On the kinetics of the development of an instability on a charged liquid surface

L. P. Gor'kov and D. M. Chernikova

L. D. Landau Institute of Theoretical Physics, Academy of Sciences of the USSR
(Submitted 18 August 1980)
Zh. Eksp. Teor. Fiz. **80**, 1940–1951 (May 1981)

A physical picture qualitatively describing the various stages of the development of an instability on a charged helium surface in a strong electric field is presented. It is shown that the nonlinear mechanisms of interaction between surface waves play an important role in the damping of the turbulent motion. The process of electron escape from the surface is interpreted in terms of the so-called "shallow-water" theory: the turbulent motions occurring during a short-wave instability generated on the surface of the liquid first gently sloping waves and then surges, on which the charge accumulates. The characteristic times of the major processes are estimated.

PACS numbers: 68.10.Jy, 67.40.Vs

1. INTRODUCTION

Several experiments[1-4] have been performed in recent years to study the instability of a charged helium surface, predicted by us in Ref. 5. It has been shown[5] that the spectrum of gravitational capillary waves on the surface of a deep liquid carrying a surface charge en_s has the form

$$\rho\omega^2(k)=k\{\rho g+\alpha k^2-k(E_1^2+E_2^2)/4\pi\}, \quad (1)$$

where ρ is the density of the liquid, α is the coefficient of surface tension, and E_1 and E_2 are the electric-field intensities respectively above and below the surface of the liquid. Experimentally, the problem normally consists in the placing of the charged surface in a capacitor; the distances h_1 and h_2 from the upper and lower plates of the capacitor are assumed in (1) to be large compared to the capillary factor $a=(\alpha/\rho g)^{1/2}$. The electric fields are connected with the surface charge density by the standard relation

$$E_1-E_2=4\pi en_s. \quad (2)$$

The term with the electric field in (1) leads to the appearance in the surface-wave dispersion law of some minimum that can, at sufficiently high fields, make the square of the frequency negative. The field-intensity values at which the frequency first vanishes, i.e., the values given by the relation

$$(E_1^2+E_2^2)_{cr}=8\pi(\rho g\alpha)^{1/2}; \quad k_0=a^{-1}, \quad (3)$$

determine the system's linear-instability threshold. At this moment the small surface perturbations with wave vector in the vicinity of k_0 begin to build up unrestrictedly, leading to a cutoff of the solution and the transition from a plane surface to some new, inhomogeneous state with a deformed surface.

It would be natural to expect the scale of the deformation to correspond with the magnitude of the capillary factor, which for helium has the order of magnitude $a\approx 0.5$ mm, and the surface deformation itself to have the character of a periodic lattice with wave vector $k_0 \approx a^{-1}$. Such a "crystalline" two-dimensional surface state (a periodically rippled surface) has indeed been observed both in experiments with electrons on a helium surface[2] and for ions[3] held by a field at the stratification boundary of the two phases of the ^{3}He–^{4}He mixture. It was, however, found that the instability-cutoff phenomenon is accompanied by a surface-discharge process[1-3] in which the charge is carried away from the surface by small charged bubbles. The departure of the charge from the surface is not a uniquely defined process: while according to (1) the time at which the instability sets in is uniquely given by the universal field combination $E_1^2+E_2^2$ in accordance with (3), the question whether the charges remain at the surface in fields higher than the stability-threshold field depends on the conditions under which the cutoff occurs. Thus, Leiderer and Wanner[3] have observed that, if the instability is attained by increasing the applied potential difference at a time when the charge source is switched on [under these conditions one of the fields, E_1 or E_2, depending on the sign of the ion charge, is equal to zero, since charge flows from the source to the surface until the appropriate field is completely screened off in accordance with (2)], then the instability cutoff is accompanied by a discharge.

If, on the other hand, some surface charge density is produced on the surface under subcritical conditions, then the source is switched off, and the applied potential difference is gradually increased, then the charge may remain on the surface. In this case the charged surface becomes, after some period of relaxation, covered with periodic ripples, retaining the whole charge residing on it. The concentration range in which the transition into the new periodic state occurs as a result of the instability (1), (3) without loss of charge is, however, not clearly indicated in Ref. 3.

Reference 2 is even less definite in this respect. In the experiment described in this paper Wanner and Leiderer showed that if the instability is attained when the electron source is switched on (i.e., the charge concentration is maximal and equal to n_{sat}, which is determined from (2) and (3) under the condition that $E_1=0$), and the source is switched off at the moment when the instability begins to grow, then some of the electrons depart, leaving behind a mean electron density equal to $\bar{n}_s\approx 0.2n_{sat}$ on the surface, which is found at this point to be periodically rippled. The "two-dimensional crystal" formed can then be "melted" and "re-

 0038-5646/81/051008-07$02.40

crystallized" by respectively decreasing and increasing the potential difference across the capacitor. The question whether we can, by using another experimental procedure, obtain a periodic solution with a large surface charge was not investigated by Wanner and Leiderer.[2] Also not investigated in the stability of the new deformed phase in the region of high fields or concentrations. It also turns out that the behavior of an electron system on a helium surface[2] is in many respects different from that of ions captured by the interface between the two phases of a ^{3}He–^{4}He mixture.

Thus, summing the above brief discussion of the experimental situation, we note, first, the possible process of charge departure from the surface as a result of the development of the instability and, second, the dependence of the final state on the experimental procedure.

The system of charges on the surface has, at the moment when the electric fields attain the critical values determined by (3), excess energy (in the general case $\sim\alpha$ per unit area), which is then realized in a chaotic turbulent motion of the surface. The characteristics of the motion arising as a result of the cutoff of a plane instability of a boundary probably have quite a general character. Owing to the low viscosity of helium at low temperatures, the turbulent state could last for a long time. In the present paper we consider the nonlinear damping mechanism (in terms of the so-called weak surface-wave turbulence[6]), and show that the departure of charge from the surface under fairly general conditions follows even from hydrodynamic considerations.

2. TURBULENT VISCOSITY

As shown in Refs. 7 and 8, the development of the instability (3) occurs in a rigid regime. In other words, in the general case the problem does not contain a parameter of smallness, and the amplitudes of the resulting deformations and chaotic motion of the surface have the order of magnitude of the capillary factor a. The estimation of the relaxation time of the nonstationary transient conditions with the aid of the expression for the damping constant of gravitational waves with wave vector $k_0 = a^{-1}$,

$$\gamma = 2\nu k_0^2 \tag{4}$$

(ν is the kinematic viscosity), yields for the transient turbulent regime a lifetime of the order of tens of seconds. Experiment yields times of the order of tenths of a second. Below we shall for the most part have in mind electrons on a helium surface, since there is reason to believe[2,9] that the mobility of ions at the interface between the ^{3}He–^{4}He phases is low. The friction between the ions and the interface could be the determining dissipation mechanism for the excess energy connected with the instability. The low mobility of the ions would in turn violate one of the basic assumptions used below, namely, the assumption that we can consider the electrostatic potential to be constant along the surface of the liquid. This approximation, which may be called the approximation of a metallic film of charge on a surface, is, for electrons, fulfilled with a huge margin to spare.

Under the indicated conditions, and with the effects of the viscosity of helium neglected, the complete system of equations describing the motion of a charged helium boundary is contained in the relation

$$\left[\rho\frac{\partial\Psi}{\partial t}+\frac{1}{2}\rho v^2-\frac{1}{8\pi}(E_1^2-E_2^2)\right]_{z=\zeta}+\rho g\zeta+\alpha\left(\frac{1}{R_1}+\frac{1}{R_2}\right)=\text{const}, \tag{5}$$

which expresses the balance of the forces on the surface of the liquid, and the kinematic condition

$$v_z|_{z=\zeta}=\frac{\partial\zeta}{\partial t}+(\mathbf{v}_t\nabla\zeta)|_{z=\zeta} \tag{6}$$

(see Refs. 10 and 11; here Ψ is the velocity potential, ρ is the helium density, and ζ is the deviation of the surface from the plane shape).

An analytical investigation of the problem is, however, possible only when the distortion of the surface is sufficiently small ($\zeta \ll a$), i.e., if the terms in Eqs. (5) and (6) can be expanded in powers of ζ. It is shown in Refs. 7 and 12 that this requirement is equivalent to the condition that the parameter

$$S=\frac{E_1^2-E_2^2}{(E_1^2+E_2^2)_{cr}}\ll 1 \tag{7}$$

be small. The relation (7) expresses, in accordance with (2), the condition for the surface concentration of the electrons to be sufficiently low.

By a Fourier expansion of the surface deformation:

$$\zeta(\mathbf{r},t)=\sum_{\mathbf{k}}\zeta_{\mathbf{k}}(t)e^{i\mathbf{kr}}, \tag{8}$$

by separating the components with wave vectors lying near the circle with radius k_0 ($|\mathbf{k}-\mathbf{k}_0|\ll|\mathbf{k}_0|$), and by writing them in the form

$$\zeta_{\mathbf{k}}={}^1\!/_2 aZ_{\mathbf{k}} \tag{9}$$

(Z is a dimensionless amplitude), we find for the latter the equations of motion in the vicinity of the stability threshold:

$$\begin{aligned}\dot{Z}_{\mathbf{k}}/gk_0=&-(k-k_0)^2Z_{\mathbf{k}}/k_0^2+2\beta Z_{\mathbf{k}}-{}^3\!/_4 S\sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3}Z_{\mathbf{k}_1}Z_{\mathbf{k}_2}\\&+{}^1\!/_2\sum_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3=\mathbf{k}}R_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}Z_{\mathbf{k}_1}Z_{\mathbf{k}_2}Z_{\mathbf{k}_3},\\R_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}=&{}^1\!/_6\{{}^1\!/_2(\mathbf{nn}_1)^2+{}^1\!/_2(\mathbf{nn}_2)^2+{}^1\!/_2(\mathbf{nn}_3)^2\\&+4(2-2\mathbf{nn}_1)^{1/2}+4(2-2\mathbf{nn}_2)^{1/2}+4(2-2\mathbf{nn}_3)^{1/2}-12\}.\end{aligned} \tag{10}$$

($\mathbf{n}_i$ is the unit vector in the direction $\mathbf{k}_i$); the parameter β determines the proximity to the threshold:

$$\beta=(E_1^2+E_2^2)/(E_1^2+E_2^2)_{cr}-1. \tag{11}$$

It is shown in Refs. 7 and 12 that the expressions (10) can be rewritten in the form

$$\dot{Z}_{\mathbf{k}}/gk_0=-\delta U/\partial Z_{-\mathbf{k}}, \tag{10'}$$

where the functional $\tilde{U}$ is related with the energy U (per unit area) of the statically deformed surface:

$$U=\alpha\tilde{U}/4. \tag{12}$$

The system (10) determines the development of the instability in the vicinity of the thresholds (i.e., for $\beta \to 0$); the second-order terms in (10), which are responsible for the strong interaction between the two unstable

modes with wave vectors deployed at an angle of 60 degrees to each other, is responsible for the rigid hysteretic character of the transition. They lead to the result that the most advantageous lattice above the threshold under equilibrium conditions is the hexagonal lattice. Referring the reader to our previous papers[7,12] for details, we emphasize that, in the absence of viscosity, the Eqs. (10) in principle describe the motion of the liquid surface arising upon the attainment of the stability threshold ($\beta=0$). As follows from these equations, the amplitude of the surface deformations is of the order of

$$\zeta(\mathbf{r},\ t)\sim aZ(\mathbf{r},\ t)\sim aS, \tag{13}$$

the characteristic rate of variation in time is roughly equal to

$$(gk_0)^{1/2}S, \tag{13'}$$

and the excess energy accumulated by the system up to the moment of attainment of the stability threshold is of the order of

$$\alpha S^4. \tag{13''}$$

The presence of the small parameter S allows us to apply to the problem, under consideration here, of the decay of the unstable plane state of the surface the concepts developed in recent years in the theory of the so-called "weak turbulence" (see Ref. 13). A typical statement of the problem usually consists in the fact that the energy is pumped into a definite region of the vibrational spectrum of the system. (In this sense the total energy of the system should be small enough to make a description in terms of the spectrum of the system's normal modes possible.) The theory of wave turbulence assumes, similarly to Kolmogorov's hypothesis for ordinary turbulence, that the nonlinear interactions between the modes transfer energy to the region of waves with increasingly large wave vectors until the viscosity effects are, finally, no more decisive.

The vibrational spectrum (1) goes over in the $k\gg k_0$ region into a capillary-wave spectrum $\omega^2\sim(\alpha/\rho)k^3$, for which the wave-turbulence problem has been solved exactly.[6]

Let us now consider Fig. 1, in which the spectrum (1) at the moment of the onset of the instability is schematically depicted. The wave-vector region $\Delta k\sim k_0S$ plays the role of a "reservoir" where the excess energy (13') associated with the turbulent motion is stored. In this region the motion is described in the first approximation by the Eqs. (10), and does not have a wave character. Nevertheless, the slow motions of the reservoir gradually build up oscillations in the wave-vector region $k\sim k_0$, where the spectrum is well defined, and $\omega(k)\sim(gk_0)^{1/2}$. The $k\sim k_0$ region is represented by the hatched regions in Fig. 1. The nonlinear interactions between the vibrations with $k\sim k_0$ transfer energy into the $k\gg k_0$ region, where Zakharov and Filonenko's[6] results are valid. Thus, it seems that, to solve the problem of the decay of the turbulence arising as a result of the instability, we only need to compute with the aid of perturbation theory the intensity of the surface waves excited in the $k\sim k_0$ region by the turbulence and then join the corresponding expressions onto the asymptotic formulas from Ref. 6.

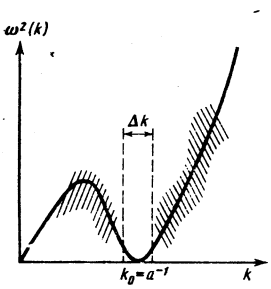

FIG. 1.

Let us rewrite (8) in the form of a Fourier integral:

$$\zeta(\mathbf{r},t)=\int\frac{d^2\mathbf{k}}{(2\pi)^2}\zeta_{\mathbf{k}}e^{i\mathbf{kr}}. \tag{8'}$$

Using perturbation theory, i.e., expanding (5) in powers of ζ, similarly to what we did in deriving (10), we obtain equations determining the amplitudes of the harmonics $\zeta_{\mathbf{k}}$ (e.g., $\mathbf{k}=\mathbf{k}_1+\mathbf{k}_2$, where $|\mathbf{k}_1|\approx|\mathbf{k}_2|\approx k_0$) in terms of $Z_{\mathbf{k}}(t)$. Limiting ourselves to the consideration of the second-order terms, and writing them in a simplified form, which is adequate for order-of-magnitude estimates, we obtain

$$\ddot{\zeta}_{\mathbf{k}}=-\omega^2(k)\zeta_{\mathbf{k}}+ASg(Z^2(\mathbf{r},\ t))_{\mathbf{k}}, \tag{14}$$

where A is an insignificant constant. Writing, in its turn,

$$\zeta_{\mathbf{k}}(t)=\eta_{\mathbf{k}}(t)e^{-i\omega(\mathbf{k})t},$$

i.e., separating out the frequency factor, we obtain for the amplitude $\eta_{\mathbf{k}}(t)$ the expression

$$\eta_{\mathbf{k}}(t)\sim\frac{Sg}{i\omega(\mathbf{k})}\int_0^t(Z^2(\mathbf{r},t'))_{\mathbf{k}}e^{i\omega(\mathbf{k})t'}\,dt'.$$

According to (10), the amplitude, $Z(\mathbf{r},t)$, of the chaotic motion that develops upon the collapse of the instability is of the order of $Z\sim S$, while the characteristic frequency scale corresponds to (13'), i.e., is much smaller than $\omega(k)\sim(gk_0)^{1/2}$. Therefore, in the developed-instability regime (i.e., for $t\to\infty$) we have

$$|\eta_{\mathbf{k}}|^2=c_0k^{-17/4}. \tag{15}$$

where Z now denotes the characteristic dimensionless amplitude of the chaotic motion of the surface in the definitions (9), (13).

The steady-state distribution in the inertial interval has the form[6]

$$\eta_{\mathbf{k}}(t)\sim aS^2Z^2, \tag{16}$$

Matching (16) and (15) at $k\sim a^{-1}$, we obtain

$$|\eta_{\mathbf{k}}|^2\sim a^8S^4Z^4(ka)^{-17/4}. \tag{17}$$

The total energy locked in the wave motion (per cm^2) is of the order of

$$\mathscr{E}_w\sim\alpha\langle(\nabla\zeta)^2\rangle\sim\frac{\alpha}{a^2}\int|\eta_{\mathbf{k}}|^2k^3\,dk,$$

from which it can be seen that the dominant contribution to the integral is made by small $k\sim a^{-1}$:

$$\mathscr{E}_w \sim \alpha S^4 Z^4. \tag{18}$$

The energy $\mathscr{P}$ (per cm^2), dissipated by the system of surface waves is proportional to

$$\mathscr{P} \sim \frac{\rho\nu}{a^2}\int \frac{(\nabla v)_k^2}{|\mathbf{k}|} d^2\mathbf{k} \sim \frac{\rho\nu}{a^2}\int \omega_k^2 |\eta_k|^2 k^2\, dk.$$

By substituting $\omega_k = (\alpha/\rho)^{1/2} k^{3/2}$ into this expression, we can see that the last integral amounts at large momenta to

$$\mathscr{P} \sim (\alpha\nu/a^2)\,(k_{max}a)^{5/4} S^4 Z^4,$$

here the cutoff wave vector k_{max} is determined in Ref. 6 from a comparison of the relative roles of the viscous and inertial terms. As applied to our problem, this yields

$$k_{max} = k_0 [S^4 Z^4 (gk_0)^{1/2}/\nu k_0^2]^{4/5}. \tag{19}$$

For $\mathscr{P}$ we obtain

$$\mathscr{P} \sim \alpha S^8 Z^8 (gk_0)^{1/2}. \tag{20}$$

The expression (20) reveals the remarkable fact, discovered by Zakharov and Filonenko,[6] that the coefficient of viscosity drops out from the expression for the dissipated energy.

For the foregoing arguments to be applicable, there should exist a broad range of wave numbers $k_0 \ll k \ll k_{max}$ for which the decisive role is played by the nonlinear wave interactions. According to (19), this is equivalent to the condition (let us recall that $Z \sim S$) that the inequality

$$S^8 (gk_0)^{1/2}/\nu k_0^2 \gg 1 \tag{21}$$

be satisfied. In the case of liquid helium we obtain for the ratio $(gk_0)^{1/2}/\nu k_0^2$ for temperatures $T \sim 4 - 1.5$ K, at which the majority of the experimental investigations[1,3] were performed, the estimate 10^3–10^4. The high power of S in (21) thus imposes some limitations from below on the smallness of S in this temperature region, where the density of the normal component is still fairly high. According to (18), for the wave description to be applicable, the condition $S^8 \ll 1$ should be fulfilled, but in order that Eqs. (10) and (14) can be used, we must have $S \ll 1$. Literally, the model under consideration is well defined when $(gk_0)^{1/2}/\nu k_0^2 \gg 1$.

After setting $\mathscr{P} = -\partial \mathscr{E}_{tur}/\partial t$, where $\mathscr{E}_{tur}$ is the excess energy concentrated in the turbulent motion, we can attempt to estimate the turbulence-damping time due to the nonlinear interaction under discussion. Thus, if we choose, in accordance with (10), (10′), and (12), $\mathscr{E}_{tur}$ in the form

$$\mathscr{E}_{tur} \sim \alpha Z^4,$$

then from (20) we find

$$Z^2 \sim S^2 (\tau_2/t)^{1/2}, \tag{22}$$

which corresponds to the boundary condition for which the amplitude of the turbulent motion $Z \sim S$ at $t \sim \tau_2$. On the whole, there is no rigorous method of estimating τ_2, the establishment time for the steady-state distribution (16), (17). Let us, however, assume that the reservoir is switched off at some moment of time. Then, following Zakharov and Filonenko,[6] we can define τ_2 as the time interval during which the energy $\mathscr{E}_w$, (18), stored in the surface waves is dissipated:

$$\mathscr{P} = -\frac{d\mathscr{E}_w}{dt} \sim \frac{\mathscr{E}_w}{\tau_2} \sim \frac{(gk_0)^{1/2}}{\alpha}\mathscr{E}_w^2,$$

which yields

$$\tau_2 \sim 1/(gk_0)^{1/2} S^8. \tag{23}$$

According to (22), the turbulent regime dies down in time according to a power law. Although the exact power in the functional dependence (22) depends, of course, on how we express $\mathscr{E}_{tur}$ in terms of Z, it can be seen that the decay law is slower than the exponential law with the time constant given by (23). The high power of S in (23) indicates that, even when $S \lesssim 1$, the time τ_2 exceeds the characteristic vibrational period of gravitational waves with frequency $\omega \sim (gk_0)^{1/2} \sim 10^{-2}$ sec^{-1}. According to (21) and (23),

$$(gk_0)^{-1/2} \ll \tau_2 < (\nu k_0^2)^{-1}. \tag{24}$$

It is pertinent to note that the departure of electrons from the surface does not follow from the above-described picture.

3. THE "SHALLOW-WATER" THEORY

Let us turn our attention again to Fig. 1. It is clear that the energy contained in the reservoir, i.e., in the wave-vector range $\Delta k \sim a^{-1}S$, is transferred not only into the region of large k, but also into the region of small $k \ll k_0$. Owing to the presence in the problem of the small parameter S, the flow components in the $k = 0$ neighborhood of width of the order of Δk can in turn be obtained independently with the aid of perturbation theory. The excitable long-wave motion corresponds to long gravitational waves, which can themselves pick up a significant portion of the energy from the reservoir and carry it off beyond the region occupied by the charge. In this section, however, we shall be interested in the other side of the matter, which is connected with the fact that, for $k \to 0$, we must take into consideration the finite depth of the liquid layer on the capacitor plates, as shown in Fig. 2.

Let us denote by $\bar{\zeta}(\mathbf{r}, t)$ the gently-sloping component of the surface deformation, i.e., that contribution to (8′) which is due to the wave vectors $k \lesssim a^{-1}S$. It is not difficult to obtain for this component an equation of the type (14), expressing the excitation of long-wave vibrations during the development of the instability. We give the answer without going into details of the essentially simple calculations:

$$gh_2 \nabla^2 \bar{\zeta} - \ddot{\bar{\zeta}} = gh_2 (1/h_1 - 1/h_2) \nabla^2 \langle \zeta^2(\mathbf{r}, t) \rangle. \tag{25}$$

In deriving (25) from (5), we assumed that $kh_1 \ll 1$ and $kh_2 \ll 1$, i.e., that

$$S \ll a/h \ll 1. \tag{26}$$

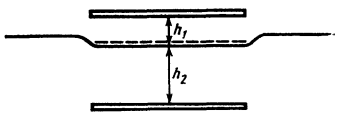

FIG. 2.

The quantity $\langle\overline{\xi}(\mathbf{r}, t)\rangle$ is, in accordance with (8), the square of the amplitude, averaged over space at distances of the order of the capillary constant, of the surface deformation due to the chaotic motion arising during the development of the instability. The characteristic scale of its variation in space is a/S, and the characteristic frequencies are given, as above, by Eq. (13′).

The left-hand side of Eq. (25) corresponds to the equation for long gravitational waves in channels with spectrum

$$\omega=(gh_2)^{1/2}k. \tag{27}$$

Thus, the spectrum of the surface oscillations in the region of the smallest $k\ll a^{-1}$ has a linear slope when allowance is made for the finite depth. (In the experiments described in Ref. 1 the widths of the gaps in the capacitor were comparable to a; in the experiments reported in Ref. 3 they attained values ~1 cm for $a\approx 0.5$ mm.)

According to (25), the amplitude of the long-wave motion

$$\xi\sim(1/h_1-1/h_2)\langle\xi^2(\mathbf{r}, t)\rangle. \tag{28}$$

From the linearized Euler equation $\rho\partial\mathbf{v}_t/\partial t=\rho g\nabla\xi$ we determine the longitudinal velocity $\mathbf{v}_t$:

$$v_t\sim(g/h_2)^{1/2}\xi. \tag{28′}$$

The most important property of the long-wave motion corresponding to the nondispersion spectrum (27) is the breaking of the front of waves of finite amplitude (see, for example, Ref. 10). The front-breaking time can be roughly estimated in the following way. In the system of coordinates moving with the phase velocity of the wave, $c_0=(gh_2)^{1/2}$, the individual points of the profile move with different velocities spread in a range of the order of the velocity, v_t, given by (28′). Taking the period $\lambda\sim k^{-1}=a/S$, and dividing it by v_t, (28′), we find

$$\tau_0\sim S^{-2}(h/a)^{1/2}(gk_0)^{-1/2} \tag{29}$$

[the time τ_0 is shorter than the time τ_2 when (26) is fulfilled].

It is very tempting to connect the breaking of the profile of the liquid surface with the discharge of the surface—the formation of charged bubbles and their departure to the lower plate of the capacitor.[1] The relations for the jumps of the level of the liquid, i.e., the generalization of the "shallow water" theory to the case of a charged surface can easily be carried out, using the standard procedure (see, for example, Ref. 10). The basic approximation of this theory is the assumption that all the quantities vary little in the longitudinal direction over distances of the order of the thickness of the liquid (it is convenient to locate the coordinate origin on the lower capacitor plate; the variable thickness of the liquid layer on the plate is denoted below by H). From this it follows that the perpendicular velocity component v_z is small, and that $\mathbf{v}\equiv\mathbf{v}_t$. The first equation

$$\partial H/\partial t+\operatorname{div} H\mathbf{v}=0 \tag{30}$$

is a consequence of the law of conservation of matter along the plate (the vector symbols have only x and y components). The pressure distribution in the liquid layer is given by the formula

$$P=P_0+\rho g(H-z)-E_1^2/8\pi+E_2^2/8\pi=P_0+\rho g(H-z)-(\Delta\Phi_1)^2/8\pi(d-H)^2+(\Delta\Phi_2)^2/8\pi H^2,$$

where $d=h_1+h_2$, and we have in the last expression written the additional terms due to the electric field in terms of the potential differences, $\Delta\Phi_1$ and $\Delta\Phi_2$, between the surface and, respectively, the top and bottom capacitor plates.

As has already been noted, it is assumed that the electron mobility is high enough for the equalization of the potentials along the surface to be possible. Substituting the expression for P into the tangential component of the Euler equations, and multiplying the latter by H, we obtain

$$H\partial\mathbf{v}/\partial t+H(\mathbf{v}\nabla)\mathbf{v}=-\nabla\overline{P}(H),$$
$$\nabla\overline{P}=H\nabla P(H)=H\nabla(\rho gH-(\Delta\Phi_1)^2/8\pi(d-H)^2+(\Delta\Phi_2)^2/8\pi H^2). \tag{30′}$$

The last transformation makes sense only in that Eqs. (30) and (30′) assume the standard—in gas dynamics—form if H is regarded as the density and $\overline{P}(H)$, the corresponding function of the pressure. Thus, we have demonstrated the inevitability of discontinuities in the profile—level jumps which appear as a result of the breaking of the profile at any finite level of nonlinearity.

The structure of the solution near the jumps is naturally not described by the shallow-water equations, but the conservation laws allow is to find a connection between the asymptotic solutions on the two sides (1, 2) of a jump. The law of conservation of matter gives

$$J=H_1v_1=H_2v_2, \tag{31}$$

where the velocities v_1 and v_2 are measured in the reference system in which the jump is at rest.

If the liquid is an ideal liquid, i.e., if the friction between it and the plate is negligible, then the longitudinal component,

$$\Pi_{tt}=\int_0^d\pi_{tt}(z)\,dz,$$

of the momentum flow is conserved. The tensor $\pi_{ik}=\pi_{ik}^{(0)}-\sigma_{ik}$ is the sum of the usual hydrodynamic contribution $\pi_{ik}^{(0)}=P+\rho v_iv_k$ and the Maxwell electromagnetic field tensor

$$-\sigma_{ik}=\frac{1}{4\pi}\left(\frac{E^2}{2}\delta_{ik}-E_iE_k\right).$$

Substituting it into the integral, we obtain

$$\Pi_{tt}=\rho v^2H+\frac{1}{2}\rho gH^2+\frac{E_1^2}{8\pi}(d-2H)+\frac{E_2^2}{4\pi}H.$$

(It can be seen, in particular, that when $E_1=E_2$, i.e., when there is no surface charge, the terms with the electric field gather into an insignificant constant $(\Delta\Phi)^2/8\pi d^2$.) From (31) and the condition that $\Pi_{tt}^{(1)}=\Pi_{tt}^{(2)}$ we can determine the final state (v_2, H_2) if we know the state (v_1, H_1). We can easily write down simple formulas for the case of "complete screening:" $\Delta\Phi_1=0, \Delta\Phi_2=\Delta\Phi$. Then we have the following generalization of the

well-known formulas[10]:

$$v_1^2=g\frac{H_1+H_2}{2}\frac{H_2}{H_1}-\frac{(\Delta\Phi)^2}{4\pi\rho H_1^2},\qquad v_2^2=g\frac{H_1+H_2}{2}\frac{H_1}{H_2}-\frac{(\Delta\Phi)^2}{4\pi\rho H_2^2}.$$

As usual, to conclude we must write down the condition expressing the fact that the kinetic energy of the liquid cannot increase inside a surge (the liquid flows from the side 1 to the side 2). The kinetic energy flux Q is equal to

$$Q=\int_0^d\left(\frac{P}{\rho}+\frac{v^2}{2}\right)\rho v\,dz=\frac{J}{2}\left\{\rho gH-\frac{(\Delta\Phi_1)^2}{4\pi(d-H)^2}+\frac{(\Delta\Phi_2)^2}{4\pi H^2}+\rho v^2\right\}.$$

The condition under which the energy is dissipated inside a surge,

$$Q_1-Q_2>0, \tag{32}$$

can easily (when $\Delta\Phi_1=0$) be reduced to the well-known condition[10]

$$\rho g(H_2-H_1)(H_1^2+H_2^2)/2H_1H_2>0. \tag{32'}$$

In other words the liquid flows to the side with the higher level. We shall not dwell on the proof that the velocity v_1 is higher and the velocity v_2 is lower than the corresponding "sound" velocities before and after the front of the surge.

As we shall now see, the break formed inevitably entails the discharge of the surface, since the law of conservation of surface charge does not, generally speaking, obtain at the front of the break. Thus, in the metallic approximation, for which the preceding formulas were derived, the surface-charge densities in front of, and behind, a surge are respectively equal to $en_{s1}=\Delta\Phi/4\pi H_1$ and $en_{s2}=\Delta\Phi/4\pi H_2$ ($E_1=0$). In the laboratory reference system the charges are stationary in the indicated approximation. Therefore, if u is the corresponding velocity of propagation of the discontinuity, the charge-density flux at the front of the discontinuity is equal to

$$I=ue(n_{s1}-n_{s2})=\frac{u\Delta\Phi}{4\pi}\left(\frac{1}{H_1}-\frac{1}{H_2}\right). \tag{33}$$

The opposite limiting case, in which the potential difference between the two sides of the discontinuity does not have time to vanish, would correspond to charges moving together with the liquid. And in this case a discontinuity in the tangential component of the velocity would not be compatible with the law of conservation of surface charge.

Thus, the charge should accumulate as time goes on at the front of the discontinuity. In the steady-state picture the charge should flow downwards. The formation of charged bubbles—"bubblons" (as they have been called by Volodin, Khaikin, and Edel'man[1])—appears to be the most natural mechanism for the discharge of the electrons. Furthermore, this process leads to a new contribution to the dissipative processes, which stabilizes the finite frontal width of a surge, i.e., facilitates the formation of bubbles. Let us recall that in the absence of dissipation the dispersion effects destroy the shock front; more exactly, they make the front too wide, since the viscosity of helium is very small. The nonlinear viscosity mechanisms discussed above clearly play an important role too, but we are not in a position to give an estimate for the nonlinearity level at which the breaking of the front occurs.

In the above-expounded picture, the appearance of the discontinuities is the result of the buildup of long-wave oscillations. As follows from (28), the role of this mechanism clearly decreases with increasing thickness of the liquid layer. If a discontinuity does form in the case in which $h\gg a$, then it is, apparently, a fairly weak discontinuity: $H_2-H_1\sim a$ (as can be estimated on the basis of simple physical arguments, the scale of the capillary constant, a, determines the size of the bubbles). In this case $v\approx(gH)^{1/2}$ in (33), and the rate at which (a unit length of) the discontinuity "cleanses" the liquid surface of charges

$$\dot{q}\sim n_s a(gH)^{1/2}/H. \tag{34}$$

4. CONCLUSION

We have shown that, at least in the model with a low surface-charge density $S\ll 1$, (7), the instability-induced process of reconstruction of the plane charged surface occurs in several stages, each of which is characterized by its own time scale. According to (10), the initial perturbations intensify over a period of time

$$\tau_1\sim 1/(gk_0)^{1/2}S,$$

by the end of which a developed turbulent regime is established at the surface. This motion generates oscillations with ever increasingly large wave vectors, as a result of which there arises a distinctive turbulent dissipation regime corresponding to the transfer of energy to the very short-wave oscillations, which largely dissipate it. This mechanism is capable of removing the system's excess energy in the time τ_2, (23). The chaotic motion excites the long-wave branch of the spectrum at the same time as it builds up the short-wave oscillations. In the case in which the liquid has a finite depth, these oscillations produce over a period of time of the order of τ_0, (29), a set of discontinuities on which, according to macroscopic arguments, the departure of charge from the surface occurs. The rate of discharge on a discontinuity in a typical situation (e.g., for $H\gtrsim a$) is given by the expression (34). In order to estimate the time during which the discharge occurs, we should, generally speaking, know the number of such discontinuities. One discontinuity, according to (33), discharges the surface during a period of time

$$\tau_3\sim(L/a)(H/g)^{1/2}, \tag{35}$$

where L is the length of the capacitor. In general, (35) gives plausible estimates for the discharge time,[1-3] although no systematic investigations of the dependence of the discharge rate on the depth of the liquid layer have been published. Notice that the discharge is another relaxation mechanism for the surface: according to (32), (32'), and (33), the energy, ΔQ, dissipated on a discontinuity is, in the first approximation, proportional to the quantity $\dot{q}$, (34).

If the electron source is not switched off above the stability threshold, then there flows through the system

a current that generates surface-wave turbulence. Taking account of the estimates made in Sec. 2, we see that the investigated system of electrons on a helium surface provides a unique opportunity for the verification of the wave-turbulence law (16).[6] According to (19), we have involved here a 3–4 orders of magnitude range of wave vectors.

Let us note in conclusion that there are probably alternatives to the above-expounded macroscopic theory of surface discharge. Thus, for $h \to \infty$, the formation of bubbles could be related to fine nonlinear effects: the formation of charged bubbles through the surmounting of the Coulomb barriers on the surging surface of the liquid.

[1]A. P. Volodin, M. S. Khaĭkin, and V. S. Édel'man, Pis'ma Zh. Eksp. Teor. Fiz. 23, 524 (1976) [JETP Lett. 23, 478 (1976)].
[2]M. Wanner and P. Leiderer, Phys. Rev. Lett. 42, 313 (1979).
[3]P. Leiderer and M. Wanner, Phys. Lett. A 73, 189 (1979).
[4]P. Leiderer, Phys. Rev. B 20, 4511 (1979).
[5]L. P. Gor'kov and D. I. Chernikova, Pis'ma Zh. Eksp. Teor. Fiz. 18, 119 (1973) [JETP Lett. 18, 168 (1973)].
[6]V. E. Zakharov and N. N. Filonenko, Zh. Prikl. Mech. Tekh. Fiz. No. 5, 62 (1967).
[7]L. P. Gor'kov and D. M. Chernikova, Dokl. Akad. Nauk. SSSR 228, 829 (1976) [Sov. Phys. Dokl. 21, 328 (1976)].
[8]D. M. Chernikova, Zh. Eksp. Teor. Fiz. 68, 249 (1975) [Sov. Phys. JETP 41, 121 (1975)].
[9]F. Vil'yams and V. B. Shikin, Fiz. Nizk. Temp. 5, 1005 (1979) [Sov. J. Temp. Phys. 5, 475 (1979)].
[10]L. D. Landau and E. M. Lifshitz, Mekhanika sploshnykh sred (Fluid Mechanics), Gostekhizdat, Moscow, 1954 (Eng. Transl., Pergamon Press, Oxford, 1959).
[11]L. D. Landau and E. M. Lifshitz, Élektrodinamika sploshnykh sred (Electrodynamics of Continuous Media), Gostekhizdat, Moscow, 1957 (Eng. Transl., Pergamon Press, Oxford, 1960).
[12]D. M. Chernikova, Fiz. Nizk. Temp. 6, 1513 (1980) [Sov. J. Low. Temp. Phys. 6, 737 (1980)].
[13]A. A. Galeev and F. Z. Sagdeev, in: Voprosy teorii plazmy (Problems of Plasma Theory), Atomizdat, Moscow, 1973, p. 54.

Translated by A. K. Agyei