O(m_{u,d}^2),\\
m_{K^\pm}^2 &= -\frac{1}{f_K^2}(m_u + m_s)\langle \bar{u}u + \bar{s}s\rangle + O(m_s^2),\\
m_{K^0}^2 &= -\frac{1}{f_K^2}(m_d + m_s)\langle \bar{d}d + \bar{s}s\rangle + O(m_s^2);\\
f_\pi &= 132 \text{ MeV}, \quad f_K = 155 \text{ MeV}.
\end{aligned} \tag{1}$$

We note that one of the methods of obtaining the values of the quark masses, cited at the beginning of this section, is to use these formulas.

If it is assumed that the remaining two isosinglet mesons also acquire mass only on account of the mass term in the Lagrangian, then the state with definite mass will be

$$\pi_1 \sim 2^{-1/2}(\bar{u}\gamma_5 u + \bar{d}\gamma_5 d) \qquad \pi_2 \sim \bar{s}\gamma_5 s. \tag{2}$$

Their masses are calculated in similar fashion and are equal to

$$\begin{aligned}
m_1^2 &\approx -\frac{1}{f_1^2}(2m_u\langle \bar{u}u\rangle + 2m_d\langle \bar{d}d\rangle) \approx m_\pi^2 = 0.02 \text{ GeV}^2,\\
m_2^2 &\approx -\frac{1}{f_2^2}4m_s\langle \bar{s}s\rangle \approx m_{K^+}^2 + m_{K^0}^2 - m_\pi^2 \approx 0.47 \text{ GeV}^2,
\end{aligned} \tag{3}$$

which differ greatly from $m_\eta^2 = 0.301$ GeV^2 and $m_{\eta'}^2 = 0.917$ GeV^2. In addition, the quark content of η and η' differs from (2). This is in fact the U_1 problem formulated by Weinberg.[20]

The error of the derivation consists in the fact that owing to the axial anomaly, the isosinglet currents are generally speaking not conserved, so that it cannot be assumed that the isosinglet η and η' mesons acquire mass only on account of the mass term. The anomaly, however, is in turn a total divergence, and the fact that it comes into play is quite nontrivial. We proceed now to a more precise examination of the question, based on the anomalous Ward identities,[17] which, on the one hand, make it possible to duplicate the results (1) for nonsinglet currents, and on the other to formulate clearly the condition under which the incorrect masses (3) do not arise.

3. ANOMALOUS WARD IDENTITIES

We begin with nonsinglet Ward identities. They are of the form[4,17]

$$\int d^4xi\langle TP^a(x)P^b(0)\rangle+\langle\bar{q}\{t^a\{t^bm\}\}q\rangle=0, \tag{4}$$

where

$$P^a=i\bar{q}\{t^am\}\gamma_5q=\partial_\mu I_{\mu5}{}^a=\partial_\mu(\bar{q}\gamma_\mu\gamma_5t^aq) \tag{5}$$

is the pseudoscalar density (t^a are the generators of the groups of SU_2 flavors, m is the mass matrix of the quarks, and $\{\,\}$ is the anticommutator). The meaning of Eq. (4) consists in the following: the propagator of the divergences of the axial currents at zero four-momentum is equal to the quark condensate. The identity (4) can be derived either by changing the variables in the functional integral [in this case it is necessary to consider the substitution $q(x)\to\exp[i\alpha^a(x)t^a\gamma_5]q(x)$] or with the aid of canonical equal-time commutation relations.[17]

We note that according to perturbation theory both terms in (4) diverge quadratically because of the free quark loop. However, the diverging part is quadratic in the quark mass. Yet, according to the principal assumption (see Sec. 2), the non-perturbative part of the condensate[1)] $\langle\bar{q}q\rangle$ is different from zero even at a zero quark mass. The propagator of the divergences (tentatively of the order of $\sim m_q^2$), should therefore contain a large contribution linear in the quark mass. Such is the contribution of the π-meson intermediate state. Defining the axial coupling constant in the usual manner:

$$\langle 0|I_{\mu5}{}^\pi|\pi\rangle=if_\pi p_\mu,\quad \langle 0|P^\pi|\pi\rangle=f_\pi m_\pi{}^2,$$

we have for the π-meson contribution to the propagator

$$\int d^4xe^{iqx}i\langle TP^\pi(x)P^\pi(0)\rangle=(f_\pi m_\pi{}^2)^2\frac{1}{m_\pi{}^2-q^2}+\ldots\xrightarrow[q\to0]{}f_\pi{}^2m_\pi{}^2+O(m_{u,d}^2),$$

from which follows relation (1).

We turn now to the singlet Ward identities. We define the currents

$$\begin{gathered}I_{1\mu5}=2^{-1/2}(\bar{u}\gamma_\mu\gamma_5u+\bar{d}\gamma_\mu\gamma_5d),\quad I_{2\mu5}=\bar{s}\gamma_\mu\gamma_5s,\\ \partial_\mu I_{1\mu5}=i2^{1/2}(m_u\bar{u}\gamma_5u+m_d\bar{d}\gamma_5d)+2^{1/2}Q\equiv P_1+2^{1/2}Q,\\ \partial_\mu I_{2\mu5}=2im_s\bar{s}\gamma_5s+2Q\equiv P_2+2Q;\\ Q=\alpha_sG\tilde{G}/8\pi.\end{gathered} \tag{6}$$

The terms with Q reflect here the presence of the axial anomaly.

The corresponding Ward identities can be derived anew by two methods: by replacing the variables in the functional integral[21] or with the aid of the canonical commutation relations.[17] We recall the second method and derive one of the Ward identities. We consider the total divergence of the correlator (it vanishes in the zero-momentum limit, since there are no massless hadrons):

$$0=\lim_{q\to0}\int d^4xe^{iqx}\partial_\mu i\langle TI_{1\mu5}(x)P_1(0)\rangle=\int d^4xi\langle TP_1(x)+2^{1/2}Q(x),P_1(0)\rangle$$
$$+\int d^4xi\delta(x_0)\langle[I_{1,05}(x),P_1(0)]\rangle. \tag{7}$$

The T-product of the operators is understood here in Dyson's sense:

$$TA(x)B(0)=\theta(x_0)AB+\theta(-x_0)BA.$$

Calculating the equal-time commutator in accordance with the canonical rules, we find that the last term is $2m_u\langle\bar{u}u\rangle+2m_d\langle\bar{d}d\rangle$. To simplify matters we shall hereafter express the correlator of two operators in momentum space in the following abbreviated form:

$$\begin{gathered}\int d^4xe^{iqx}i\langle TA(x)B(0)\rangle\equiv\langle AB\rangle_q,\\ \int d^4xe^{iqx}\partial_\mu i\langle TA(x)B(0)\rangle\equiv-iq_\mu\langle AB\rangle_q,\end{gathered} \tag{8}$$

etc. If the momentum q is zero, we shall write simply $\langle AB\rangle$.

In this notation, the Ward identity (4) for the π^0-meson channel is rewritten in the form

$$\langle P^{\pi^0}P^{\pi^0}\rangle+2m_u\langle\bar{u}u\rangle+2m_d\langle\bar{d}d\rangle=0, \tag{4'}$$

and the identity (7) in the form

$$\langle P_1P_1\rangle+2^{1/2}\langle QP_1\rangle+2m_u\langle\bar{u}u\rangle+2m_d\langle\bar{d}d\rangle=0. \tag{9}$$

We can similarly derive four more independent identities:

$$\langle P_2P_2\rangle+2\langle QP_2\rangle+4m_s\langle\bar{s}s\rangle=0, \tag{10}$$
$$\langle P_1P_2\rangle+2^{3/2}\langle QP_2\rangle=0, \tag{11}$$
$$\langle QP_1\rangle+2^{3/2}\langle QQ\rangle=0, \tag{12}$$
$$\langle QP_2\rangle+2\langle QQ\rangle=0. \tag{13}$$

We note immediately that the identity (4′), from which relation (1) follows, differs from identity (9) only by the correlator $\langle QP_1\rangle$ or, according to (12), by the correlator $\langle QQ\rangle$. If $\langle QQ\rangle=0$ were to be satisfied, we would obtain from identities (9) and (10) the "pseudo-Goldstone" masses (3) just as we obtained relation (1) from the non-singlet identities (4). Thus, to prevent the appearance of incorrect masses (3), the very fact that anomalous divergences exist is not sufficient, and it is necessary that at zero momentum the correlator $\langle QQ\rangle$ be different from zero. This is a rather nontrivial condition, inasmuch as the correlator vanishes in any order of perturbation theory.

In fact, as will be shown in the next section, the correlator $\langle QQ\rangle$ in the identities (12) and (13) should be taken to mean the limit $q_\mu q_\nu\langle K_\mu K_\nu\rangle_{q\to0}$, where K_μ is the gauge-invariant gluon current:

$$\begin{gathered}K_\mu=\frac{\alpha_s}{4\pi}\varepsilon_{\mu\alpha\beta\gamma}A_\alpha{}^a\left(\partial_\beta A_\gamma{}^a+\frac{g_s}{3}f^{abc}A_\beta{}^bA_\gamma{}^c\right),\\ \partial_\mu K_\mu=\frac{\alpha_s}{8\pi}G\tilde{G}\equiv Q.\end{gathered} \tag{14}$$

Thus, to ensure masses for the η and η' mesons it is necessary that the correlator $\langle K_\mu K_\nu\rangle$ have a pole as $q^2\to0$; e.g.,

$$\langle K_\mu K_\nu\rangle_{q\to0}=\frac{q_\mu q_\nu}{q^4}\text{const}\quad\text{or}\quad\langle K_\mu K_\nu\rangle_{q\to0}=\frac{g_{\mu\nu}}{q^2}\text{const}. \tag{15}$$

This condition seems even more surprising than the "equivalent" condition $\langle QQ\rangle\neq0$. How does this pole arise? What is its physical nature? To what conclusion concerning QCD, outside the framework of perturbation theory, does the "experimental," as we see, fact of its existence lead? This pole does by itself not appear in the real spectrum of the hadrons, since the current K_μ is not gauge-invariant, and is therefore not observable, but to which observable consequences does it lead?

The sections that follow will be devoted to a discussion of these questions.

4. CONNECTION BETWEEN THE DIFFERENT DEFINITIONS OF THE CORRELATORS AND THE VACUUM ENERGY

In view of the fundamental importance of the conclusion that a pole is present in the correlator $\langle K_\mu K_\nu\rangle$, we shall spend some time on a more detailed determination of the correlators, and at the same time ascertain their connection with the energy of the vacuum as a function of the known parameter[2)] θ.

We introduce into the QCD Lagrangian the θ-term:

$$\mathscr{L}_\theta=-{}^1/_4G_{\mu\nu}{}^aG_{\mu\nu}{}^a+\bar{q}(i\hat{\nabla}-m)q+\theta Q,\quad Q=\partial_\mu K_\mu. \tag{16}$$

The energy density ε of vacuum can be defined, following Feynman, with the aid of the continual integral

$$\exp(-i\varepsilon VT)=\int DA_\mu{}^aD\bar{q}Dq\exp\left(i\int\mathscr{L}_\theta d^4x\right)\equiv Z_\theta. \tag{17}$$

Taking logarithms of both sides the differentiating with respect to θ, we get

$$\frac{\partial^2\varepsilon}{\partial\theta^2}=\frac{i}{VT}\left[\frac{1}{Z}\frac{\partial^2 Z}{\partial\theta^2}-\frac{1}{Z^2}\left(\frac{\partial Z}{\partial\theta}\right)^2\right]=-\int d^4xi[\langle TQ(x)Q(0)\rangle-\langle Q(0)\rangle^2]. \tag{18}$$

The right-hand side of (18) is the connective part of the two-particle correlator of the densities of the topological charges at zero momentum, and we shall hereafter designate it$-\langle QQ\rangle^W$. The superscript W denotes time ordering after Wick. The weak T-ordering (which, generally speaking differs from the Dyson T-ordering, see below) is defined with the aid of a functional integral. It is understood, in particular, that all the differentiation operators are applied after the calculation of the convolutions of the fields (the latter is performed in accordance with Feynman's rules). The Wick T-product can be understood as an analytic continuation of the Green's functions defined in Euclidean space. We have thus obtained

$$\partial^2\varepsilon/\partial\theta^2=-\langle QQ\rangle^W. \tag{19}$$

We shall show below that the anomalous Ward identities (12) and (13) contain precisely $\langle QQ\rangle^W$, and note for the present only that the difference of $\langle QQ\rangle^W$ from zero, which is needed for the solution of the U_1 problem, means automatically a nontrivial dependence of the energy density of the vacuum, and also probably of other quantities in the theory, on the parameter θ. In view of the importance of relation (19) we shall derive it by another method, which will also enable us to establish the connection between the Dyson and Wick T-products.

We calculate $\varepsilon(\theta)$, following Witten,[18] by the Hamiltonian method in the gauge $A_0=0$. In this gauge

$$\mathscr{L}_\theta=\frac{1}{2}\dot{\mathbf{A}}^2-\frac{1}{2}\mathbf{H}^2+\bar{q}(i\hat{\nabla}-m)q+\theta Q,\qquad Q=\frac{\alpha_s}{2\pi}\dot{\mathbf{A}}\mathbf{H}, \tag{20}$$

where $\mathbf{H}^a$ is the chromomagnetic intensity. We define the canonical momentum

$$\boldsymbol{\pi}^a=\frac{\partial\mathscr{L}_\theta}{\partial\dot{\mathbf{A}}^a}=\dot{\mathbf{A}}^a+\frac{\alpha_s}{2\pi}\theta\mathbf{H}^a \tag{21}$$

and construct the Hamiltonian

$$\begin{aligned}\mathscr{H}_\theta&=\boldsymbol{\pi}\dot{\mathbf{A}}-\mathscr{L}_\theta=\frac{1}{2}\boldsymbol{\pi}^2+\frac{1}{2}\mathbf{H}^2+q^+[i\alpha\nabla+\beta m]q\\&\quad-\theta\frac{\alpha_s}{2\pi}\boldsymbol{\pi}\mathbf{H}+\theta^2\left(\frac{\alpha_s}{2\pi}\mathbf{H}\right)^2,\\ Q&=\frac{\alpha_s}{2\pi}\left(\boldsymbol{\pi}\mathbf{H}-\frac{\alpha_s}{2\pi}\theta\mathbf{H}^2\right).\end{aligned} \tag{22}$$

We calculate the vacuum energy density, which we expand in a perturbation-theory series in θ up to second order. We have

$$\begin{aligned}\varepsilon(\theta)&=\varepsilon(0)+\frac{\theta^2}{2}\left\langle 0\left|\left(\frac{\alpha_s}{2\pi}\mathbf{H}\right)^2\right|0\right\rangle\\&\quad+\theta^2\sum_n\left|\left\langle 0\left|\frac{\alpha_s}{2\pi}\boldsymbol{\pi}\mathbf{H}\right|n\right\rangle\right|^2\frac{V}{E_0-E_n}+O(\theta^3)\\&=\varepsilon(0)+\frac{\theta^2}{2}\left\langle 0\left|\left(\frac{\alpha_s}{2\pi}\mathbf{H}\right)^2\right|0\right\rangle-\frac{\theta^2}{2}\langle QQ\rangle^D+O(\theta^3).\end{aligned} \tag{23}$$

We obtained here the Dyson T-product, since it is precisely this (and not the Wick!) product which can be represented as a sum over the intermediate states. Thus,

$$\left.\frac{\partial^2\varepsilon}{\partial\theta^2}\right|_{\theta=0}=\left\langle\left(\frac{\alpha_s\mathbf{H}}{2\pi}\right)^2\right\rangle-\langle QQ\rangle^D=-\langle QQ\rangle^W. \tag{24}$$

This equation is the sought connection between the Wick and Dyson T-products.

We shall show now that the Ward identities (12) and (13) contain in fact $\langle QQ\rangle^W$. To this end, we consider the total divergence of the density correlator of the topological charge Q with a current made up, e.g., of same-quarks:

$$\begin{aligned}&\lim_{q\to 0}\int d^4xe^{iqx}\partial_\mu i\langle TI_{2\mu5}(x)Q(0)\rangle^D\\&=\langle P_2+2Q,Q\rangle^D+\int d^4xi\langle[I_{2,05}(x),Q(0)]\rangle\delta(x_0).\end{aligned} \tag{25}$$

Before we calculate the equal-time commutator, we note that $I_{2,05}(x)=s^+(x)\gamma_5 s(x)$ is a poorly defined object. For an accurate definition, we carry out a gauge-invariant separation after Schwinger:

$$\begin{aligned}I_{2,05}(x)&=\lim_{\varepsilon\to 0}s^+\left(x+\frac{\varepsilon}{2}\right)\gamma_5 P\exp\left(ig_s\int_{x-\varepsilon/2}^{x+\varepsilon/2}A_\mu dx_\mu\right)s\left(x-\frac{\varepsilon}{2}\right)\\&\approx\lim_{\varepsilon\to 0}s^+\left(x+\frac{\varepsilon}{2}\right)\gamma_5 s\left(x-\frac{\varepsilon}{2}\right)+2K_0.\end{aligned}$$

The two terms here are not gauge-invariant, but their sum is. The commutator of the first term with Q yields obviously zero; the commutator of the second, however, can be calculated in the gauge $A_0=0$ by using the canonical commutation relations. We have

$$\int d^3xi\langle[2K_0(x),Q(0)]\rangle=-2\langle(\alpha_s\mathbf{H}/2\pi)^2\rangle. \tag{26}$$

Substituting this expression in (25) we obtain in place of (13)

$$\langle P_2Q\rangle^D+2\langle QQ\rangle^D-2\langle(\alpha_s\mathbf{H}/2\pi)^2\rangle=0. \tag{27}$$

Comparing with Eq. (24), we see that Eq. (13) remains in force if the correlator $\langle QQ\rangle$ in it is understood in the sense of Wick [the same holds also for Eq. (12)].

The connection between the correlators $\langle K_\mu K_\nu\rangle$ and $\langle QQ\rangle$ (accurate to possible Schwinger terms which constitute a polynomial in $\mathbf{q}^2$) is of the form

$$q_\mu q_\nu\langle K_\mu K_\nu\rangle_q{}^D=-iq_\mu\langle K_\mu Q\rangle_q{}^D=\langle QQ\rangle_q{}^D-\langle(\alpha_s\mathbf{H}/2\pi)^2\rangle=\langle QQ\rangle_q{}^W. \tag{28}$$

This relation is, on the one hand, almost obvious from the definition of the Wick T product, and on the other it can be obtained by a canonical method similar to the derivation of Eqs. (25)–(27).

We have thus confirmed the statement made at the end of the preceding section, that the masses of the isosinglet mesons can differ from their pseudo-Goldstone values (3) only if the propagator $\langle K_\mu K_\nu \rangle$ has a pole. At the same time, relation (19) shows that the residue of this pole depends seriously on the quark masses. In fact, in a theory with at least one massless quark all the quantities, as is well known, do not depend on the parameter θ, so that $\partial^2\varepsilon/\partial\theta^2 \to 0$ as $m_q \to 0$. Therefore the residue of the poles should vanish as $m_q \to 0$.

We note, finally, that strictly speaking it would be necessary to take into account in (28) the possible Schwinger terms which are not controllable by canonical commutation relations. [We note that the Schwinger terms appear if terms proportional to the derivatives of $\delta^{(3)}(x)$ appear in the equal-time commutators; in the momentum representation they are therefore polynomials in $\mathbf{q}^2$.] To cause them to vanish it suffices to choose a reference frame $q_\mu = (\omega, 0)$. Then relation (28) takes the form

$$\omega^2 \langle K_0 K_0 \rangle = \langle QQ \rangle^W \underset{\omega\to 0}{\longrightarrow} -\frac{\partial^2 \varepsilon}{\partial \theta^2} \neq 0 \tag{29}$$

or

$$\lim_{\omega\to 0} \omega^2 \int dt\, e^{i\omega t} i\langle TX(t)X(0)\rangle = -\partial^2 E/\partial\theta^2, \tag{30}$$

where

$$X(t) = \int d^3\mathbf{x}\, K_0(t, x),$$

$E = \varepsilon V$ is the energy of the vacuum, and V is the volume of the universe.

Thus, strictly speaking, to solve the U_1 problem the propagator

$$\int dt\, e^{i\omega t} i\langle TX(t)X(0)\rangle,$$

must have a pole $1/\omega^2$ whose residue vanishes as $m_q \to 0$. In the next section we discuss the meaning of this requirement and explain at the same time the physical meaning of the pole.

5. PHYSICAL INTERPRETATION OF THE GHOST POLE

We digress for a while from the real world and consider pure gluodynamics (there are no quarks or they are infinitely heavy). In the gauge $A_0^a = 0$, the gluodynamics Hamiltonian is of the form [cf. (22)]

$$\begin{aligned} \mathcal{H} &= \frac{1}{2}\int d^3\mathbf{x} (\boldsymbol{\pi}^2(x) + \mathbf{H}^2(x)), \\ H_k^a &= \varepsilon_{ijk}\left(\partial_i A_j^a + \frac{g_s}{2} f^{abc} A_i^b A_j^c\right), \end{aligned} \tag{31}$$

and the corresponding stationary Schrödinger equation $\mathcal{H}\Psi = \mathscr{E}\Psi$ is obtained by making the substitution $\pi^{ia}(x) \to i\delta/\delta A_i^a(x)$:

$$\int d^3\mathbf{x} \left[\frac{1}{2}\left(-\frac{\delta}{\delta A_i^a(x)}\right)^2 + \frac{\mathbf{H}^2(x)}{2}\right]\Psi[A] = \mathscr{E}\Psi[A]. \tag{32}$$

Furthermore, an additional condition, the analog of $\operatorname{div}\mathbf{E} = 0$, must be imposed on the physical wave function, (see, e.g., Ref. 22):

$$(\delta^{ab}\partial_j + g_s f^{acb} A_j^c(x))\,\delta\Psi/\delta A_j^b(x) = 0. \tag{33}$$

The meaning of the condition is the following: the wave function must not be changed by infinitely small gauge transformations. This condition allows us to assume that the wave function Ψ, as well as the potential energy

$$V = \frac{1}{2}\int \mathbf{H}^2\, d^3x$$

depends on the generalized coordinates $X, Y, \ldots$, which are functionals of $A_i^a(x)$ that are invariant to small gauge transformations. In particular, special interest attaches to the generalized coordinate

$$X = \int d^3\mathbf{x}\, K_0(x), \quad K_0 = \frac{\alpha_s}{4\pi}\varepsilon_{ijk} A_i^a \left(\partial_j A_k^a + \frac{g_s}{3} f^{abc} A_j^b A_k^c\right). \tag{34}$$

Under the gauge transformation

$$\hat{A}_i \to S\hat{A}_i S^+ - (i/g)(\partial_i S) S^+$$

the quantity X transforms as follows:

$$X \to X - \frac{1}{24\pi^2}\int d^3\mathbf{x}\, \varepsilon_{ijk}\, \mathrm{Sp}[(\partial_i S) S^+ (\partial_j S) S^+ (\partial_k S) S^+] = X + n. \tag{35}$$

In the last equation we used the remarkable fact (see, e.g., Ref. 23) that the integral in (35) is either equal to zero (for "nontopological" S transformations) or to an integer n equal to the "topological charge" of the transformation. At the same time, the potential energy V is not changed by any gauge transformation. This means that the potential energy V is a periodic function of the generalized coordinate X, with unity period.[24] We note that this is true also for chromodynamics with quarks. The remaining generalized coordinates $Y, \ldots$, on which V depends are of no interest to us now—there is no periodicity in these coordinates.

Thus, the situation turns out to be similar to the problem of an electron in the periodic field of a crystal. We know that a band spectrum is produced, the state energy E is periodically dependent on the quasimomentum k with a period 2π, and at small quasimomenta (e.g., near the bottom of the first band) the electron behaves like a free particle ($E = k^2/2m^*$) with effective mass m^* determined by the penetrability of the barriers. In particular, its Green's function corresponds to free propagation:

$$\int dt\, e^{i\omega t} i\langle Tx(t)x(0)\rangle \underset{\omega\to 0}{\longrightarrow} -\frac{1}{\omega^2 m^*} = -\frac{1}{\omega^2}\frac{\partial^2 E}{\partial k^2}\bigg|_{k=0}. \tag{36}$$

The meaning of this formula is the following: after a long time the electron can move arbitrarily far away (on account of tunneling).

Formula (36) has a striking similarity with the chromodynamic formula (30), especially if attention is called to the fact that the quantity X in (30) coincides in the gauge $A_0 = 0$ with the generalized coordinate (34) with respect to which periodicity is present! This means that the wave function of the vacuum in QCD is not concentrated in the vicinity of one of the wells with respect to the generalized coordinate X, but is smeared out over all of X-space. The pole (30) responsible for the solution of the

U_1 problem means that the system moves freely along X, i.e., that the potential barriers are penetrable.

This should not surprise us, since we know that there exist classical below-barrier trajectories with finite action (instantons).[16] The finite character of the action means in fact penetrability of the barrier. We emphasize, however, that our conclusion that the system moves freely along the generalized coordinate X does not invoke the quasiclassical approach; we mention instantons only for the sake of illustration.

It is of interest to note that if we now include the quarks and let one of the quark masses go to zero, then according to (30) the residue at the pole $1/\omega^2$ vanishes (the effective mass tends to infinity). This means that the barriers become effectively impenetrable. The last phenomenon can be traced also in the case of instantons as a concrete example of a below-the-barrier transition—the amplitude of the transition vanishes because of the zero modes of the massless quarks in the instanton field. It appears, however, that instantons lead literally to an incorrect dependence of the transition amplitude on such free parameters of the theory as the number N_c of the colors, of the number of light quarks L, and of the mass of the quarks.[17]

To conclude this section, we show that the parameter θ [see (16)] plays the role of the quasimomentum. We recall for this purpose that in the presence of periodicity the solution of the Schrödinger equation (32) is sought in the form of a Bloch wave function (k is the quasimomentum):

$$\Psi[A]=\exp(ikX[A])U_k[A]. \qquad (37)$$

Substituting (37) in Eq. (32) and using the remarkable fact that

$$\delta X/\delta A_i^a(x)=(\alpha_s/2\pi)H_i^a(x),$$

we obtain an equation for the amplitude of the Bloch function:

$$\int d^3\mathbf{x}\left[\frac{1}{2}\left(-i\frac{\delta}{\delta A_i^a(x)}+k\frac{\alpha_s}{2\pi}H_i^a(x)\right)^2+\frac{\mathbf{H}^2}{2}\right]U_k=\mathscr{E}U_k, \qquad (38)$$

at the same time, we introduce the θ term in the Lagrangian [see (20)], construct a Hamiltonian [see (22), but without quarks], and write down the corresponding Schrödinger equation:

$$\int d^3\mathbf{x}\left[\frac{1}{2}\left(-i\frac{\delta}{\delta A_i^a(x)}+\theta\frac{\alpha_s}{2\pi}H_i^a(x)\right)^2+\frac{\mathbf{H}^2}{2}\right]\Psi_\theta=\mathscr{E}\Psi_\theta. \qquad (39)$$

We see that the resultant equation is identical with (38) if we put $k=\theta$ for the quasimomentum. It follows from (35) that the potentials $V(X,\ldots)$ is periodic in X with unity period. Thus, all the quantities of the theory should be periodic in θ with period 2π.

We note that the Hamiltonian (22) with the θ term has exactly the same spectrum as the Hamiltonian (31) without the θ term, since the corresponding Schrödinger equations transform into one another when the wave function is multiplied by the inessential phase factor (37). In particular, with or without the θ term in the Lagrangian, the ground state of the system (vacuum) corresponds to the bottom of the band.

In this connection we wish to make more precise the meaning of the quantity $\partial^2\varepsilon/\partial\theta^2$ calculated in the preceding section. The quantity ε was called there, not quite precisely, the energy density of the vacuum, it being tacitly understood that the energy of the ground state of the system can depend on the θ term in the Lagrangian. We see now that this is not the case: the energy of the ground state is independent of θ. Only the minimum energy of the state with given quasimomentum θ, depends on this θ. It is precisely this relation which determines

$$\partial^2\varepsilon/\partial\theta^2=-\langle QQ\rangle^W$$

[see (29)]. In the real world, at any rate, the value of the quasimomentum is $|\theta|\lesssim 10^{-9}$.[25]

We note finally, by way of a curiosity, that Eq. (32) has formally an exact nontrivial solution with zero energy:

$$\Psi[A]=\exp(\pm(2\pi/\alpha_s)X[A]).$$

Unfortunately, this solution increases along certain directions in the space of the potentials $A_i^a(x)$, and therefore cannot serve as a wave function of the vacuum.

6. DIAGONALIZATION OF PSEUDO-GOLDSTONE STATES

Thus, the potential energy in chromodynamics is periodic in the generalized coordinate

$$X=\int d^3\mathbf{x}K_0(x),$$

and the potential barriers are penetrable, so that the system behaves like a free particle with respect to this coordinate and after a long time it moves arbitrarily far away along this coordinate. This indeed is the physical interpretation of the pole (30) that is the necessary condition for the solution of the U_1 problem. We ask now, is it sufficient to obtain the correct masses of the η and η' mesons? In this section we describe a Veneziano construction[19] (to be sure, using somewhat different terms than in Ref. 19, see Refs. 26 and 27), which allows us not only to obtain the correct masses of the η and η' mesons, but also to calculate the amplitudes of the different physical processes in which they take part.

We start from the imagined pure gluon world (all the quarks, if they exist at all, are infinitely heavy). We note first that although we know, strictly speaking, only that the correlator $\langle K_0K_0\rangle$ has a pole $1/q_0^2$ (at $\mathbf{q}=0$), it is natural to assume by virtue of the Lorentz-invariance that in an arbitrary reference frame the correlator $\langle K_\mu K_\nu\rangle$ has a ghost pole as $q^2\to 0$, so that [see (28)]

$$\lim_{q\to 0} q_\mu q_\nu\langle K_\mu K_\nu\rangle^D=\langle QQ\rangle^W=-\partial^2\varepsilon/\partial\theta^2=-\lambda^4\neq 0. \qquad (40)$$

We emphasize that all the physical result could be derived also in the system $\mathbf{q}=0$; we have preferred here the Lorentz-invariant technique only from esthetic considerations.

In what follows it will be convenient to introduce formally the ghost in the form of a lower intermediate state in the propagator $\langle K_\mu K_\nu\rangle$. We write

$$\langle K_\mu K_\nu\rangle_q=\sum_p\langle 0|K_\mu|a^p\rangle\frac{1}{-q^2}\langle a^p|K_\nu|0\rangle+\text{possible gluon contributions}, \qquad (41)$$

where Σ_p denotes summation over the polarization of the ghost. We put (ε_μ^p is the polarization vector of the ghost)

$$\langle 0|K_\mu|a^p\rangle=\lambda^2\varepsilon_\mu{}^p,\quad \langle a^p|K_\nu|0\rangle=\lambda^2\varepsilon_\nu{}^{p*}. \tag{42}$$

Then (41) is rewritten as follows

$$\langle K_\mu K_\nu\rangle=\lambda^4\sum_p\frac{\varepsilon_\mu{}^p\varepsilon_\nu{}^{p*}}{-q^2}=\lambda^4\langle a_\mu a_\nu\rangle_0, \tag{43}$$

where we have introduced the propagator $\langle a_\mu a_\nu\rangle_0$ of the ghost. Substituting (43) in (40), we obtain a condition on the propagator:

$$q_\mu q_\nu\langle a_\mu a_\nu\rangle_0=-1\quad\text{or}\quad q_\mu q_\nu\sum_p\varepsilon_\mu{}^p\varepsilon_\nu{}^{p*}=q^2. \tag{44}$$

The actual form of the propagator of the ghost depends, naturally, on the gauge of the gluon field [the invariant condition is (44)]. For practical calculations it is convenient to use a gauge in which the polarization vector of the ghost is of the form[3)]

$$\varepsilon_\mu{}^p=q_\mu/(q^2)^{1/2},\quad \langle a_\mu a_\nu\rangle_0=-q_\mu q_\nu/q^4. \tag{45}$$

We return now to the real world with three light quarks: $m_u\sim m_d\ll m_s$. Without allowance for the annihilation diagrams (with gluon intermediate states), there are two isosinglet pseudo-Goldstone states $\pi_{1,2}$ with masses $m_{1,2}$, given by Eq. (3). It is easily conceivable that, because of the axial anomaly, the quark-antiquark states $\pi_{1,2}$ must of necessity have a nonzero amplitude of transition into a ghost, with an order of magnitude $\mu\sim\lambda^2/f$ (for a more accurate value see below). We introduce the following transition amplitudes:

$$\langle a^p|\pi_{1,2}\rangle=-iq_\mu\varepsilon_\mu{}^p\mu_{1,2},\quad \langle\pi_{1,2}|a^p\rangle=iq_\nu\varepsilon_\nu{}^{p*}\mu_{1,2} \tag{46}$$

(we note that in the SU_3 symmetry limit, when $m_{u,d,s}\ll M_{\rm str}$, we have $\mu_1=\mu_2\sqrt{2}$. We shall not assume that this is satisfied). It is convenient, after separating the polarization vector ε_μ^p and assigning it to the adjacent ghost propagator, to define the transition amplitude in slightly different form:

$$\langle a_\nu|\pi_{1,2}\rangle=-iq_\nu\mu_{1,2},\quad \langle\pi_{1,2}|a_\nu\rangle=iq_\nu\mu_{1,2}\ . \tag{47}$$

We add also the bare propagators of the states $\pi_{1,2}$:

$$\langle\pi_i\pi_j\rangle_0=\delta_{ij}/(m_i^2-q^2). \tag{48}$$

The diagonalization of the three states a_μ and $\pi_{1,2}$, which corresponds to solution of the coupled Dyson equations with bare propagators (48) and (45) and with transition amplitudes (47), leads to the following exact propagators:

$$\begin{gathered}
\langle a_\mu a_\nu\rangle=-\frac{q_\mu q_\nu}{q^4}\frac{(m_1^2-q^2)(m_2^2-q^2)}{z(q^2)},\quad \langle\pi_1\pi_2\rangle=-\frac{\mu_1\mu_2}{z(q^2)},\\
\langle\pi_1\pi_1\rangle=\frac{m_2^2+\mu_2^2-q^2}{z(q^2)},\quad \langle\pi_2\pi_2\rangle=\frac{m_1^2+\mu_1^2-q^2}{z(q^2)},\\
\langle a_\mu\pi_1\rangle=\frac{iq_\mu}{q^2}\mu_1\frac{m_2^2-q^2}{z(q^2)},\quad \langle a_\mu\pi_2\rangle=\frac{iq_\nu}{q^2}\mu_2\frac{m_1^2-q^2}{z(q^2)},\\
z(q^2)=q^4-q^2(m_1^2+m_2^2+\mu_1^2+\mu_2^2)\\
+m_1^2m_2^2+\mu_1^2m_2^2+\mu_2^2m_1^2=(m_+^2-q^2)(m_-^2-q^2).
\end{gathered} \tag{49}$$

The roots of the denominator $z(q^2)$ determine the masses of the physical η and η' mesons:

$$m_\pm^2={}^1/_2\{m_1^2+m_2^2+\mu_1^2+\mu_2^2\pm[(m_1^2+\mu_1^2-m_2^2-\mu_2^2)^2+4\mu_1^2\mu_2^2]^{1/2}\}\ . \tag{50}$$

To find the connection between the propagators (49) and the observable ones we define

$$\begin{gathered}
\langle 0|Q|a_\mu\rangle=-iq_\alpha\langle 0|K_\alpha|a_\mu\rangle=-iq_\mu\lambda^2,\\
\langle a_\nu|Q|0\rangle=iq_\nu\lambda^2;
\end{gathered} \tag{51}$$

$$\begin{gathered}
\langle 0|P_i|\pi_j\rangle=\delta_{ij}f_im_i^2,\quad P_1=i\sqrt{2}(m_u\bar u\gamma_5u+m_d\bar d\gamma_5d),\\
P_2=2im_s\bar s\gamma_5s
\end{gathered} \tag{52}$$

(in the limit of SU_3 symmetry we have $f_1=f_2=f_\pi$; we, however, do not assume this). Using (51), (52), and the propagators (49), it is easy to find the correlators of the gauge-invariant quantities:

$$\begin{gathered}
\langle QQ\rangle_q{}^W=-\lambda^4(m_1^2-q^2)(m_2^2-q^2)/z(q^2),\\
\langle P_1P_2\rangle_q=-f_1f_2m_1^2m_2^2\mu_1\mu_2/z(q^2),\\
\langle P_1P_1\rangle_q=f_1^2m_1^4(m_2^2+\mu_2^2-q^2)/z(q^2),\\
\langle P_2P_2\rangle_q=f_2^2m_2^4(m_1^2+\mu_1^2-q^2)/z(q^2),\\
\langle P_1Q\rangle_q=\lambda^2f_1m_1^2\mu_1(m_2^2-q^2)/z(q^2),\\
\langle P_2Q\rangle_q=\lambda^2f_2m_2^2\mu_2(m_1^2-q^2)/z(q^2).
\end{gathered} \tag{53}$$

The coupling constants $f_{1,2}$, $\mu_{1,2}$ and λ^2 introduced by us are not independent, but are connected with one another as a result of the Ward identities (9)–(13). We obtain

$$2^{-1/2}f_1\mu_1=f_2\mu_2=2\lambda^2; \tag{54}$$

$$\begin{gathered}
f_1^2m_1^2=-2(m_u+m_d)\langle\bar uu+\bar dd\rangle=f_\pi^2m_\pi^2,\\
f_2^2m_2^2=-4m_s\langle\bar ss\rangle.
\end{gathered} \tag{55}$$

These relations were in fact obtained by Veneziano[19] under the assumption of exact SU_3 symmetry ($f_1=f_2$, $\mu_1=\mu_2\sqrt{2}$, ...). The SU_3 symmetry is actually broken quite strongly: we recall, e.g., that $f_\pi=0.132$ GeV, $f_k=0.155$ GeV, $f_K/f_\pi=1.18$, and this turns out to be of importance for the numerical calculations (see Sec. 7).

We note that according to (53)

$$\partial^2\varepsilon/\partial\theta^2=-\langle QQ\rangle_{q=0}^{W}=\lambda^4m_1^2m_2^2/m_+^2m_-^2. \tag{56}$$

In the limit of infinitely heavy quark masses (the case of pure gluodynamics) we have $m_1^2m_2^2=m_+^2m_-^2$ [see (50)] and $\partial^2\varepsilon/\partial\theta^2=\lambda^4$. At small quark masses $\partial^2\varepsilon/\partial\theta^2$ vanishes linearly with any of the masses, as expected.

7. MASSES AND COUPLING CONSTANTS OF η AND η' MESONS

We obtain first the coupling constants f_2, $\mu_{1,2}$ and λ^2 ($f_1=f_\pi$). To determine f_2 we can use the premises of chiral perturbation theory.[28] In the linear approximation in m_s/M_{str} we can write

$$f_2=f_\pi+2(f_K-f_\pi)=0.178\text{ GeV},\quad f_2/f_1=f_2/f_\pi=1.35, \tag{57}$$

which fixes by virtue of (54) the ratio $\mu_1/\mu_2=1.91$. To obtain a second relation between $\mu_{1,2}$ we can specify, e.g., the sum $m_\eta^2+m_{\eta'}^2=1.218$ GeV^2 [then the difference $m_\eta^2-m_{\eta'}^2$ will be obtained from Eq. (50)]. We obtain

$$\begin{gathered}
\mu_1^2=0.57\text{ GeV}^2,\quad \mu_2^2=0.16\text{ GeV}^2,\\
m_\eta^2=0.307\text{ GeV}^2\ (0.301).\quad m_{\eta'}^2=0.912\text{ GeV}^2\ (0.917)
\end{gathered} \tag{58}$$

(the quantities in the parentheses are the experimental masses). We can also find the singlet-octet mixing angle, defined in accordance with the equations

$$\eta'=\cos\theta|1\rangle+\sin\theta|8\rangle,\quad \eta=-\sin\theta|1\rangle+\cos\theta|8\rangle.$$

It turns out to equal $-9°$ ($-10°$ in experiment), the sign following from the theory. From (56) we obtain the value of $\partial^2\varepsilon/\partial\theta^2$ in a world without light quarks, i.e.,

$$\partial^2\varepsilon/\partial\theta^2|_{\rm wlq}=\lambda^4=f_\pi^2\mu_1^2/8=(0.188\text{ GeV})^4. \tag{59}$$

We note that this quantity can be connected with $\langle G^2\rangle$ in the real world by using certain plausible reasoning.[7,26] The obtained number is then in good agreement with the value of $\langle G^2\rangle$ obtained from the dispersion sum rules.[6]

From expressions (53) it is easy to find all the residues of the η and η' mesons (all the quantities are in units of GeV^3):

$$\begin{aligned}
\langle 0|Q|\eta\rangle &= \left[\frac{\lambda^4(m_2^2-m_\eta^2)(m_\eta^2-m_1^2)}{m_{\eta'}^2-m_\eta^2}\right]^{1/2}=0.010\ (0.008),\\
\langle 0|Q|\eta'\rangle &= \left[\frac{\lambda^4(m_{\eta'}^2-m_2^2)(m_{\eta'}^2-m_1^2)}{m_{\eta'}^2-m_\eta^2}\right]^{1/2}=0.028\ (0.026),\\
\langle 0|P_1|\eta\rangle &= \left[\frac{f_1^2m_1^4(m_2^2+\mu_2^2-m_\eta^2)}{m_{\eta'}^2-m_\eta^2}\right]^{1/2}=1.9\cdot10^{-3}\ (1.5\cdot10^{-3}),\\
\langle 0|P_1|\eta'\rangle &= \left[\frac{f_1^2m_1^4(m_{\eta'}^2-m_2^2-\mu_2^2)}{m_{\eta'}^2-m_\eta^2}\right]^{1/2}=1.8\cdot10^{-3}\ (2.6\cdot10^{-3}),\\
\langle 0|P_2|\eta\rangle &= -\left[\frac{f_2^2m_2^4(m_1^2+\mu_1^2-m_\eta^2)}{m_{\eta'}^2-m_\eta^2}\right]^{1/2}=-0.056\ (-0.049),\\
\langle 0|P_2|\eta'\rangle &= \left[\frac{f_2^2m_2^4(m_{\eta'}^2-m_1^2-\mu_1^2)}{m_{\eta'}^2-m_\eta^2}\right]^{1/2}=0.062\ (0.049).
\end{aligned}\tag{60}$$

The quantities in the parentheses are the residues in the limit of exact (SU_3) symmetry, which is realized when the quark masses tend to zero ($m_{1,2}\ll\mu_1=\mu_2\sqrt{2}$). In this limit, η is a member of an octet with mass $(m_1^2+m_2^2)/2$, and η' is a pure SU_3 singlet with mass $(\mu_1^2+\mu_2^2)/2$ [see (50)]. We see that the real world does not differ excessively from such an idealization.

By way of one of the applications of the obtained residues we consider the ratio of the widths of the radiative decays $\psi\to\eta'\gamma$ and $\psi\to\eta\gamma$. According to the standard logic, these decays proceed via emission of a photon and two gluons in the state 0^-, which later go over into η or η'. Then

$$\frac{\Gamma(\psi\to\eta'\gamma)}{\Gamma(\psi\to\eta\gamma)}=\left|\frac{\langle 0|Q|\eta'\rangle}{\langle 0|Q|\eta\rangle}\right|^2\frac{\mathbf{p}_{\eta'}^3}{\mathbf{p}_\eta^3}=6.4.$$

The experimentally obtained[29] ratio is[4)] 5.9 ± 1.5.

Using the residues (60), we can also calculate the widths of the two-photon decays η and η' (for more details see Ref. 27):

$$\Gamma(\eta\to2\gamma)=515(324\pm46)\ \text{eV},\quad \Gamma(\eta'\to2\gamma)=5.1(5.8\pm1.2)\ \text{keV}.$$

8. DISCUSSION

We see thus that introduction of the ghost pole in the gauge-invariant correlator $\langle K_\mu K_\nu\rangle$ and the subsequent diagonalization of the ghost and of the pseudo-Goldstone states leads to a perfectly successful description of the masses and matrix elements of the η and η' mesons (especially if the SU_3 symmetry assumption is disregarded: $f_\pi\neq f_K$ etc.). Further development of high-energy physics of η and η' mesons consists in the construction of effective chiral Lagrangians, that describe the entire nonet of pseudoscalar mesons and satisfy the anomalous Ward identities.[30]

The presence of a pole as $q^2\to0$ in the correlator $\langle K_\mu K_\nu\rangle$ is a rather nontrivial property of exact QCD. We have shown in Sec. 5 that this pole is the consequence of the periodicity of the potential energy in QCD. It is the mixing of gapless excitation with a singlet Goldstone state ($\sim\bar u\gamma_5u+\bar d\gamma_5d+\bar s\gamma_5s$) which leads to the appearance of a massive particle (η' meson) with a mass proportional to the mixing amplitude.

It is interesting to note that in two-dimensional models (massive spinor electrodynamics[31,26] and the CP^{N-1} model[32]) the presence of an analogous pole solves not only the U_1 problem, but also the confinement problem. It is possible that in QCD the color confinement is also connected with the existence of a pole in the correlator $\langle K_\mu K_\nu\rangle$.

We find it useful to draw an analogy between the considered mechanism and the Higgs mechanism. In spontaneous violation of gauge invariance, the Goldstone boson, owing to scattering by the condensate, has a nonzero amplitude of transition into the longitudinal component of the guage field. The propagator of the latter component has a ghost pole, the origin of which is obvious: the longitudinal component is a cyclic degree of freedom in gauge theory. As a result of mixing, the Goldstone mode vanishes and is transformed into a massive third component of a vector boson, whose mass is equal to the mixing amplitude. In contrast to the Higgs model in chromodynamics, the degree of freedom connected with the ghost pole is strictly speaking not cyclic. However, the dependence of the potential energy on this degree of freedom is periodic, which in our case does not play a principal role, since motion along the corresponding coordinates is at any rate practically free, a fact manifest in the existence of a massless pole in the correlator $\langle K_\mu K_\nu\rangle$.

At the same time, $\langle K_\mu K_\nu\rangle$ is a gauge-noninvariant quantity, and it would be useful to reformulate all the concrete results of Secs. 6 and 7 in terms of gauge-noninvariant quantities, etc.

Let us consider, e.g., the correlator $\langle QQ\rangle^W=q_\mu q_\nu\times\langle K_\mu K_\nu\rangle$. The dispersion representation for this quantity calls obviously for two subtractions (at high energy s the imaginary part behaves like s^2 in accordance with the asymptotically free gluon loop). The existence of the ghost is thus equivalent to the statement that the subtraction constant differs from zero, which by itself is not surprising.[5)] For example, in a scalar gluon channel an analogous subtraction constant is connected with the energy density of the vacuum $\langle G^2\rangle$ because of the renormalizability of the theory.[33] Further, by allowing a transition of the pseudo-Goldstone states $\pi_{1,2}$ into a ghost [see (46), (47)] we have actually introduced a point gauge-invariant transition amplitude[6)]

$$\langle\pi_i|\pi_j\rangle=\langle\pi_i|a_\mu\rangle\langle a_\mu a_\nu\rangle_0\langle a_\nu|\pi_j\rangle=-\mu_i\mu_j.\tag{61}$$

Diagonalization, of course, yields in this case the previous formulas (49).

Introducing also the "direct" matrix elements

$$\langle 0|Q|\pi_i\rangle=\lambda^2\mu_i$$

and the correlator $\langle QQ\rangle_{wlc}=-\lambda^4$ as the subtraction constant in a world without light quarks, so that in the real world

$$\langle QQ\rangle=\langle QQ\rangle_{\text{wlc}}+\sum_{i,j}\langle 0|Q|\pi_i\rangle\langle\pi_i\pi_j\rangle\langle\pi_j|Q|0\rangle,$$

we satisfy the Ward identities and duplicate all the results without invoking the ghost concept.

Thus, in the language of gauge-invariant quantities, the ghost is simply a method of writing down the contribution of the intermediate gluon states with very large masses. (We can point out here a certain analogy with Compton scattering by a nucleon. The Thomson limit of the amplitude $-e^2/m_N$ is, on the one hand, a subtraction constant in the dispersion relation, and on the other, the contribution of a one-nucleon intermediate state.)

Despite the formal possibility of getting along without mentioning the ghost, this concept seems to us exceptionally useful, since it stems from the most important aspects of chromodynamics.

We are grateful to A. I. Vaĭnshteĭn, V. N. Gribov, G. S. Danilov, I. T. Dyatlov, V. I. Zakharov, V. A. Novikov, V. Yu. Petrov, L. L. Frankfurt, and M. A. Shifman for interesting and helpful discussions.

1) Hereafter, $\langle \bar{q}q\rangle$ will be taken to mean precisely the finite nonperturbative part which does not vanish in the chiral limit.

2) More accurately speaking, we are dealing with the energy of a state characterized by a quasimomentum equal to θ, see the next section.

3) This raises the interesting question: To what gauge of the gluon field does this correspond? Does such a gauge exist? Fortunately, the physical results do not depend on the concrete choice of $\langle a_\mu a_\nu\rangle$, provided it satisfies the condition (44).

4) The experimental situation is not quite settled here. The older measurements yield for this ratio a value ≈ 3.

5) The appearance of the pole $1/q^2$ as a result of the divergence of the theory at short distances was emphasized many times by V. N. Gribov using various examples of field theory. We take the opportunity to thank him for numerous helpful discussions.

6) The nontrivial manifestations of the ghost are, first, in the sign of the amplitude (61) and, second, in its factorization. The contribution of the intermediate states with large mass to this amplitude could be written in factorized form $\sim x_j^+(0) x_j(0) \sim f_1 f_j$. This, however, is not the factorization that we need here, since Ward's identity leads to (54): $f_1 f_j \sim \mu_i \mu_j$, and not $\mu_i \mu_j$.

[1] H. Fritzsch and M. Gell-Mann, Proc. XVI Conf. on High Energy Physics, Chicago, 1972, Vol. 2, p. 135. H. Fritzsch, M. Gell-Mann, and H. Leutwyler, Phys. Lett. 47B, 265 (1973).
[2] Yu. L. Dokshitzer, D. I. Dyakonov, and S. I. Troyan, Phys. Reports 58C, 269 (1980). Ya. I. Azimov, Yu. L. Dokshitzer, and V. A. Khoze, Usp. Fiz. Nauk 132, 443 (1980) [Sov. Phys. Usp. 23, 732 (1980)].
[3] M. Gell-Mann, R. J. Oakes, and B. Renner, Phys. Rev. 175, 2195 (1968).
[4] S. Glashow and S. Weinberg, Phys. Rev. Lett. 20, 224 (1968).
[5] V. A. Novikov *et al.*, Phys. Reports 41C, 1 (1978).
[6] M. A. Shifman, A. I. Vaĭnshteĭn, and V. I. Zakharov, Nucl. Phys. B147, 385, 448, 519 (1979).
[7] A. I. Vaĭnshteĭn, V. I. Zakharov, V. A. Novikov, and M. A. Shifman, Elementary Particle Physics, Materials of XV Winter School, Leningrad Inst. Nucl. Phys., 1980, p. 5; Preprints ITEF-87 and 88, 1980.
[8] S. Adler and R. F. Dashen, Current Algebras and Applications to Particle Physics, Benjamin, 1967.
[9] V. De Alfaro, S. Fubini, G. Furlan, and C. Rosetti, Currents in Hadron Physics (Russ. transl.), Mir, 1976.
[10] S. Weinberg, Phys. Rev. Lett. 31, 494 (1973).
[11] S. Weinberg, Proc. XVII Intern. Conf. on High Energy Physics, London, 1974, Vol. 3, p. 59; Phys. Rev. D11, 3583 (1975).
[12] S. Glashow, Hadrons and Their Interactions, Academic, 1968, p. 83. S. Glashow, R. Jackiw, and S.-S. Shei, Phys. Rev. 187, 1916 (1969).
[13] J. Kogut and L. Susskind, Phys. Rev. D11, 3583 (1975).
[14] G. 't Hooft, Phys. Rev. Lett. 37, 8 (1976); Phys. Rev. D14, 3432 (1976).
[15] J. Schwinger, Phys. Rev. 82, 664 (1951). S. L. Adler, Phys. Rev. 177, 2426 (1969). J. S. Bell and R. Jackiw, Nuovo Cimento 60A, 47 (1969).
[16] A. A. Belavin, A. M. Polyakov, A. S. Schwartz, and Yu. S. Tyupkin, Phys. Lett. 59B, 85 (1975).
[17] R. J. Crewther, Phys. Lett. 70B, 349 (1977); Preprint TH. 2646, CERN, 1978.
[18] E. Witten, Nucl. Phys. B156, 269 (1979).
[19] G. Veneziano, Nucl. Phys. B159, 213 (1979).
[20] S. Weinberg, in: Festschrift for I. I. Rabi, Lloyd Motz, ed., NY, 1978.
[21] K. Fujikawa, Phys. Rev. Lett. 44, 1733 (1980).
[22] G. S. Danilov, Nucl. Phys. B145, 429 (1978).
[23] S. Coleman, The Uses of Instantons, Preprint HUTP-78/A004, 1978.
[24] V. N. Gribov, 1974 (unpublished). A. M. Polyakov, 1974 (unpublished), R. Jackiw and C. Rabbi, Phys. Rev. Lett. 37, 172 (1976). C. Gallan, R. Dashen, and D. Gross, Phys. Lett. 63B, 334 (1976); Phys. Rev. D16, 2526 (1977).
[25] M. A. Shifman, A. I. Vaĭnshteĭn, and V. I. Zakharov, Preprint ITEP-64, 1979; R. J. Crewther, P. DiVecchia, G. Veneziano, and E. Witten, Phys. Lett. 88B, 123 (1979).
[26] D. I. Dyakonov and M. I. Eides, Preprint LNPI-639, 1981.
[27] D. I. D'yakonov and M. I. Éĭdes, Elementary particle Physics. Materials of 16th Winter School, Leningrad Inst. Nucl. Phys., 1981, p. 123.
[28] H. Pagels, Phys. Reports 16C, 219 (1975); P. Langacker and H. Pagels, Phys. Rev. D19, 2070 (1979).
[29] Crystal Ball Collab. R. Partridge *et al.*, Preprint SLAC-PUB-2430, 1979.
[30] P. Di Vecchia and G. Veneziano, Nucl. Phys. B171, 253 (1980). C. Rosenzweig, J. Schechter, and G. Trahern, Phys. Rev. D21, 2388 (1980). P. Nath and R. Arnowitt, Preprint TH. 2818-CERN, 1980. P. Di Vecchia, F. Nicodemi, R. Pettorino, and G. Veneziano, Preprint TH. 2898-CERN, 1980.
[31] S. Coleman, Ann. Phys. (N.Y.) 101, 239 (1976).
[32] A. D'Adda, P. Di Vecchia, and M. Luscher, Nucl. Phys. B152, 125 (1979). E. Witten, Nucl. Phys. B149, 285 (1979). P. Di Vecchia, Phys. Lett. 85B, 357 (1979).
[33] V. A. Novikov, M. A. Shifman, A. I. Vaĭnshteĭn, and V. I. Zakharov, Phys. Lett. 86B, 347 (1979). Nucl. Phys. B165, 167 (1980).

Translated by J. G. Adashko