Quasiparticle Lagrangian for the binding energies and self-consistent fields of nuclei in the Fermi-liquid approach

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The problem of calculating the binding energy and self-consistent field of a nucleus in terms of the effective interaction of quasiparticles at the Fermi surface is solved. It is shown that for this one can go over from the system of N Fermi particles to a system of N interacting quasiparticles described by an effective quasiparticle Lagrangian L_q. It is shown that the corresponding quasiparticle energy is equal to the ground-state energy of the system. The connection between the parameters of the effective Lagrangian and the constants of the quasiparticle interaction introduced in the theory of finite Fermi systems is established.

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1. INTRODUCTION

The present paper is devoted to the problem of calculating the binding energies and self-consistent fields of nuclei on the basis of the Fermi-liquid approach. In the theory of finite Fermi systems,[2] which is constructed on the principles of the theory of a Fermi liquid,[1] it is possible to describe many nuclear phenomena and, in particular, solve numerous problems that have no analog in an infinite system. The price to be paid for this success is the introduction of a relatively small number of phenomenological parameters, which are universal for all nuclei and characterize the self-consistent field of a nucleus and the interaction of the quasiparticles at the Fermi surface.

Until recently, the theory of finite Fermi systems, like the theory of a Fermi liquid, was concerned solely with problems related in some manner to the change in the distribution of the quasiparticles at the Fermi surface. It was believed that the calculation of, say, the ground-state energy E_0 or the self-consistent potential of a nucleus is quite impossible in the framework of such an approach because the result necessarily contains sums over states far from the Fermi surface, which cannot be eliminated by means of the standard renormalization procedure.[1] However, this is not the case, since in systems in equilibrium under the influence of only internal forces the pressure is zero. It follows from this directly that the binding energy E_0 is uniquely related to the single-particle spectrum of the excitations of the systems near the Fermi surface, i.e., it is uniquely related to the properties of the quasiparticles. The connection is especially simple in a macroscopic system, whose energy is $E_0 = \mu_0 A$, where A is the number of particles and μ_0 is the chemical potential, which is equal to the limiting energy of the quasiparticles. Thus, E_0 is quite independent of the spectrum of single-particle excitations of the macroscopic system far from the Fermi surface. It is true that in nuclei, in which the number of particles is not particularly large, the expression for the binding energy $E_0(N, Z)$ is somewhat complicated by the need to take into account the terms with the following powers of the parameter $A^{-1/3}$ in the liquid-drop part of E_0, and also the irregular shell corrections, but this does not affect the essence of the matter. The problem of calculating the binding energy of the nucleus reduces to finding the chemical potentials μ_n and μ_p, or rather the quasiparticle spectrum near the Fermi surface. Since the properties of the quasiparticles near the Fermi surface are the same as those of the real single-particle excitations, this circumstance makes it possible to go over from the real system of A interacting particles to a system of A quasiparticles whose equations of motion are obtained from the Dyson equation. To describe the quasiparticle system, one can introduce an effective Lagrangian L_q of the quasiparticles, requiring the Lagrange equations to be identical to the equations of motion of the quasiparticles (Sec. 3). It can be shown that the energy of the system of interacting quasiparticles, calculated in accordance with the usual rules on the basis of the Lagrange function L_q, is equal to the total binding energy E_0 (Sec. 4). This assertion is analogous to the well-known Landau–Luttinger theorem on the equality of the number of particles and quasiparticles.

Thus, the effective quasiparticle Lagrangian L_q determines both the spectrum of single-particle excitations at the Fermi surface and the binding energy E_0 of the nucleus. In Sec. 5, we establish the connection between this Lagrangian and the amplitude F of the local quasiparticle interaction introduced in the theory of finite Fermi systems; this means that it is also possible to calculate in terms of L_q the response function of the system when an external field is applied, i.e., one can solve the same problems as in the theory of finite Fermi systems. The upshot is that, specifying a few parameters, which are universal for all nuclei and characterize the interaction Lagrangian of the quasiparticles, we obtain the possibility of calculating the basic characteristics of nuclei, namely, the binding energy, single-particle spectra, static moments, transition probabilities, and so forth.

2. QUASIPARTICLE DESCRIPTION OF FINITE FERMI SYSTEMS

The quasiparticle formalism of the theory of a Fermi liquid is generally employed to describe the single-particle states near the Fermi surface and to find the response of a system to a long-wavelength external field.[1,2] However, the possibilities of the quasiparticle description are much richer. As we shall see, in problems involving the calculation of the total binding energy of the system and its self-consistent field we can also in fact replace the real system of A interacting particles by a system of A quasiparticles with effective two-

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body interaction that is uniquely related to the quasiparticle scattering amplitude at the Fermi surface. One can here use the principle of least action, applying it to the quasiparticles and requiring that the corresponding Lagrange equations be identical to the equations of motion of the quasiparticles.

These equations are usually obtained from the Dyson equation by expanding the particle mass operator $\Sigma(\mathbf{r}, \mathbf{p}, \varepsilon)$ near the Fermi surface in powers of $\varepsilon - \mu$ and $p^2 - p_F^2$, the contribution of the quadratic and following terms being ignored, i.e., the mass operator Σ_q of a quasiparticle in nuclear matter is given by

$$\Sigma_q(\mathbf{p}, \varepsilon) = \Sigma(p_F, \mu_0) + (\varepsilon - \mu_0)\left(\frac{\partial \Sigma}{\partial \varepsilon}\right)_0 + (p^2 - p_F^2)\left(\frac{\partial \Sigma}{\partial p^2}\right)_0. \quad (1)$$

The subscript 0 indicates that the derivatives are taken at the point $\varepsilon = \mu_0, p = p_F$. The Fermi momentum p_F and the chemical potential μ_0 of equilibrium nuclear matter are related by

$$p_F^2/2m + \Sigma(p_F, \mu_0) = \mu_0. \quad (2)$$

The exact Green's function is $G = G_R + G_q$, where G_R is the regular part of G and $G_q = (\varepsilon - \varepsilon_p^0 - \Sigma_q)^{-1}$ is the pole part. The poles of G_q determine the quasiparticle spectrum $\varepsilon_p = (p - p_F) \cdot p_F/m^*$, which near the Fermi surface is, as is well known, the same as the spectrum of the real single-particle excitations of the infinite system. The quantity

$$a = \left[1 - \left(\frac{\partial \Sigma}{\partial \varepsilon}\right)_0\right]^{-1}$$

determines the weight of the quasiparticle in the exact single-particle state, and the effective mass of the quasiparticle is

$$m^* = ma^{-1}\left[1 + 2m\left(\frac{\partial \Sigma}{\partial p^2}\right)_0\right]^{-1}.$$

In a finite system, the mass operator Σ also becomes a function of the coordinates. Note that because the operators r and p do not commute, it is necessary to arrange them in an order that does not violate the Hermiticity of the mass operator Σ_q, and where nothing is said to the contrary a product written in the form $p^2 f(\mathbf{r})$ is to be understood as the operator $\mathbf{p} f(\mathbf{r}) \mathbf{p}$. With allowance for this, the analog of (1) for a finite system is the expression

$$\Sigma_q(\mathbf{r}, \mathbf{p}, \varepsilon) = \Sigma_0(\mathbf{r}) + \mathbf{p}\Sigma_1(\mathbf{r})\mathbf{p} + \varepsilon\Sigma_2(\mathbf{r}). \quad (3)$$

The quantity $\Sigma_2(\mathbf{r})$ is related to the renormalization factor $a(\mathbf{r})$ of the Green's function: $\Sigma_2(\mathbf{r}) = 1 - a^{-1}(\mathbf{r})$. Essentially, Eq. (3) is a consequence of the short range or, as one often says, locality of the nuclear forces. The expansion parameter, which permits the retention of the terms $\sim p^2$ but the neglect of the remainder, is the ratio of the range of the forces to the characteristic distance over which the nuclear density changes appreciably. For the interior regions of the nucleus, this ratio is $\sim A^{-1/3}$. On the surface of the nucleus, it is of order unity, and in the surface components of Σ it is in principle necessary to take into account higher powers of the operator p, i.e., terms of the type $p_\alpha p_\beta \nabla_\alpha n \nabla_\beta n$, etc. Their contribution to the single-particle energies is small ($\sim A^{-1/3}$), and in nuclear matter it is further suppressed for numerical reasons, in particular, because of the weakness of the velocity forces of the quasiparticle interaction at the Fermi surface. In principle, it is not difficult to take into account these terms, but since they play a small part, we shall, to simplify the exposition, initially omit them altogether. Then the equation for the pole part G_q of the Green's function has the form

$$[\varepsilon - \varepsilon_p^0 - \Sigma_q(\mathbf{r}, \mathbf{p}, \varepsilon)] G_q(\mathbf{r}, \mathbf{r}', \varepsilon) = \delta(\mathbf{r} - \mathbf{r}'), \quad (4)$$

where $\Sigma_q(\mathbf{r}, \mathbf{p}, \varepsilon)$ is given by Eq. (3).

The solution of this equation can be expressed in terms of the eigenfunctions ψ_λ of the homogeneous equation:

$$G_q(\mathbf{r}, \mathbf{r}', \varepsilon) = \sum_\lambda \frac{\psi_\lambda^*(\mathbf{r})\psi_\lambda(\mathbf{r})}{\varepsilon - \varepsilon_\lambda - i\delta\,\mathrm{sgn}(\varepsilon_\lambda - \mu)}. \quad (5)$$

The equation for the quasiparticle wave functions ψ_λ differs from the Schrödinger equation in that the single-particle potential depends on the energy:

$$i\frac{\partial \psi_\lambda(\mathbf{r}, t)}{\partial t} = \left(\frac{p^2}{2m} + \Sigma_0(\mathbf{r}) + \mathbf{p}\Sigma_1(\mathbf{r})\mathbf{p} + i\Sigma_2(\mathbf{r})\frac{\partial}{\partial t}\right)\psi_\lambda(\mathbf{r}, t). \quad (6)$$

Because of this, the eigenfunctions $\psi_\lambda(\mathbf{r})$ of this equation are orthogonal with weight $a^{-1}(\mathbf{r})$:

$$\int \psi_\lambda^*(r) a^{-1}(\mathbf{r}) \psi_{\lambda'}(\mathbf{r})\, d\mathbf{r} = \delta_{\lambda\lambda'}. \quad (7)$$

By the Landau–Luttinger theorem, the number of occupied states is equal to the number of particles:

$$\sum_\lambda n_\lambda = A.$$

3. LAGRANGIAN OF A SYSTEM OF INTERACTING QUASIPARTICLES

The system of equations (6) can be obtained from the principle of least action for quasiparticles. The action

$$S_q = \int L_q(t)\, dt$$

can be expressed in terms of the effective quasiparticle Lagrangian

$$L_q(t) = \int \mathcal{L}_q(\mathbf{r}, t)\, d\mathbf{r},$$

which is determined by the variational formula (derived in Appendix 1)

$$\delta L_q = \int d\mathbf{r}\left(i\frac{\partial}{\partial \tau} - \varepsilon_p^0 - \Sigma_q\left(\mathbf{r}, \mathbf{p}, i\frac{\partial}{\partial \tau}\right)\right)\delta G_q(\mathbf{r}, \mathbf{r}, \tau)|_{\tau \to -0}. \quad (8)$$

It follows that the Lagrangian L_q can be written in the form $L_q = L_q^0 + L_q'$, where L_1^0 is the free Lagrangian, whose density is given by

$$\mathcal{L}_q^0(\mathbf{r}) = \left(i\frac{\partial}{\partial \tau} - \varepsilon_p^0\right) G_q(\mathbf{r}, \mathbf{r}, \tau)|_{\tau \to -0} = \nu_2 - \nu_1. \quad (9)$$

Here,

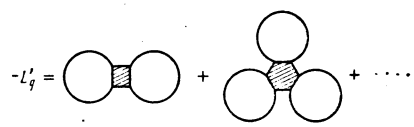
$$\nu_1(\mathbf{r}) = \frac{1}{2m}\sum_\lambda n_\lambda (\nabla \psi_\lambda^*(\mathbf{r}))(\nabla \psi_\lambda(\mathbf{r})), \quad (10)$$

$$\nu_2(\mathbf{r}) = \sum_\lambda n_\lambda \frac{1}{2i}(\psi_\lambda^* \dot{\psi}_\lambda - \dot{\psi}_\lambda^* \psi_\lambda). \quad (11)$$

In accordance with (8), the interaction Lagrangian L_q' is related to the quasiparticle mass operator:

$$\Sigma_q(1, 2) = -\frac{\delta L_q'}{\delta G_q(1, 2)}. \quad (12)$$

This variational formula can be illustrated graphically. We represent $-L_q'$ as a sum over double, triple, fourfold, etc., collisions of the quasiparticles:



The continuous lines here correspond to propagators G_q, and the hatched $2n$-sided polygon, which is the appropriately symmetrized effective potential, corresponds to the amplitude of the local interaction of the n quasiparticles. In the graphical language, the variation of L_q' reduces to the breaking of one of the continuous lines. The resulting set of graphs is the single-particle potential, the quasiparticle mass operator Σ_q.

Further simplifications are associated with the specific form of $\Sigma_q(\mathbf{r}, \mathbf{p}, \varepsilon)$. If the mass operator Σ_q is given by formula (3), then from (12) we obtain

$$\delta L_q = -\int \Sigma_i(\mathbf{r})\,\delta\nu_i(\mathbf{r})\,d\mathbf{r}. \tag{13}$$

Here, ν_1 and ν_2 are determined by Eqs. (10) and (11), and

$$\nu_0(\mathbf{r}) = \sum_\lambda n_\lambda \psi_\lambda^*(\mathbf{r})\,\psi_\lambda(\mathbf{r}). \tag{14}$$

The quantity ν_0 is the quasiparticle contribution to the total density $\rho(\mathbf{r})$ and differs only in the factor $a(\mathbf{r})$ from the quasiparticle density $n(\mathbf{r})$ normalized to the number of particles. Since the expression (13) for δL_q contains only the variations $\delta\nu_i$, this means that in the considered approximation both the Lagrangian L_q and the mass operator Σ_q depend on only three functions: ν_0, ν_1, and ν_2. The functions $\Sigma_i(\nu_k)$ are the variational derivatives of L_q':

$$\Sigma_i = -\frac{\delta L_q'}{\delta \nu_i} \tag{15}$$

$$\frac{\delta L_q'}{\delta \nu_i} = \frac{\partial \mathscr{L}_q}{\partial \nu_i} - \nabla \frac{\partial \mathscr{L}_q}{\partial(\nabla \nu_i)} + \Delta \frac{\partial \mathscr{L}_q}{\partial(\Delta \nu_i)} - \dots. \tag{16}$$

Therefore, they are connected by definite relations which follow from the equality of the mixed derivatives of L_q'.

It is obvious that the variation with respect to ψ_λ^* of the Lagrangian L_q determined from (9) and (15) leads, with allowance for the relations

$$\frac{\delta \nu_0}{\delta \psi_\lambda^*} = n_\lambda \psi_\lambda, \quad \frac{\delta \nu_1}{\delta(\nabla_\alpha \psi_\lambda^*)} = \frac{n_\lambda \nabla_\alpha \psi_\lambda}{2m}, \quad \frac{\delta \nu_2}{\delta \psi_\lambda^*} = \frac{i n_\lambda \psi_\lambda}{2} \quad \frac{\delta \nu_2}{\delta \psi_\lambda} = -\frac{i n_\lambda \psi_\lambda}{2},$$

to the equation of motion (6) of the quasiparticles.

Let us now consider what happens if we include the surface components of Σ_q which depend on the momentum $\mathbf{p}$. The principal one of these is the spin-orbit part $\sim[\sigma \times \mathbf{p}]\nabla\nu_0(\mathbf{r})$ linear in the momentum. The inclusion of the spin–orbit forces leads to the appearance in the variation of the Lagrangian of the additional term

$$L_{sl}' = -\Sigma_s^\alpha \delta \nu_s^\alpha;$$
$$\nu_s^\alpha = \sum_\lambda n_\lambda \psi_\lambda^* [\sigma \times \mathbf{p}]_\alpha \psi_\lambda, \quad \Sigma_s^\alpha \sim \nabla_\alpha \nu_0(\mathbf{r}).$$

The other p-dependent surface corrections to the mass operator Σ_q can be taken into account similarly. The Lagrangian L_q' is a functional of the new densities ν_i, and the corresponding components Σ_i are determined by Eq. (15).

The Lagrangian approach makes it possible to find not only the spectrum of single-particle excitations of a nucleus but also, as we shall now see, to calculate the binding energy of the nucleus, including the shell corrections.

4. EQUALITY OF THE ENERGIES OF THE PARTICLE AND QUASIPARTICLE SYSTEMS

The ground-state energy of a macroscopic system maintained in equilibrium by internal forces is related to the chemical potential μ_0 by the simple euqation $E_0 = \mu_0 A$, which follows from the vanishing of the external pressure. Thus, in the macroscopic limit the energy E_0 is equal to the quasiparticle energy $E_q = \mu_q A$, since, by definition, the chemical potential μ_0 is equal to the limiting quasiparticle energy μ_q. It is clear that a similar assertion is also valid for an atomic nucleus with a sufficiently large number of particles, and the only question is that of its accuracy.

We define the quasiparticle energy of the nucleus by the usual relation

$$E_q(N, Z) = \int \mathscr{H}_q(\mathbf{r})\,d\mathbf{r} = \int \left(\nu_2 \frac{\delta L_q}{\delta \nu_2} - \mathscr{L}_q\right) d\mathbf{r}. \tag{17}$$

Here, we have used the circumstance that L_q depends on ψ only through ν_2. It follows from (17) that the difference between the ground-state energies of systems of $A+1$ and A quasiparticles is exactly equal to the energy ε_{λ_0} of the last occupied quasiparticle level (see Appendix 1). The binding energy $E_0(N, Z)$ of the nucleus is the sum of the Weizsäcker drop energy

$$E_0^W(N, Z) = \alpha A + \beta A^{2/3} + \gamma (N-Z)^2/A + \dots$$

and the shell correction δE_0^S, which varies rapidly and irregularly with the particle number. Although δE_0^S does not exceed 10 MeV (and the binding energy of heavy nuclei is $E_0 \sim 1.5$ GeV), it is necessary to take into account the shell effects, for it is only these that offer hope of finding new islands of nuclear stability. The magnitude of the shell correction is entirely determined by the behavior of the single-particle levels at the Fermi surface. It is maximal for magic nuclei, in whose single-particle excitation spectrum there is a magic gap.

The energy E_q of the quasiparticle system is also a sum of the drop energy

$$E_q^W = \alpha_q A + \beta_q A^{2/3} + \gamma_q (N-Z)^2/A + \dots$$

and the shell correction δE_q^S. The values of δE_q^S and δE_0^S are almost equal, since the quasiparticle spectrum near the Fermi surface is nearly the same as the spectrum of the real single-particle excitations. The drop energies E_0^W and E_q^W also differ only slightly. The difference between them can be estimated by comparing the chemical potentials

$$\mu^W = E_0^W(A+1) - E_0^W(A) = \alpha + \frac{2}{3}\beta A^{-1/3} + \dots, \quad \mu_q^W = \alpha_q + \frac{2}{3}\beta_q A^{-1/3} + \dots.$$

The chemical potential μ is determined implicitly by the exact relation

$$(\chi_{\lambda_0}(\mathbf{r}),\ [\varepsilon_p^0 + \Sigma(\mathbf{r}, \mathbf{p}, \mu) - \mu]\chi_{\lambda_0}(\mathbf{r})) = 0,$$

where χ_{λ_0} is the wave function of the last occupied single-particle state belonging to the system of functions that diagonalize the exact mass operator $\Sigma(\mu)$. The chemical potential μ_q of the quasiparticles is determined by the analogous relation

$$(\psi_{\lambda_0}(\mathbf{r}),\ [\varepsilon_p^0+\Sigma_q(\mathbf{r},\ \mathbf{p},\ \mu_q)-\mu_q]\psi_{\lambda_0}(\mathbf{r}))=0.$$

The functions χ_{λ_0} and ψ_{λ_0} differ to the extent that Σ and Σ_q are not identical to each other.

Suppose that in Σ_q all the necessary p-dependent surface corrections have already been taken into account. Then the difference between Σ and Σ_q is due to the contribution of the second and higher derivatives of Σ with respect to ε and p^2, and therefore

$$\Sigma(\mu)-\Sigma_q(\mu_q)\sim\left(\frac{\partial^2\Sigma}{\partial\varepsilon^2}\right)(\mu-\mu_0)^2+2\left(\frac{\partial^2\Sigma}{\partial\varepsilon\,\partial p^2}\right)(\mu-\mu_0)(p^2-p_F^2)+\left(\frac{\partial^2\Sigma}{\partial(p^2)^2}\right)(p^2-p_F^2)^2.$$

Since $(\mu-\mu_0)\sim\varepsilon_F A^{-1/3}$, it is readily seen that the contribution of the second derivative $(\partial^2\Sigma/\partial\varepsilon^2)_0$ to the difference $\mu-\mu_q$ is of order $\varepsilon_F A^{-2/3}$. The contributions of the terms $(\partial^2\Sigma/\partial\varepsilon\partial p^2)_0$, $[\partial^2\Sigma/\partial(p^2)^2]_0$ are of the same order. Thus, if we do not include in the volume part $\Sigma_q(p,\varepsilon)$ the terms quadratic in ε and ε_p^0, a difference between μ^W and μ_q^W appears in the terms of order $\varepsilon_F A^{-2/3}$, so that the three main coefficients α, β, and γ in the Weizsäcker expressions for the particles and quasiparticles are the same. If we need to increase the accuracy (for example, to calculate the contribution of the curvature terms $\sim A^{-2/3}$), we must include in Σ_q the terms with the second derivatives of Σ.

Thus, using the quasiparticle picture we can calculate the binding energy of the Fermi system, and also the spectrum of single-particle excitations near the Fermi surface. For this, it is necessary to specify phenomenologically the effective quasiparticle Lagrangian L_q. As we shall see in the following section, the constants in L_q are uniquely related to the parameters of the amplitude of the local quasiparticle interaction at the Fermi surface, which determines the response of the system to an external field.

5. CONNECTION BETWEEN THE QUASIPARTICLE LAGRANGIAN AND THE AMPLITUDE OF THE QUASIPARTICLE INTERACTION

The amplitude $\Lambda(1,2,3,4)$ of the local quasiparticle interaction can be introduced as the variational derivative $\delta^2 L_q/\delta G_q(1,2)\delta G_q(3,4)$ or, equivalently, by the variational formula

$$\delta\Sigma_q(1,2)=\int\Lambda(1,2,3,4)\delta G_q(3,4)\,d\tau_3\,d\tau_4. \tag{18}$$

Graphically, the amplitude Λ is obtained from L_q in the diagrams given above if two continuous lines are broken. We introduce the notation $\xi_0=1$, $\xi_1=p^2$, and $\xi_2=\varepsilon=i\partial/\partial t$. Then the quasiparticle mass operator (3) can be represented in the form $\Sigma_q=\xi_i\Sigma_i(\mathbf{r})$, and the amplitude Λ corresponding to it as $\Lambda=\xi_i(1)\xi_k(2)\Lambda_{ik}(\mathbf{r}_1,\mathbf{r}_2)$. From (18), we obtain

$$\delta\Sigma_i(\mathbf{r})=\int\Lambda_{ik}(\mathbf{r},\mathbf{r}')\delta\nu_k(\mathbf{r}')\,d\mathbf{r}'. \tag{18'}$$

In conjunction with (15), this expression enables us to determine readily the components of the local amplitude Λ from given quasiparticle Lagrangian L_q. We now establish how Λ is related to the local amplitude $\mathcal{U}$ introduced in the theory of finite Fermi systems. This can be done in several ways. One of them uses a consistency condition.[3]

This condition is a consequence of the spontaneous breaking of translation invariance in a system maintained in equilibrium by internal forces: The displacement operator p commutes with the total Hamiltonian of the system but not with the mass operator Σ. As a result, in the identity for the derivatives of the Green's function based on the Galilean transformation[4] terms containing the commutator

$$[\mathbf{p},\Sigma]\sim\left(\frac{\partial}{\partial\mathbf{r}_1}+\frac{\partial}{\partial\mathbf{r}_2}\right)\Sigma(\mathbf{r}_1,\mathbf{r}_2,\varepsilon)$$

appear. This leads to the consistency condition[3]:

$$\left(\frac{\partial}{\partial\mathbf{r}_1}+\frac{\partial}{\partial\mathbf{r}_2}\right)\Sigma(\mathbf{r}_1,\mathbf{r}_2,\varepsilon)=\int d\mathbf{r}_3 d\mathbf{r}_4 d\mathbf{r}_5 d\mathbf{r}_6\int_C\frac{d\varepsilon'}{2\pi i}\mathcal{U}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4;\varepsilon,\varepsilon')\times G(\mathbf{r}_3,\mathbf{r}_5,\varepsilon')\left[\left(\frac{\partial}{\partial\mathbf{r}_5}+\frac{\partial}{\partial\mathbf{r}_6}\right)\Sigma(\mathbf{r}_5,\mathbf{r}_6,\varepsilon')\right]G(\mathbf{r}_6,\mathbf{r}_4,\varepsilon'), \tag{19}$$

or, in symbolic form,

$$\frac{\partial\Sigma}{\partial\mathbf{r}'}=\mathcal{U}GG\frac{\partial\Sigma}{\partial\mathbf{r}}. \tag{20}$$

In (19), the contour C is closed, as usual, in the upper half-plane of ε'. The 4-pole block $\mathcal{U}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4;\varepsilon,\varepsilon')$, which is irreducible in the particle-hole channel, is a short-range local function of the coordinates and universal for all nuclei. The local component B is also present in the product G_qG_q, but its pole part also contains a long-range interaction, which varies in a non-universal manner from nucleus to nucleus.[2]

In this paper, we shall use not only the coordinate representation but also a mixed representation. In it, the block $\mathcal{U}(\mathbf{r}_1,\mathbf{p}_1,\mathbf{r}_2,\mathbf{p}_2;\varepsilon,\varepsilon')$ is a smooth function of the momenta $\mathbf{p}_1$ and $\mathbf{p}_2$.

The direct solution of Eq. (20) is impossible, because the theory is as yet incapable of calculating quantitatively the contribution of the regions of integration far from the Fermi surface. But, in contrast to the other relations between the mass operator Σ and the two-particle interaction, Eq. (20) has an important feature—it can be renormalized. This operation, proposed by Landau,[1] consists of decomposing the product GG into the sum

$$GG=A+B.$$

The term A can be calculated theoretically. It contains the product G_qG_q of the pole parts and is a peaked δ-functional function of the variables ε, p_1, and p_2 in the neighborhood of the Fermi surface. The term B, which like the block $\mathcal{U}$, cannot be calculated in the theory, is regular at the Fermi surface.

Once the decomposition (21) has been made, the original equation can be transformed by simple algebraic operations in such a way that it contains only one combination of the two universal functions $\mathcal{U}$ and B which cannot be calculated in the theory. For this, we substitute (21) in (20) and take the term $\mathcal{U}B\partial\Sigma/\partial\mathbf{r}$ to the left-hand side. Then, after application of the operator $(1-\mathcal{U}B)^{-1}$ to both sides of the resulting equation, and

denoting the amplitude of the two-particle interaction which arises as a result of the renormalization by the same letter Λ, we arrive at the equation

$$\frac{\partial\Sigma}{\partial\mathbf{r}}=\Lambda A\frac{\partial\Sigma}{\partial\mathbf{r}}. \tag{22}$$

The amplitude Λ satisfies an equation of the same type as the amplitude Γ^{ω} in the theory of Fermi liquids:

$$\Lambda=\mathcal{U}+\mathcal{U}B\Lambda. \tag{23}$$

It has the same properties as $\mathcal{U}$, i.e., it is regular in the neighborhood of the Fermi surface and, like $\mathcal{U}$, is universal for all nuclei.

Further, since the propagator A is, by definition, a δ-like function of ε and p in the neighbhorhood of the Fermi surface, the integration over the intermediate momenta in (22) takes place only in this region. Therefore, on the right-hand side of (22) the true mass operator Σ can be replaced by the quasiparticle Σ_q. If, further, the initial 4-momenta are near the Fermi surface, a similar substitution can be made on the left-hand side of this equation. Finally, if we ignore the small surface corrections in Σ_q which depend on the direction of the vector $\mathbf{p}$, we can integrate the right-hand side of (22) over the directions of the intermediate momenta. Then in (22) there remain only the zeroth harmonics of Λ and A with respect to the angle between the momenta $\mathbf{p}_1$ and $\mathbf{p}_2$. After these remarks, we can write the renormalized consistency condition in the form

$$\frac{\partial\Sigma_q(\mathbf{r},p,\varepsilon)}{\partial\mathbf{r}}=\int_c\frac{d\varepsilon_1}{2\pi i}\int d\mathbf{r}_1d\mathbf{r}_2\frac{p_1^2dp_1p_2^2dp_2}{(2\pi)^6}\Lambda^0(\mathbf{r},p,\mathbf{r}_1,p_1;\varepsilon,\varepsilon_1)\times A^0(\mathbf{r}_1,p_1,\mathbf{r}_2,p_2;\varepsilon_1)\frac{\partial\Sigma_q(\mathbf{r}_2,p_2,\varepsilon_1)}{\partial\mathbf{r}_2}. \tag{24}$$

We again write

$$\Sigma_q=\xi_i\Sigma_i(\mathbf{r}),\quad \Lambda=\xi_i(1)\xi_k(2)\Lambda_{ik}(\mathbf{r},\mathbf{r}').$$

Then, integrating Eq. (24) by parts, we obtain

$$\frac{d\Sigma_i}{d\mathbf{r}}=\int\Lambda_{ik}^0(\mathbf{r},\mathbf{r}_1)A_{kl}^0(\mathbf{r}_1,\mathbf{r}_2)\frac{d\Sigma_l}{d\mathbf{r}_2}d\mathbf{r}_1d\mathbf{r}_2, \tag{25}$$

where

$$A_{kl}^0(\mathbf{r}_1,\mathbf{r}_2)=\int_c\frac{d\varepsilon_i}{2\pi i}\int\frac{p_1^2dp_1p_2^2dp_2}{(2\pi)^6}\xi_k\xi_lA^0(\mathbf{r}_1,p_1,\mathbf{r}_2,p_2;\varepsilon). \tag{26}$$

If we have substitute the product G_qG_q of the pole Green's functions as A^0, then (25) is transformed to

$$\frac{d\Sigma_i}{d\mathbf{r}}=\int\Lambda_{ik}^0(\mathbf{r},\mathbf{r}')\frac{d\nu_k}{d\mathbf{r}'}d\mathbf{r}'. \tag{27}$$

A rigorous method for decomposing GG into A and B corresponding to this is given in Appendix 2. Exactly the same relation is obtained from the variational formula (18). Thus, the use of the consistency condition makes it possible to establish rather easily the connection between the local amplitude Λ of the quasiparticle interaction and the irreducible block $\mathcal{U}$ of two-particle scattering [see (23)]. From this, it is easy to obtain the connection between Λ and Γ^{ω}. The amplitude Γ^{ω} satisfies an equation of the same type as Λ, but with a somewhat different decomposition of GG into A and B. The propagator $A_0^0(\mathbf{r}_1,p_1,\mathbf{r}_2,p_2;\varepsilon)$ used in the theory of finite Fermi systems has the form[2]

$$A_0^0=\frac{4(2\pi)^7i}{p_1p_2}\delta(p_1^2-p_F^2(\mathbf{r}_1))\delta(p_2^2-p_F^2(\mathbf{r}_2))\delta(\varepsilon-\mu)\cdot\int_c\frac{d\varepsilon'}{2\pi i}G_q(\mathbf{r}_1,\mathbf{r}_2\ \varepsilon')\times G_q(\mathbf{r}_2,\mathbf{r}_1,\varepsilon'). \tag{28}$$

It is easy to derive an equation relating Λ to Γ^{ω}:

$$\Lambda=\Gamma^{\omega}+\Gamma^{\omega}(A-A_0^0)\Lambda. \tag{29}$$

The difference $A-A_0^0$, in contrast to A_0^0, no longer contains any long-range components, and therefore both Λ and Γ^{ω}, we can express this amplitude in terms of Λ and thus calculate Γ^{ω} on the basis of the quasiparticle Lagrangian. It can then be used in all the traditional problems of the theory of finite Fermi systems, namely, to calculate the static moments, transition probabilities, and so forth. Thus, virtually all the phenomena in low-energy nuclear physics can be described by means of the quasiparticle Lagrangian L_q.

In constructing the quasiparticle Lagrangian of a nucleus we shall, as is customary in the theory of finite Fermi systems, assume that the quasiparticle interaction amplitude depends on the density ν_0 linearly and does not depend on ν_1 and ν_2. Then L_q' can be written as

$$L_q'=-\Big[{}^1/_2\int\lambda_{ik}(\mathbf{r}_1,\mathbf{r}_2)\nu_i(\mathbf{r}_1)\nu_k(\mathbf{r}_2)d\mathbf{r}_1d\mathbf{r}_2+{}^1/_6\int\gamma(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)\nu_0(\mathbf{r}_1)\nu_0(\mathbf{r}_2)\nu_0(\mathbf{r}_3)d\mathbf{r}_1d\mathbf{r}_2d\mathbf{r}_3\Big], \tag{30}$$

where λ_{ik} and γ are local functions of the coordinate differences. For the majority of applications, it is sufficient to restrict ourselves in λ_{00} and γ to the effective-range approximation:

$$\lambda_{00}(\mathbf{r}_1,\mathbf{r}_2)=\lambda_{00}(1+r_p^2\Delta_1)\delta(\mathbf{r}_1-\mathbf{r}_2),\qquad \gamma(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)=\gamma[1+r_t^2(\Delta_1+\Delta_2+\Delta_3)/2]\delta(\mathbf{r}_1-\mathbf{r}_2)\delta(\mathbf{r}_2-\mathbf{r}_3). \tag{31}$$

In the remaining less important terms, we need retain only the first δ-functional term. Hence, using (15), we find

$$\Sigma_0=\lambda_{0k}\nu_k+\gamma\nu_0^2/2+\lambda_{00}r_p^2\Delta\nu_0+\gamma r_t^2[\nu_0\Delta\nu_0+(\nabla\nu_0)^2/2],\qquad \Sigma_1=\lambda_{1k}\nu_k,\quad \Sigma_2=\lambda_{2k}\nu_k. \tag{32}$$

The principal component of the amplitude Λ is

$$\Lambda_{00}(\mathbf{r}_1,\mathbf{r}_2)=(\lambda_{00}+\gamma\nu_0+\gamma r_t^2\Delta\nu_0)\delta(\mathbf{r}_1-\mathbf{r}_2)+\gamma r_t^2(\nabla\nu_0)\nabla\delta(\mathbf{r}_1-\mathbf{r}_2)+(\lambda_{00}r_p^2+\gamma r_t^2\nu_0)\Delta\delta(\mathbf{r}_1-\mathbf{r}_2), \tag{33}$$

and the remaining components are

$$\Lambda_{ik}(\mathbf{r}_1,\mathbf{r}_2)=\lambda_{ik}\delta(\mathbf{r}_1-\mathbf{r}_2). \tag{33'}$$

Finally, the quasiparticle energy E_q, calculated in accordance with formula (17) with the Lagrangian (30), has the form

$$E_q=\int d\mathbf{r}[\lambda_{00}(\nu_0^2-r_p^2(\nabla\nu_0)^2)/2+\lambda_{01}\nu_0\nu_1+\lambda_{11}\nu_1^2/2-\lambda_{22}\nu_2^2/2+\gamma\nu_0^3/6-\gamma r_t^2\nu_0(\nabla\nu_0)^2/2]. \tag{34}$$

To simplify this expression, we have omitted the isotopic dependence of the coefficients, and we have also ignored the spin–orbit and Coulomb forces, which can be readily taken into account.

If the quasiparticle Lagrangian L_q is specified, then the functional dependence $\Sigma_i(\nu_k)$ is also known. Then the characteristics of nuclei can be calculated iteratively. For this, it is necessary to specify the original mass operator $\Sigma_q^{(0)}$, solve Eq. (6) with it, construct the set of quasiparticle wave functions $\psi_\lambda^{(0)}$ and, using (8), (9), and (10), calculate the densities $\nu_i(\mathbf{r})$. Then, using

the connection between Σ_i and ν_k found from (18), it is necessary to find the following approximation $\Sigma_q^{(1)}$ for the mass operator Σ_q, after which the entire procedure must be repeated. As can be seen from preliminary calculations, the iterative process converges fairly rapidly, and as a result we obtain the quasiparticle mass operator Σ_q and the densities $\nu_i(\mathbf{r})$, and with them we calculate the total binding energy of the given nucleus by means of (34).

The proposed approach is based on the natural assumption of a local connection between the mass operator Σ_q and the quasiparticle density matrix. Then the number of parameters introduced in the theory is small. In general, this assumption is invalid if the system is near a phase transition point. Then the amplitude Λ, which is assumed to be local, contains a long-range interaction, so that the Lagrangian L_q also becomes nonlocal. As an example, we can take liquid ^{3}He, in which the existence of such long-range interaction is indicated by the slow decrease of the harmonics Γ_L^ω, which can be attributed to the paramagnon singularity in the transverse channel.[5] Similar, albeit small effects arise in nuclear physics too if one takes into account the contribution of the collective degrees of freedom. Then the prescription for constructing L_q must be made more precise and it is necessary to separate the long-range interaction explicitly, after which the remaining part of the Lagrangian can be parametrized in the usual manner.

6. DISCUSSION

We have constructed a quasiparticle description suitable for the theoretical investigation of many nuclear phenomena—the calculation of the masses and self-consistent fields of nuclei, and also the response of systems to an external field—and we have constructed a quasiparticle Lagrangian whose variation gives the equations of motion of the quasiparticles. The quasiparticle energy E_q corresponding to it is equal to the total binding energy of the system. This assertion is analogous to the Landay–Luttinger theorem which states that the number of particles is equal to the number of quasiparticles. In the usual manner of introducing quasiparticles, when only the term linear in $\varepsilon - \mu$ and $p^2 - p_F^2$ are taken into account in the quasiparticle mass operator Σ_q, the equality of E_q and E_0 holds to accuracy $\sim A^{-2/3}$. The accuracy of the theory can be increased by including in the definition of Σ_q the following terms of the expansion. In principle, this increase in the accuracy of the theory should also permit calculation of the quasiparticle damping $\sim (\varepsilon - \mu)^2$, and, when the theory is generalized to temperatures $T \neq 0$, the calculation of the contributions nonanalytic in T to the specific heat and other characteristics of the system.

In the limiting case when the quasiparticle interaction does not depend on the energy, the renormalization factor a becomes equal to unity, and the results of the present approach are identical to the analogous results of the Hartree–Fock method with effective forces.[6] In the form proposed in Ref. 6, this method, in contrast to the ordinary Hartree–Fock method, has no theoretical foundation, since one seeks the minimum of a quantity that is not the Hamiltonian of the system. That the nuclear masses are well reproduced in this approach is largely due to the special choice of the constants, which compensates the inaccuracy of the theory, but when attempts are made to describe other phenomena with the same parameters results that contradict the experiments are often obtained. The reason for these contradictions is in the neglect of the energy dependence of the effective forces.

We note that methods similar to the Hartree–Fock method with effective forces are frequently used in not only nuclear physics but also in solid state theory and other fields. Our analysis explains why such methods frequently lead to reasonable results when the energy dependence of the effective interaction is weak, and it gives the possibility of taking into account this dependence correctly when it is important.

We now discuss the determination of the parameters of the effective quasiparticle Lagrangian from experiments. For this, besides the single-particle spectra and the energies of the nuclei, we can use the entire set of nuclear phenomena studied generally in the theory of finite Fermi systems. The corresponding experimental data are being augmented continuously. The significant improvement in experimental techniques achieved in recent years has led to a veritable revolution in nuclear spectroscopy. Where until recently the probabilities of electromagnetic transitions were known only in a few cases, tens of differential cross sections for the scattering of fast electrons and protons with the excitation of individual states $|s\rangle$ have now been measured with high accuracy and the transition densities

$$\rho_{sr}(\mathbf{r}) = \langle s | \psi^+(\mathbf{r}) \psi(\mathbf{r}) | 0 \rangle -$$

extracted. These are functions of the coordinates and contain immeasurably richer information about nuclear structure. Thus, the finding of the few parameters of the quasiparticle Lagrangian is a perfectly realistic problem.

The method developed here can also be applied to other Fermi systems, for example, liquid ^{3}He and quantum crystals.

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APPENDIX 1

In this Appendix, we consider the problem of the existence of the quasiparticle Lagrangian L_q, and we also compare the Lagrangian and Hamiltonian formulations of the problem.

We begin with the quasiparticle Hamiltonian H_q. Using the methods of the theory of finite Fermi systems, we calculate the change in the ground-state energy $E_0 = \langle \Phi_0 | H | \Phi_0 \rangle$ when the number of particles is changed. In accordance with the general formula, the change in the expectation value of the operator Q on the addition of

a particle to the state λ_0 near the Fermi surface is given by

$$\delta Q=(\mathcal{T}(Q)\delta_0 G_q), \tag{A.1}$$

where $\delta_0 G_q$ is the change in the pole part G_q of the Green's function G associated with the change in the rules of pole avoidance. When a quasiparticle appears in the state λ_0,

$$\delta_0 G_q=2\pi i\psi_{\lambda_0}^*(\mathbf{r}_1)\psi_{\lambda_0}(\mathbf{r}_2)\delta(\varepsilon-\varepsilon_{\lambda_0}),$$

and the vertex part is $\mathcal{T}(Q)=Q+\mathcal{U}GG\mathcal{T}(Q)$.

The vertex part $\mathcal{T}(H)$ was calculated earlier in Ref. 7:

$$\mathcal{T}(H)=\varepsilon_p^0+\Sigma(\mathbf{r},\mathbf{p},\varepsilon)-\varepsilon(\partial\Sigma/\partial\varepsilon)_0. \tag{A.2}$$

Therefore, in accordance with (A.1), we can write

$$\delta E_0=\sum_\lambda\left(\psi_\lambda,\left[\varepsilon_p^0+\Sigma(\varepsilon_\lambda)-\varepsilon_\lambda\left(\frac{\partial\Sigma}{\partial\varepsilon}\right)_0\right]\psi_\lambda\right)\delta n_\lambda$$

and, ignoring the terms of order $A^{-2/3}$, we can obtain

$$\delta E_0=\sum_\lambda(\varphi_\lambda,h\varphi_\lambda)\delta n_\lambda; \tag{A.3}$$

$$\varphi_\lambda=a^{1/2}\psi_\lambda,\quad h=a^{1/2}\varepsilon_p^0 a^{1/2}+a^{1/2}\Sigma(\mu)a^{1/2}-a\mu(\partial\Sigma/\partial\varepsilon)_0.$$

Using the formulas of Sec. 2, we can verify that φ_λ are orthonormalized eigenfunctions of the single-particle Hamiltonian h:

$$h\varphi_\lambda=\varepsilon_\lambda\varphi_\lambda. \tag{A.4}$$

Using (A.4), we can rewrite (A.3) in the coordinate representation:

$$\delta E_0=\int h(\mathbf{r},\mathbf{p})\delta n(\mathbf{r},\mathbf{r}')|_{\mathbf{r}'\to\mathbf{r}}d\mathbf{r}, \tag{A.5}$$

where δn is the variation of the density matrix

$$n(\mathbf{r},\mathbf{r}')=\sum n_\lambda\varphi_\lambda^*(\mathbf{r})\varphi_\lambda(\mathbf{r}')$$

due to the redistribution of the quasiparticles at the Fermi surface:

$$\delta n(\mathbf{r},\mathbf{r}')=\sum_\lambda\varphi_\lambda^*(\mathbf{r})\varphi_\lambda(\mathbf{r}')\delta n_\lambda+\sum_\lambda n_\lambda\delta(\varphi_\lambda^*(\mathbf{r})\varphi_\lambda(\mathbf{r}')). \tag{A.6}$$

The variational relation (A.5) can be extended to arbitrary variations of the population numbers of the quasiparticles or their wave functions. We shall call the corresponding functional E_q, determined by the relation

$$\delta E_q=\int\delta\mathcal{H}_q(\mathbf{r})\,d\mathbf{r}=\int h(\mathbf{r},\mathbf{p})\delta n(\mathbf{r})\,d\mathbf{r} \tag{A.7}$$

for arbitrary variations of the quasiparticle density matrix $n(\mathbf{r})$, the quasiparticle energy of the system. It can be seen from (A.7) that the minimum of the functional E_q is attained on the quasiparticle wave functions φ_λ determined by (A.4). Note also that if we add a quasiparticle to some state λ_1, i.e., $\delta n=\delta_{\lambda\lambda_1}$, then it follows from (A.7) that

$$\delta E_q=h_{\lambda_1\lambda_1}=\varepsilon_{\lambda_1},$$

from which we obtain for the difference between the quasiparticle energies of neighboring nuclei, which determines the chemical potential of the nucleus,

$$E_q(N+1)-E_q(N)=\mu_q=\varepsilon_{\lambda_0},$$

where ε_{λ_0} is the energy of the last occupied level.

To construct E_q on the basis of (A.7), it is necessary to know the functional dependence of h on n. It is here that the main shortcoming of the Hamiltonian formalism is revealed, namely, the consistency condition does not reduce to a simple formula that permits the finding of an analytic connection between h and n. However, for the mass operator Σ_q and the density γ_0 an analytic connection can be established relatively easily, as we have seen. This makes the Lagrangian formulation of the problem preferable.

To construct the quasiparticle Lagrangian L_q, we write down the change in the expectation value of the Lagrangian

$$L=\int\mathcal{L}\,d\mathbf{r}$$

of the system with respect to the ground state as produced by a variation of the quasiparticle population:

$$\delta\langle L\rangle=(\mathcal{T}(L)\delta_0 G_q). \tag{A.8}$$

The vertex part $\mathcal{T}(L)$ can be calculated in the same way as $\mathcal{T}(H)$. We then obtain the result

$$\mathcal{T}(L)=\varepsilon-\varepsilon_p^0-\Sigma. \tag{A.9}$$

Using the fact that the matrix elements of $\mathcal{T}(L)$ vanish for physical states, we can rewrite $\delta\langle L\rangle$ for variations of G_q near the Fermi surface in the form

$$\begin{aligned}\delta\langle L\rangle&=\int_C\frac{\partial\varepsilon}{2\pi i}d\mathbf{r}(\varepsilon-\varepsilon_p^0-\Sigma_q(\mathbf{r},\mathbf{p},\varepsilon))\delta G_q(\mathbf{r},\mathbf{r}',\varepsilon)|_{\mathbf{r}'\to\mathbf{r}}\\&=\int d\mathbf{r}\{(1-\Sigma_2)\delta\nu_2-(1+\Sigma_1)\delta\nu_1-\Sigma_0\delta\nu_0\}.\end{aligned} \tag{A.10}$$

It follows that the coefficients of $\delta\nu_0$, $\delta\nu_1$, and $\delta\nu_2$ are not arbitrary but must be connected by the relations which follow from the equality of the mixed derivatives. The fulfillment of these equalities makes it possible to extend formula (A.10) to arbitrary variations $\delta\nu_1$, the quasiparticle Lagrangian L_q being determined by the same equation (A.10) but for arbitrary variations.

If the quasiparticle Lagrangian L_q is known, we can readily calculate for it the current density, the energy-momentum tensor, and so forth. At the same time, using the expressions for the variation $\delta\nu_1$ from Sec. 3 and the definition (16), we can write down for the current density vector

$$\begin{aligned}j_\alpha&=i\sum_\lambda\left(\psi_\lambda^*\frac{\delta L_q}{\delta(\nabla_\alpha\psi_\lambda^*)}-\psi_\lambda\frac{\delta L_q}{\delta(\nabla_\alpha\psi_\lambda)}\right)\\&=\frac{\delta L_q}{\delta\nu_1}\frac{i}{2m}\sum_\lambda n_\lambda(\psi_\lambda^*\nabla_\alpha\psi_\lambda-\psi_\lambda\nabla_\alpha\psi_\lambda^*)=\frac{\delta L_q}{\delta\nu_1}j_\alpha^0,\end{aligned} \tag{A.11}$$

or its fourth component (the "charge" density)

$$n(\mathbf{r})=i\sum_\lambda\left(\psi_\lambda^*\frac{\delta L_q}{\delta\psi_\lambda^*}-\psi_\lambda\frac{\delta L_q}{\delta\psi_\lambda}\right)=\nu_0(\mathbf{r})\frac{\delta L_q}{\delta\nu_0}, \tag{A.12}$$

and, finally, for the density of the quasiparticle Hamiltonian corresponding to the Lagrangian L_q

$$\mathcal{H}_q(\mathbf{r})=\sum_\lambda\left(\psi_\lambda^*\frac{\delta L_q}{\delta\psi_\lambda^*}+\psi_\lambda\frac{\delta L_q}{\delta\psi_\lambda}\right)-\mathcal{L}_q=\frac{\delta L_q}{\delta\nu_0}\nu_0-\mathcal{L}_q.$$

It is obvious that the Hamiltonian $\mathcal{H}_q$ defined in this manner is identical to the Hamiltonian introduced above.

APPENDIX 2

We here renormalize the consistency condition, i.e., the transition from (20) to (27), more rigorously. In

Sec. 5, this transition is made under the assumption that the dependence of the interaction amplitude Λ on $\xi_i(1)$ and $\xi_k(2)$ is linear. As we shall now show, the same result can be obtained in the general case if the propagator A in the decomposition (21) of the product GG into $A+B$ is appropriately defined. We recall that (22) contains the zeroth harmonic A^0 of A with respect to the angle between the momenta $\mathbf{p}_1$ and $\mathbf{p}_2$. The usual definition (28) for A^0 given in the theory of finite Fermi systems can be regarded as the first term in the expansion of the product G_qG_q in a series in δ functions of $\varepsilon-\mu$ and $p^2-p_F^2(\mathbf{r})$ and their derivatives. To obtain Eq. (27), it is necessary to include in A^0 the following terms of the expansion. To avoid cumbersome expressions, we shall regard ξ_i as the components of the "vector" $\xi=(1,p^2,\varepsilon)$. We also introduce the vector $\xi^0=[0,p_F^2(\mathbf{r}),\mu]$ and the vector δ function

$$\delta(\xi-\xi)^0=1\cdot\delta(p^2-p_F^2(\mathbf{r}))\cdot\delta(\varepsilon-\mu).$$

Finally, we symmetrize the product G_qG_q formally with respect to the energies and the momenta, denoting

$$G_qG_q=K_0(\mathbf{r}_1,\mathbf{p}_1,\mathbf{r}_2,\mathbf{p}_2;\varepsilon_1,\varepsilon_2)2\pi i\delta(\varepsilon_1-\varepsilon_2)=K_0(1;2). \quad \text{(A.14)}$$

We now define A^0 as follows:

$$A^0(1;2)=\sum_{i,k=0,1,2} z_{ik}\bar{A}_{ik}^0(\mathbf{r}_1,\mathbf{r}_2)\frac{\partial}{\partial\xi_i(1)}\frac{\partial}{\partial\xi_k(2)} \times[\delta(\xi(1)-\xi^0(1))\delta(\xi(2)-\xi^0(2))], \quad \text{(A.15)}$$

where $\partial/\partial\xi_0\equiv 1$, and z_{ik} is a sign factor, equal to $z_{01}=z_{10}=z_{02}=z_{20}=1$ and $z_{ik}=1$ for other i,k. The coefficients $\bar{A}^0_{i,k}$ in (A.15) are

$$\bar{A}_{ik}^0(\mathbf{r}_1,\mathbf{r}_2)=\int K_0(1;2)(\xi_i(1)-\xi_i^0(1))(\xi_k(2)-\xi_k^0(2)) \cdot\frac{d\mathbf{p}_1d\mathbf{p}_2}{(2\pi)^6}\frac{d\varepsilon_1d\varepsilon_2}{(2\pi i)^2} \quad \text{(A.16)}$$

They are related to the functions $A^0_{ik}(\mathbf{r}_1,\mathbf{r}_2)$ defined by Eq. (26) by

$$\bar{A}_{ik}^0(\mathbf{r}_1,\mathbf{r}_2)=A_{ik}^0(\mathbf{r}_1,\mathbf{r}_2)-\xi_i^0(1)A_{0k}(\mathbf{r}_1,\mathbf{r}_2) -A_{i0}^0(\mathbf{r}_1,\mathbf{r}_2)\xi_k^0(2)+\xi_i^0(1)A_{00}(\mathbf{r}_1,\mathbf{r}_2)\xi_k^0(2). \quad \text{(A.17)}$$

Substitution of (A.15) in (22) gives

$$\frac{\partial\Sigma(1)}{\partial\mathbf{r}_1}=\int\left(\frac{\partial\Lambda^0(1;2)}{\partial\xi_i(2)}\right)_0\bar{A}_{ik}(\mathbf{r}_2,\mathbf{r}_3)\frac{\partial}{\partial\mathbf{r}_3}\left(\frac{\partial\Sigma(3)}{\partial\xi_k(3)}\right)_0 d\mathbf{r}_2\,d\mathbf{r}_3, \quad \text{(A.18)}$$

where the subscript 0 means that the derivative is taken at $\varepsilon=\mu$ and $p^2=p_F^2(\mathbf{r})$. Simple calculations lead to the relation

$$\int\bar{A}_{ik}^0(\mathbf{r}_1,\mathbf{r}_2)\frac{\partial}{\partial\mathbf{r}_2}\left(\frac{\partial\Sigma(2)}{\partial\xi_k(2)}\right)_0 d\mathbf{r}_2=\frac{\partial v_i(\mathbf{r}_1)}{\partial\mathbf{r}_1}-\xi_i^0(1)\frac{\partial v_0(\mathbf{r}_1)}{\partial\mathbf{r}_1}, \quad \text{(A.19)}$$

by means of which we can readily obtain from (A.18) the required equation (27), in which the coefficients Λ^0_{ik} are expressed in terms of the zeroth harmonic $\Lambda^0(1,2)$ of the interaction amplitude and its derivative at the Fermi surface as follows:

$$\Lambda_{00}(\mathbf{r}_1,\mathbf{r}_2)=\sum_{i,k}z_{ik}\xi_i^0(1)\left(\frac{\partial^2\Lambda(1;,2)}{\partial\xi_i(1)\partial\xi_k(2)}\right)_0\xi_k^0(2),$$

$$\Lambda_{10}(\mathbf{r}_1,\mathbf{r}_2)=\Lambda_{01}(\mathbf{r}_2,\mathbf{r}_1)=-\sum_k z_{1k}\left(\frac{\partial^2\Lambda(1;2)}{\partial\xi_1(1)\partial\xi_k(2)}\right)_0\xi_k^0(2),$$

$$\Lambda_{20}(\mathbf{r}_1,\mathbf{r}_2)=\Lambda_{02}(\mathbf{r}_2,\mathbf{r}_1)=-\sum_k z_{2k}\left(\frac{\partial^2\Lambda(1;2)}{\partial\xi_2(1)\partial\xi_k(2)}\right)_0\xi_k^0(2),$$

$$\Lambda_{11}(\mathbf{r}_1,\mathbf{r}_2)=\left(\frac{\partial^2\Lambda(1;2)}{\partial\xi_1(1)\partial\xi_1(2)}\right)_0,\quad \Lambda_{22}(\mathbf{r}_1,\mathbf{r}_2)=\left(\frac{\partial^2\Lambda(1;2)}{\partial\xi_2(1)\partial\xi_2(2)}\right)_0$$

$$\Lambda_{12}(\mathbf{r}_1,\mathbf{r}_2)=\Lambda_{21}(\mathbf{r}_2,\mathbf{r}_1)=\left(\frac{\partial^2\Lambda(1;2)}{\partial\xi_1(1)\partial\xi_2(2)}\right)_0.$$

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