Oscillations and instability of a multicomponent gravitating medium

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It is shown that in an n-component homogeneous gravitating medium there are not only acoustic (Jeans) oscillations, which are synphasic, but also a new class of "asynphasic" oscillations, which are related to Langmuir plasma oscillations or optical vibrations of a crystal lattice. In contrast to the acoustic branches, the new branches remain oscillatory for arbitrarily large wavelengths, which changes the traditional picture of unavoidable gravitational condensation of all long-wavelength perturbations. Two theorems concerning the stability of a heterogeneous gravitating medium are formulated. In accordance with the first theorem, which holds in the special case when the components are at rest, there are $n-1$ different asynphasic branches of oscillations; the Jeans's instability can develop only in the synphasic branch. The second theorem is formulated for the general case of components moving relative to each other, and it determines the number of stream instabilities as a function of the relationships between the stationary parameters of the components.

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§1. GENERAL COMMENTS

Gravitational condensation is the reason for the formation of the basic astrophysical objects, i.e., stars and systems of them (from star clusters to clusters of galaxies). In his classic work Ref. 1, Jeans described the criterion of gravitational condensation of perturbations in a homogeneous medium. The essence of this criterion is that in a homogeneous gravitating medium oscillations with wavelength λ greater than some critical λ_{cr} cannot exist. According to Jeans, all perturbations with $\lambda > \lambda_{cr}$ collapse. The meaning of this criterion is extremely clear, namely, an increase in the scale of a perturbation increases the mass and, therefore, the perturbed gravitational force and decreases the pressure gradient. As a result, the first force is predominant, which leads to collapse of the perturbations.

We should note that cases of a homogeneous gravitating medium for which the above picture of gravitational condensation is valid are extremely rare.[1)] For example, our Galaxy consists of several star and gas components, i.e., it is a heterogeneous system. Thus, if heterogeneity is a necessary property of an astrophysical system, it is natural to consider what is the criterion of gravitational condensation for a heterogeneous system.

Hitherto, this problem has been solved trivially—the Jeans criterion for a homogeneous sytem has been used. The reason for this "inductive approach" is obvious in the present case; the Jeans criterion is so simple, and the physical mechanism underlying it—the balance of two forces, the gravitational and the pressure forces—is so universal that the possibility of using the criterion for more complicated systems was not doubted.

As is shown in the present paper, the picture described by Jeans is qualitatively changed in a heterogeneous medium. Even in the simplest case of components at rest (considered in Sec. 2) it is possible to have oscillations with wavelength appreciably exceeding the greatest of the Jeans wavelengths of the individual components. It is necessary to consider how perturbations can exist in the form of oscillations and without collapsing when they have a scale so large that the elastic force—the pressure gradient for each of the components—becomes much less than the gravitating force of contraction.[2)]

The existence in a heterogeneous system of characteristic oscillations with arbitrarily large wavelength is due to the establishment of phase relations between the oscillations of the individual components. As a result, the total amplitude of the density wave is less than the amplitudes of each of the oscillating components. For example, in a two-component gravitating medium the oscillations of the density in the two different components are antiphased, i.e., the density minima in one of the components coincide with the density maxima in the other. On the one hand, an increase in the ratio λ/λ_{cr} leads to an increase in the ratio F_{grav}/F_{press} (and it can be made much greater than unity for each of the components). On the other hand, the increase in λ/λ_{cr} leads to mutual suppression of the density perturbations, which compensates the first effect. Of course, such a mechanism is impossible in a homogeneous system.

The asynphasic oscillations investigated in the paper are related to Langmuir oscillations in a plasma,[5] optical vibrations of a crystal lattice,[6] the vibrations of molecules,[7] and so forth.

In Sec. 3, we investigate stream instabilities whose conditions of excitation are identical to the analogous condition of the plasma two-stream instability.[8] The gravitational stream instabilities develop on the background of the ordinary Jeans instability of the entire medium in the same range of wavelengths. For this reason, they are not so readily distinguished from the Jeans instability; for this, as is shown in Ref. 9, a specially prepared gravitating system is required.

§2. THE CASE OF COMPONENTS AT REST

If a spatially unbounded medium is heterogeneous and consists, for example, of a hot and a cold component, then in this case there are two Jeans scales, one for the cold component, λ_c, and one for the hot, λ_h. Let us consider how the perturbations with different wavelengths behave in such a heterogeneous medium. We shall restrict ourselves to the case of one-dimensional perturbations propagating along the x axis. Thus, all the perturbed quantities depend only on x, and the stationary parameters are spatially homogeneous. We shall later generalize the relations that we obtain for a two-component medium to the arbitrary case of n components.

For the cold component of the medium, the linearized equations of motion and continuity are

$$\frac{\partial v_c}{\partial t}=-\frac{\partial\psi}{\partial x}-\frac{1}{\rho_{0c}}\frac{\partial P_c}{\partial x},\quad \frac{\partial\rho_c}{\partial t}+\frac{\partial}{\partial x}(\rho_{0c}v_c)=0.$$

Here, v_c is the x component of the velocity, ρ_c is the density, P_c is the pressure, and ψ is the gravitational potential. We shall distinguish the stationary quantities from the perturbed quantities by the subscript 0.

For the hot component, we have an analogous system of equations:

$$\frac{\partial v_h}{\partial t}=-\frac{\partial\psi}{\partial x}-\frac{1}{\rho_{0h}}\frac{\partial P_h}{\partial x},\quad \frac{\partial\rho_h}{\partial t}+\frac{\partial}{\partial x}(\rho_{0h}v_h)=0.$$

The two systems of equations are related by the Poisson equation

$$\frac{\partial^2\psi}{\partial x^2}=4\pi G(\rho_c+\rho_h).$$

We assume that the perturbations are adiabatic and that the pressure $P=P(\rho)$ is related linearly to the density for the perturbed quantities:

$$P=c_s^2\rho,$$

where $c_s^2=\partial P_0/\partial\rho_0$ is the square of the velocity of sound. Obviously, there is an equation of state for each of the components of the medium.

Since the medium is spatially homogeneous, we choose the perturbed functions in the form

$$\psi(x,t)=\sum_k\psi_{\omega,k}e^{ikx-i\omega t},$$

and since the equations are linear, it is sufficient to consider an arbitrary harmonic of the Fourier series. Thus

$$\psi(x,t)=\psi_{\omega,k}e^{ikx-i\omega t}.$$

Omitting in what follows the subscripts ω and k of the perturbed quantities, we readily arrive at the system of equations

$$\rho_c=\rho_{0c}\frac{k^2\psi}{\omega^2-k^2c_{sc}^2},\quad \rho_h=\rho_{0h}\frac{k^2\psi}{\omega^2-k^2c_{sh}^2},$$

from which we obtain the required dispersion relation

$$\frac{\omega_{0c}^2}{\omega^2-k^2c_{sc}^2}+\frac{\omega_{0h}^2}{\omega^2-k^2c_{sh}^2}=-1.$$

Here, $\omega_{0c}^2=4\pi G\rho_c$, $\omega_{0h}^2=4\pi G\rho_h$. Denoting the left-hand side of this equation by $f(\omega^2)$, we represent this function schematically in Fig. 1(a). The roots of the dispersion equation correspond to the points of intersection of the curves $f(\omega^2)$ with the straight line $f=-1$. It can be seen

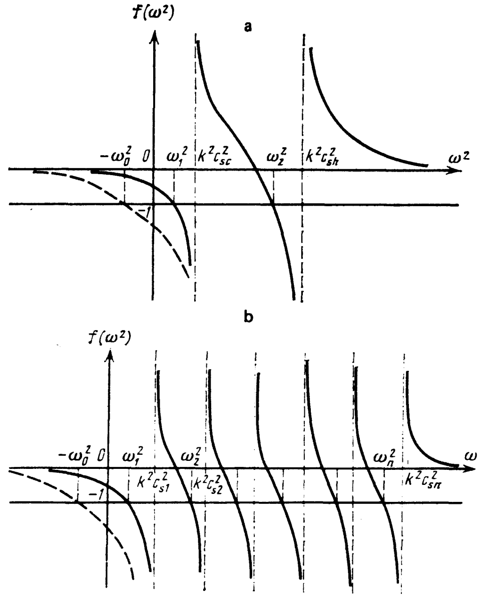

FIG. 1. a) The case of a 2-component medium ($v_{0i}=0$); b) the case of an n-component medium ($v_{0i}=0$).

from Fig. 1(a) that the dispersion relation can have only one negative root $\omega^2=-\omega_0^2$, which is when the left-hand curve $f(\omega^2)$ has a behavior similar to the broken curve (intersects the straight line $f=-1$ at negative values of ω^2). But if the left-hand curve $f(\omega^2)$ has a behavior similar to the continuous curve, then both roots are positive, $\omega_{1,2}^2>0$.

The above analysis of the roots of the dispersion relation for the two-component system can be readily generalized to the case of an n-component system. For an arbitrary i-component gas, its perturbed density is

$$\rho_i=\rho_{0i}\frac{k^2\psi}{\omega^2-k^2c_{si}^2},$$

and, therefore, the dispersion relation for the n-component system has the form

$$\sum_{i=1}^{n}\frac{\omega_{0i}^2}{\omega^2-k^2c_{si}^2}=-1,$$

where $\omega_{0i}^2=4\pi G\rho_{0i}$. We again denote the left-hand side of this equation as $f(\omega^2)$; we have plotted $f(\omega^2)$ schematically in Fig. 1(b). It can be seen from Fig. 1(b) that the dispersion relation for the n-component system can have only one negative root ($\omega^2<0$), which is when the extreme left-hand curve $f(\omega^2)$ has a behavior similar to that of the broken curve. All the remaining roots are positive.

Thus, the following theorem can be formulated.[3)]

In a heterogeneous system consisting of n homogeneous components, only a single aperiodic instability can develop, this occurring when

$$\sum_{i=1}^{n}\frac{\omega_{0i}^2}{k^2c_{si}^2}>1.$$

The remaining $n-1$ collective modes are asynphasic oscillations.

We must now consider what wavelengths participate in the aperiodic instability of a heterogeneous system.

To establish this, we restrict ourselves to the case of a two-component medium.

The solution of the dispersion relation for the two-component medium is

$$\omega^2=\frac{1}{2}\{-(\omega_h^2+\omega_c^2)\pm((\omega_h^2-\omega_c^2)^2+4\omega_{0h}^2\omega_{0c}^2)^{1/2}\},$$

where $\omega_h^2=\omega_{0h}^2-k^2c_{sh}^2$, $\omega_c^2=\omega_{0c}^2-k^2c_{sc}^2$. We consider the following four limiting cases:

1) The wavelength of the perturbation is less than the Jeans length in both the hot and the cold components, $\lambda\ll\lambda_c, \lambda_h$.

From the equivalent inequalities $k^2c_{sc}^2\gg\omega_{0c}^2$ and $k^2c_{sh}^2\gg\omega_{0h}^2$ we obtain $\omega_c^2\approx-k^2c_{sc}^2$, $\omega_h^2\approx-k^2c_{sh}^2$, which after substitution in the solution of the dispersion relation leads to two oscillation branches,

$$\omega_1^2\approx k^2c_{sc}^2, \quad \omega_2^2\approx k^2c_{sh}^2,$$

each of which is determined by the parameters of the hot and cold components of the medium, respectively.

2) The wavelength of the perturbation is less than the Jeans length in the cold component and greater than the Jeans length in the hot component, $\lambda_c\gg\lambda\gg\lambda_h$.

The equivalent inequalitites $k^2c_{sc}^2\gg\omega_{0c}^2$ and $k^2c_{sh}^2\ll\omega_{0h}^2$ can be written together as $\omega_{0h}^2\gg k^2c_{sh}^2\gg k^2c_{sh}^2\gg\omega_{0c}^2$. We see that in this case the density of the hot component by many times the density of the cold component, $\omega_{0h}^2\gg\omega_{0c}^2$.

From the system of inequalities, we obtain $\omega_h^2\approx\omega_{0h}^2$, $\omega_c^2\approx-k^2c_{sc}^2$. Using these values of the squares of the characteristic frequencies, we find the following solutions of the dispersion relation:

$$\omega_1^2\approx-\omega_{0h}^2, \quad \omega_2^2\approx k^2c_{sc}^2.$$

The first of these solutions describes a Jeans instability whose growth rate is determined by the density of the hot component. The second solution describes asynphasic oscillations in the heterogeneous medium with velocity equal to the velocity of sound in the cold component by itself.

3) The wavelength of the perturbation is greater than the Jeans length in the cold component and less than it in the hot component, $\lambda_h\gg\lambda\gg\lambda_c$.

From the equivalent inequalities $k^2c_{sc}^2\ll\omega_{0c}^2$ and $k^2c_{sh}^2\gg\omega_{0h}^2$ we find $\omega_h^2\approx-k^2c_{sh}^2$, $\omega_c^2\approx\omega_{0c}^2$. The solutions of the dispersion relation in this case are

$$\omega_1^2\approx-\omega_{0c}^2, \quad \omega_2^2\approx k^2c_{sh}^2.$$

The first solution describes an aperiodic instability of the heterogeneous system whose growth rate is equal to the Jeans growth rate of the cold component of the medium by itself. The second solution describes asynphasic oscillations propagating in the heterogeneous system with frequency equal to the acoustic frequency of the oscillations in just the hot component.

4) The wavelength of the perturbation is greater than the Jeans lengths in both the cold and the hot component, $\lambda\gg\lambda_c, \lambda_h$.

From the equivalent system of inequalities $k^2c_{sc}^2\ll\omega_{0c}^2$ and $k^2c_{sh}^2\ll\omega_{0h}^2$ we find $\omega_h^2\approx\omega_{0h}^2$, $\omega_c^2\approx\omega_{0c}^2$. The corresponding solutions of the dispersion relation are

$$\omega_1^2\approx-\omega_0^2, \quad \omega_2^2\approx(k^2c_{sh}^2\omega_{0c}^2+k^2c_{sc}^2\omega_{0h}^2)/\omega_0^2,$$

where $\omega_0^2\equiv\omega_{0h}^2+\omega_{0c}^2$. The first solution describes the Jeans instability of a heterogeneous system for which the square of the growth rate is equal to the sum of the squares of the Jeans growth rates of the instabilities that would each develop by itself in the corresponding component of the medium alone. The second solution describes the propagation of asynphasic oscillations.

Obviously, it is not difficult to generalize these results to the case of an n-component medium. In summarizing, it is just such a system that we shall have in mind.

Thus, 1) perturbations with wavelength smaller than the smallest of the n Jeans lengths lead to the occurrence of n different oscillation branches, each of which is determined by the parameters of its corresponding subsystem; 2) perturbations with wavelength greater than the Jeans lengths of m components of the medium and less than the Jeans lengths of $n-m$ components lead to instability of the system, the square of the eigenfrequency being equal to the sum of the squares of the m Jeans frequencies taken with the minus sign,

$$\omega^2\approx-\sum_{i=1}^{m}\omega_{0i}^2,$$

and to the occurrence of $n-1$ branches of asynphasic oscillations. For $m=n$, we arrive at the last case, 4, considered above.

The existence of oscillations with arbitrarily large wavelength in the heterogeneous system is explained by the fact that the density oscillations in the different components compensate each other. Indeed, it can be seen from Figs. 1(a) and 1(b) that with the exception of the extreme left-hand root, which can correspond to instability, and other root i lies between $k^2c_{si}^2$ and $k^2c_{s(i+1)}^2$, i.e., $k^2c_{si}^2<\omega^2<k^2c_{s(i+1)}^2$. As follows from the expressions for the perturbed densities, it is precisely the terms ρ_i and ρ_{i+1} that make the largest contribution for root i. Since $\omega^2-k^2c_{si}^2>0$ but $\omega^2-k^2c_{s(i+1)}^2<0$, the signs of ρ_i and ρ_{i+1} are opposite.

§3. THE CASE OF MOVING COMPONENTS, $v_{0i}\neq0$

The picture described above changes qualitatively if the components of the medium have velocities relative to one another with a magnitude exceeding the corresponding velocities of sound. Then nongrowing oscillations occur at wavelengths less than the Jeans length (see Sec. 2, Case 1). In the opposite case (Case 4 in Sec. 2), when the wavelength of the perturbation exceeds the Jeans lengths, asynphasic oscillations are excited, and in this case n different instabilities develop in the system.

If the unperturbed velocity of the cold component is v_{0c}, and that of the hot v_{0h}, then the perturbed densities of each of the components are now

$$\rho_c=\rho_{0c}\frac{k^2\psi}{(\omega-kv_{0c})^2-k^2c_{sc}^2}$$

$$\rho_h=\rho_{0h}\frac{k^2\psi}{(\omega-kv_{0h})^2-k^2c_{sh}^2}$$

and the corresponding dispersion relation is

$$\frac{\omega_{0c}^2}{(\omega-kv_{0c})^2-k^2c_{sc}^2}+\frac{\omega_{0h}^2}{(\omega-kv_{0h})^2-k^2c_{sh}^2}=-1,$$

and for n components

$$\sum_{i=1}^{n}\frac{\omega_{0i}^2}{(\omega-kv_{0i})^2-k^2c_{si}^2}=-1.$$

The roots of this equation determine in general form the solution of the problem we have posed.

We consider first the simplest example when the density and pressure of the cold and hot components are equal: $\omega_{0c}^2=\omega_{0h}^2\equiv\omega_0^2/2$, $c_{sc}^2=c_{sh}^2\equiv c_s^2$. In the frame in which $v_{0c}=-v_{0h}\equiv v_0$, we have the dispersion relation

$$\frac{\omega_0^2}{(\omega-kv_0)^2-k^2c_s^2}+\frac{\omega_0^2}{(\omega+kv_0)^2-k^2c_s^2}=-2.$$

The solution of this equation is

$$\omega^2=k^2(v_0^2+c_s^2)-\omega_0^2/2\pm(4k^4v_0^2c_s^2-2\omega_0^2k^2v_0^2+\omega_0^4/4)^{1/2}.$$

We take three different limiting cases: a) $\omega_0^2\gg k^2v_0^2$, $k^2c_s^2$; b) $\omega_0^2\ll k^2v_0^2$, $k^2c_s^2$; c) $k^2v_0^2\gg\omega_0^2\gg k^2c_s^2$.

In case a), we obtain two roots:

$$\omega^2\approx-\omega_0^2,\quad \omega^2\approx-k^2(v_0^2-c_s^2).$$

The first root describes the Jeans instability, the second a two-stream instability for $|v_0|>c_s$. As we see, the necessary condition for the two-stream instability for the heterogeneous gravitating medium is identical with the analogous condition in a plasma.[8]

In case b), both roots are positive, which corresponds to an oscillatory regime.

In case c), we have

$$\omega^2=k^2(v_0^2+c_s^2)\pm i\sqrt{2}\omega_0kv_0.$$

Thus, two roots describe growing solutions, and the other two damped solutions.

We represent the dispersion relation obtained above for the case of streams of equal densities and equal (in modulus) velocities in the form

$$f(\omega)=-2.$$

The function $f(\omega)$ is shown graphically in Figs. 2(a) and 2(b) for the cases $v_0^2\ll c_s^2$ and $v_0^2\gg c_s^2$, respectively.

As follows from Fig. 2(a), in the case $v_0^2\ll c_s^2$ it is only possible to have the Jeans instability (under the condition $\omega_0^2>k^2c_s^2$, which is represented by the broken curve, the two roots ω_2 and ω_3 on the real axis are absent). When $\omega_0^2<k^2c_s^2$, all four roots are real.

In the case $v_0^2\gg c_s^2$, as can be seen from Fig. 2(b), all the roots are real when $\omega_0^2<2k^2c_s^2$, while for $\omega_0^2>2k^2c_s^2$ they are all imaginary. If the latter condition is satisfied, the two-stream instability develops in the system as well as the Jeans instability.

We now turn to the formulation of a theorem about the number of unstable roots for the general case of n moving components.

We label the components of the heterogeneous system in order of decrease of the numbers $(v_0+c_s)_i$ corresponding to them:

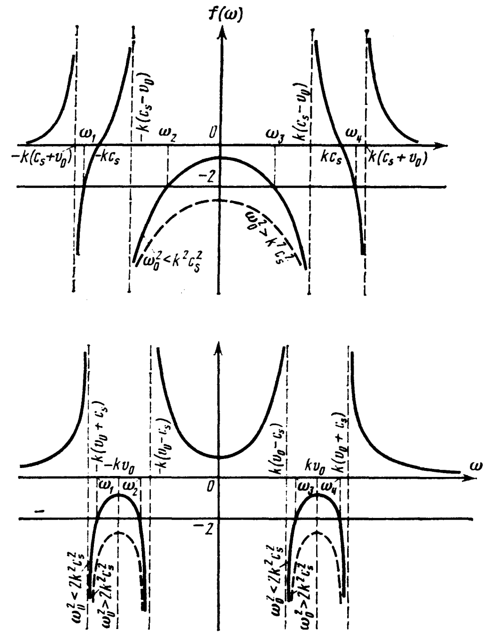

FIG. 2. The case of two beams of equal (in modulus) densities and velocities: a) $v_0^2\ll c_s^2$, b) $v_0^2\gg c_s^2$.

$$(v_0+c_s)_1>(v_0+c_s)_2>\ldots>(v_0+c_s)_n.$$

Each of these numbers determines a flow. We shall say that two arbitrary flows i and j $(i>j)$ are coupled if

$$\Delta v_{ij}\equiv v_{0i}-v_{0j}<c_i+c_j.$$

We shall distinguished *coupled* flows by brackets:

We shall put a stroke through brackets that are completely "covered" by some other brackets (1–2, 2–3, i–j in the given example). The remaining brackets determine *independent elements*. We also include the *uncoupled flows* (flow 5 in the given example) among the independent elements. We now have the following theorem.

The number of instabilities of a heterogeneous system with moving homogeneous flows is equal to the number of independent elements.

[1)]If the neutrino rest mass is assumed to be nonzero,[2] then there are no such cases at all.[3,4]

[2)]We recall that the pressure gradient is inversely proportional to the wavelength of the perturbation, while the gravitational force is directly proportional to it.

[3)]The content of this theorem for the case of two components was first stated by Ya. B. Zel'dovich during a seminar at the P. K. Shternberg State Astronomical Institute; the possibility of a generalization of it to the case of an arbitrary number of components was communicated to us by L. P. Grishchuk (for more details, see Ref. 10).

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