Broadening of spectral lines due to optical transitions of an atom in a three-dimensional oscillator well

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A calculation is made of the profile of an absorption line of an atom trapped in a three-dimensional oscillator well. The radiative and Stark broadening effects are considered and the influence of the orientation of the incident wave relative to the symmetry axes of the well is discussed. When the frequency of vibrations of an ion in a well is much greater than the homogenous line width, a resonance with a homogeneous width appears at the center of the absorption line.

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A resonance absorption line of a gas usually experiences Doppler broadening. The present authors and Chebotaev have demonstrated[1] that in the case of a particle trapped in a potential well a resonance with a homogeneous width appears at the center of the absorption line. Our treatment applies to the case when the wavelength of the absorbed radiation is considerably less than the dimensions of the well, i.e., to the optical and shorter range of wavelengths. Dicke considered[2] a homogeneous-width resonance of a trapped particle in the rf range, when the wavelength is comparable with the well size.

Sufficiently effective ion-trapping methods are now available[3,4] and these make it possible to confine an ion in a trap even for a few hours. In principle, the width of a resonance is governed by the radiative lifetime of the energy levels. In the case of a trapped ion it should be possible to achieve spectral line resolution better than the Doppler width.

Our paper is concerned with the profile of a line due to optical transitions in an atom trapped in a three-dimensional oscillator well. The profile of an absorption line is an equidistant array of resonances which appear against the background of a Doppler profile because of periodic motion of the atom. The radiative broadening of such resonances is found in §3. The two (in our opinion, main) mechanisms of inhomogeneous broadening of a line of a trapped particle are considered in §§4, 5. One of these broadening mechanisms is associated with the Stark effect since the confinement potential is created by using an electric field. Different electric fields act on an atom at different points in a well and this gives rise to an inhomogeneous broadening of the line components. The second mechanism is associated with the oscillator anharmonicity, i.e., with the deviation of the potential from the rigorous harmonic oscillator form. This allowance for anharmonicity causes disappearance of periodic arrays of an ideal three-dimensional oscillator in the Doppler profile. However, the resonance at the center of the absorption line is retained and it has a homogeneous width. The dependence of the magnitude of the resonance on the direction of incidence of a light wave on a three-dimensional oscillator is analyzed in §6.

The motion of an atom in a three-dimensional oscillator well will be considered quantum-mechanically. This approach provides the simplest solution of the problem formulated here and it makes it possible to explain the existence of a resonance at the center of an absorption line. The same results can be obtained assuming that a particle moves along a classical trajectory. However, in this case the derivation is more cumbersome because we need to solve the quantum transport equation for the density matrix in order to allow for the recoil and Stark effects.

§1. HAMILTONIAN OF THE SYSTEM

The wave function of an atom depends on the coordinates of its center of inertia r and on the set of internal coordinates. We shall write the Hamiltonians in the form

$$H_0=\frac{\mathbf{p}^2}{2M}+\sum_i\frac{M\Omega_i^2x_i^2}{2}+H, \tag{1}$$

where p is the momentum operator of the atom; M is the atomic mass; x_i is the projection of r on the axis i; $x_i=(X, Y, Z)$; Ω_i is the frequency of a linear oscillator along the coordinate i and the frequency along the Z axis is assumed to be $\Omega_3\equiv\Omega$; H is the part of the Hamiltonian which is associated with the internal degrees of freedom and which determines the energy levels of an atom E_ν and its eigenfunctions $|\nu\rangle$ related by $H|\nu\rangle=E_\nu|\nu\rangle$.

We shall denote the eigenfunctions of H_0 by

$$|\nu; n_1n_2n_3\rangle=|\nu\rangle|n_1n_2n_3\rangle,\ |n_1n_2n_3\rangle=|n_1\rangle|n_2\rangle|n_3\rangle,$$

where $|n_i\rangle$ are the eigenfunctions of the harmonic oscillator:

$$\left(\frac{p_i^2}{2M}+\frac{M\Omega_i^2x_i^2}{2}\right)|n_i\rangle=E_i|n_i\rangle;\ E_i=\hbar\Omega_i\left(n_i+\frac{1}{2}\right),\ n_i=0,1,2,\ldots.$$

The eigenvalues of H_0 are

$$E^{(\nu)}_{n_1n_2n_3}=\sum_i\hbar\Omega_i(n_i+{}^1/_2)+E_\nu. \tag{2}$$

An atom in a three-dimensional oscillator well (trap) is acted upon by an electric field which gives rise to a Stark shift of the levels that we shall assume to be quadratic in respect of the field. The interaction associated with this effect will be allowed for in the form

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of a small correction to the Hamiltonian:

$$V=-\alpha E^2(\mathbf{r})/2,$$

where α is the operator that applies to the internal quantum numbers and whose matrix element $\alpha^{(\nu)} = \langle \nu | \alpha | \nu \rangle$ is the polarizability of an atom at a level ν. In the expansion of V as a Taylor series near the bottom of a well we shall retain the principal term expressed in the form

$$V=-\alpha E_0^2 \sum_i \left(\frac{x_i^2}{a_i^2}\right),$$

where E_0 is the characteristic intensity of the electric field in a trap and a_i are the characteristic dimensions of the trap. The energy correction associated with this perturbation is

$$\Delta E^{(\nu)}_{n_1n_2n_3} = \langle \nu; n_1n_2n_3 | V | \nu; n_1n_2n_3 \rangle$$
$$= -\alpha^{(\nu)} E_0^2 \sum_i \frac{\hbar}{M\Omega_i a_i^2}\left(n_i + \frac{1}{2}\right). \qquad (3)$$

§2. ABSORPTION-LINE PROFILE

Let us assume that an atom in a trap is acted upon by an electromagnetic wave

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E} e^{-i\omega t + ikz} + \text{c.c.},$$

whose frequency ω is close to the transition frequency $\omega_{21} = (E_2 - E_1)/\hbar$, where E_2 is the energy of the upper level and E_1 is the energy of the lower level [the index ν in Eq. (2) assumes two values: $\nu = 1, 2$], and k is the wave vector. The Hamiltonian of the interaction of the wave with the atom is

$$V_1 = -\langle 2 | \mathbf{d} | 1 \rangle \mathbf{E}(\mathbf{r}, t) + \text{c.c.},$$

where $\langle 2 | \mathbf{d} | 1 \rangle$ is the matrix element of the dipole moment of an atom between the states $|\nu\rangle$.

According to Ref. 5, the probability of a transition per unit time from a state $i = (1; n_1, n_2, n)$ to all the states $f = (2; m_1, m_2, m)$ is

$$w_i = \frac{2\pi}{\hbar} |\mathbf{d}_{21}\mathbf{E}|^2 \sum_{m_1m_2m} B_{fi} \delta(E_f - E_i - \hbar\omega), \qquad (4)$$

where

$$B_{fi} = |\langle m_1m_2m | e^{ikz} | n_1n_2n \rangle|^2.$$

The probability w_i is independent of the indices n_1 and n_2 of the initial state and, therefore, we can describe $w_n \equiv w_i$ by

$$w_n = \frac{W_0}{\Omega} \sum_{m=0}^{\infty} B_{n,m-n} \delta(m - n - x), \qquad (5)$$

where $W_0 = 2\pi |\mathbf{d}_{21}\mathbf{E}/\hbar|^2$, $x = (\omega - \omega_{21})/\Omega$, and the values of $B_{n,m-n}$ can be calculated using formulas 7.388 (6), (7) in Ref. 6:

$$B_{n,m-n} = e^{-\lambda} \lambda^{m-n} (L_n^{m-n}(\lambda))^2 n!/m!,$$

$\lambda = \Delta/\Omega$, $\Delta = \hbar k^2/2M$, and $L_n^{m-n}(\lambda)$ are the generalized Laguerre polynomials.

We shall now find the profile of an absorption line for a gas of atoms with a temperature T. We shall do this by averaging w_n using a distribution function $\rho_n = \beta e^{-\beta n}$, where $\beta = \hbar\Omega/T \ll 1$, T is the absolute temperature in energy units, and

$$\sum_{n=0}^{\infty} \rho_n = 1.$$

Using the results in the Appendix, we find that the total probability of absorption as a result of the $1 \to 2$ transition is

$$w = \sum_{n=0}^{\infty} \rho_n w_n = \frac{W_0}{\Omega} \mathcal{L}(x), \qquad (6)$$

$$\mathcal{L}(x) = \sum_{l=-\infty}^{\infty} D_l \delta(l - x), \quad D_l = \frac{1}{\sqrt{\pi}\mu} \exp\left[-\frac{(l-\lambda)^2}{\mu^2}\right],$$

where $\mu = \omega_D/\Omega \gg 1$, $\omega_D = kv_0$ is the Doppler width, and $v_0 = (2T/M)^{1/2}$. It should be noted that

$$\int_{-\infty}^{\infty} \mathcal{L}(x)\, dx = 1.$$

Equation (6) shows that the absorption differs from zero only at the incident field frequencies satisfying $\omega = \omega_{21} + l\Omega$, where $l = 0, \pm 1, \ldots$. The probability of absorption at these frequencies is proportional to the function D_l, which describes a "discrete" Doppler profile shifted, as expected, by a value equal to the recoil energy Δ.

§3. RADIATIVE LINE WIDTH

The line profile (6) can be derived ignoring the factors causing broadening of the individual components, i.e., broadening of the δ functions. It is easiest to allow for the radiative decay of the levels 1 and 2. This is done by the following substitution in Eq. (6):

$$\delta(x - k) \to \Gamma/\pi[\Gamma^2 + (x-k)^2],$$

where $\Gamma = (\gamma_1 + \gamma_2)/2\Omega$ is the homogeneous half-width of a line in units Ω; γ_1 and γ_2 are the reciprocal lifetimes of the levels 1 and 2. Instead of Eq. (6), we now have

$$\mathcal{L}(x) = \sum_{k=-\infty}^{\infty} D_k \Gamma/\pi[\Gamma^2 + (x-k)^2],$$

i.e., the line profile is now a sum of the Lorentzian profiles whose amplitudes are proportional to D_k. If $\mu \gg \Gamma$, then D_k can be taken outside the summation sign at the point $k = x$, i.e.,

$$\mathcal{L}(x) = D_x f(x), \qquad (7)$$

where

$$f(x) = \frac{1}{\pi} \sum_{k=-\infty}^{\infty} \frac{\Gamma}{\Gamma^2 + (x-k)^2}.$$

After summation we obtain

$$f(x) = \operatorname{sh} 2\pi\Gamma / (\operatorname{ch} 2\pi\Gamma - \cos 2\pi x).$$

If $\Gamma \ll 1$, this function reduces-as expected-to a sum of nonoverlapping Lorentzian profiles of Eq. (7). If $\Gamma \gg 1$, then

$$f(x) = 1 + 2e^{-2\pi\Gamma} \cos 2\pi x.$$

This formula shows that at high values of Γ, we have $f(x) \to 1$, i.e., $\mathcal{L}(x)$ in Eq. (7) is the usual Doppler profile.

§4. STARK EFFECT

Allowance for a small correction to the Hamiltonian in Eq. (1) will be made using perturbation theory. Then, the matrix elements B_{fi} are still given by Eq. (4), i.e., we shall use unperturbed oscillator wave functions and allow for the perturbation by modifying E_f and E_i. The shifts of these levels will be denoted by

$$\Delta E_f=\Delta E^{(2)}_{m_1m_2m}, \quad \Delta E_i=\Delta E^{(1)}_{n_1n_2n}.$$

The probability of a transition in the field of a traveling wave is still given by Eq. (4), which should now be averaged over the initial states n_1, n_2, n with a distribution function

$$\rho_{n_1n_2n}=\beta_1\beta_2\beta\exp[-(\beta_1n_1+\beta_2n_2+\beta n)],$$

where $\beta_1=\hbar\Omega_1/T\ll 1$, and $\beta_2=\hbar\Omega_2/T\ll 1$. The total probability of absorption can then be described by

$$w=W_0\mathcal{L}(x)/\Omega, \tag{8}$$

$$\left.\begin{aligned}\mathcal{L}(x)&=\beta_1\sum_{n_1=0}^{\infty}e^{-\beta_1n_1}\beta_2\sum_{n_2=0}^{\infty}e^{-\beta_2n_2}\beta\sum_{n=0}^{\infty}e^{-\beta n}\sum_{k=-\infty}^{\infty}B_{nk}(\lambda)\delta(x-k-Q_{k;n_1n_2n}),\\ Q_{k;n_1n_2n}&=(\Delta E^{(2)}_{n_1n_2,n+k}-\Delta E^{(1)}_{n_1n_2n})/\hbar\Omega,\\ B_{nk}&=\lambda^ke^{-\lambda}\frac{n!}{\Gamma(n+k+1)}(L_n^k(\lambda))^2,\end{aligned}\right\} \tag{9}$$

where we still have

$$\int_{-\infty}^{\infty}\mathcal{L}(x)dx=1.$$

We can allow for the influence of the Stark effect on the absorption line profile by using the expressions for the energy correction (3). Substituting Eq. (3) in Eq. (9), we obtain

$$Q_{k;n_1n_2n_3}=\sum_i\gamma_i\beta_in_i,$$

where

$$\gamma_i=\frac{\Delta\omega}{\Omega}\frac{L_i^2}{a_i^2}, \quad |\gamma_i|\ll 1,$$

$$\Delta\omega=(\alpha^{(1)}-\alpha^{(2)})\frac{E_0^2}{2\hbar}$$

is the difference between the Stark shifts of the levels 2 and 1, and $L_i=v_0/\Omega_i$ is a characteristic size associated with the amplitude of thermal vibrations of the i-th oscillator. The term containing $k\alpha^{(2)}$ in Q will be omitted because it results in just a slight renormalization of Ω.

Bearing in mind that

$$\delta(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\tau x}d\tau,$$

we can reformulate Eq. (8) as follows:

$$\mathcal{L}(x)=\sum_{k=-\infty}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}d\tau e^{-i(x-k)\tau}\times\beta_1\sum_{n_1=0}^{\infty}\exp[-\beta_1n_1(1-i\gamma_1\tau)]\beta_2\sum_{n_2=0}^{\infty}\exp[-\beta_2n_2(1-i\gamma_2\tau)]\times\beta\sum_{n=0}^{\infty}\exp[-\beta n(1-i\gamma_3\tau)]B_{nk}(\lambda).$$

The summation over n_1 and n_2 is performed in an elementary manner; the summation over n is carried out using the results in the Appendix. Assuming, as in §3, that D_k is a slowly varying function of k ($\mu\gg|\gamma_i|$), we obtain

$$\mathcal{L}(x)=D_kf(x),$$

$$f(x)=\sum_{k=-\infty}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\exp[-i(x-k)\tau]d\tau}{(1-i\gamma_1\tau)(1-i\gamma_2\tau)(1-i\gamma_3\tau)^{1/2}}.$$

In the derivation of $f(x)$ we have made the substitution $\sin\beta_i\gamma_i\tau/\beta_i\to\gamma_i\tau$, since $\beta_i\ll 1$ ($\beta_i\sim 10^{-5}$ at $T=1$°K and for $\Omega\sim 10^5$ sec^{-1}).

The function $f(x)$ is periodic and the period is unity. Therefore, we shall consider it in the interval $0\leqslant x<1$. In the case when $\gamma_1=\gamma_2=\gamma_3=\gamma>0$, calculation of the integral gives

$$f(x)=\frac{4}{3\sqrt{\pi}\gamma}\sum_{k=0}^{\infty}\left(\frac{x+k}{\gamma}\right)^{1/2}e^{-(x+k)/\gamma}.$$

If $\gamma\ll 1$, then

$$f(x)=\frac{4}{3\sqrt{\pi}\gamma}\left(\frac{x}{\gamma}\right)^{1/2}e^{-x/\gamma}.$$

The Stark effect shifts the maximum of an individual component by $3\gamma/2$. The width at half-maximum is then 3.2γ. In the limit $\gamma\to 0$, we have $f(x)\to\delta(x)$.

In the case when $\gamma_1=\gamma_2=\gamma>0$ and $|\gamma_3|\ll\gamma$, integration and summation with respect to k gives

$$f(x)=\frac{1}{\gamma^2}\frac{e^{-(x-1)/\gamma}}{e^{1/\gamma}-1}\left(\frac{1}{e^{1/\gamma}-1}+x\right), \quad 0\leqslant x<1.$$

If $\gamma\ll 1$, then

$$f(x)=xe^{-x/\gamma}/\gamma^2.$$

Thus, the Stark effect produces the same broadening and shift of all the line components. Naturally, the periodicity of $f(x)$ is retained, exactly as in the case of radiative broadening (§3). The shift and broadening of a single component is of the order of $\Delta\omega L_i^2/a_i^2$. The factor $L_i^2/a_i^2\sim T$ allows for the reduction in the Stark shift as a result of cooling since the characteristic size associated with the amplitude of thermal vibrations becomes less than the trap size. We shall now obtain a numerical estimate. For a trap[4] of linear size $\sim 250\mu$, we have $E_0\sim 10^4$ V/cm, $\Omega=2.4$ MHz. If this trap is to confine ions characterized by $L_i\sim a_i$ and $|\alpha^{(1)}-\alpha^{(2)}|\sim 10^{-23}$ cm^3, then

$$\gamma=\gamma\Omega\sim 10 \text{ MHz}.$$

This means that it is possible to resolve the individual line components either for transitions for which $|\alpha^{(1)}-\alpha^{(2)}|\ll 10^{-23}$ cm^3 or by cooling the trapped ions.

§5. OSCILLATOR ANHARMONICITY

In a real trap the potential to which an ion is subjected differs from the oscillator form. This means that the potential near the bottom of the well expanded as a Taylor series contains not only the terms

$$\sum_i\frac{M\Omega_i^2x_i^2}{2},$$

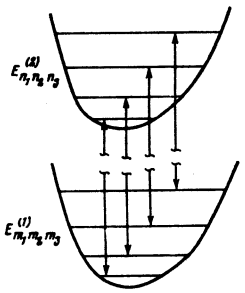

FIG. 1. Arrows identify transitions of frequency $\omega_{21}=(E_2-E_1)/\hbar$.

but also the corrections associated with the anharmonicity. We shall show that the deviation of the potential from the harmonic-oscillator form flattens the periodic array in the line profile with the exception of the resonance at the line center. It should be noted that in the case of an arbitrary potential the energy of a particle excited at the upper and lower levels is written in the form

$$E^{(2)}_{n_1n_2n_3}=E_{n_1n_2n_3}+E_2, \quad E^{(1)}_{m_1m_2m_3}=E_{m_1m_2m_3}+E_1,$$

where n_1, n_2, and n_3 are the quantum numbers representing the state of the center of gravity of an atom experiencing this potential.

There are many transitions (of the order of $\bar{n}=\beta^{-1}$) with $n_1=m_1$, $n_2=m_2$, and $n_3=m_3$, which have the same frequency $\omega_{21}=(E_2-E_1)/\hbar$. This sequence of transitions gives rise to a resonance at the line center (see also Fig. 1). In the case of different quantum numbers of the upper and lower levels we do not generally find a sequence of transitions with identical frequencies, i.e., there are no resonances at other frequencies. Thus, in the case of an arbitrary potential the line absorption profile is in resonance with a homogeneous width at the background of the Doppler profile. The magnitude of this resonance depends on the actual form of the potential.

§6. THREE-DIMENSIONAL OSCILLATOR

We shall now find the dependence of the absorption line profile on the direction of propagation of an electromagnetic wave specified by the angles θ and φ in a spherical coordinate system. The wave vector has the projections

$$k_x=k\sin\theta\cos\varphi, \quad k_y=k\sin\theta\sin\varphi,$$

$$k_z=k\cos\theta.$$

Calculations similar to those carried out in §2 give, for the case of an arbitrary incidence of a light wave relative to the symmetry axes of an oscillator well, the following line profile:

$$\mathcal{L}(x)=\sum_{m,n,l=-\infty}^{\infty}\exp[-(\mu_1^2+\mu_2^2+\mu_3^2)/2]I_m(\mu_1^2/2)I_n(\mu_2^2/2)I_l(\mu_3^2/2)\times\Gamma/\pi[\Gamma^2+(x-m\nu_1-n\nu_2-l)^2], \tag{10}$$

where I_n is a Bessel function with an imaginary argument,

$$\mu_1=\mu k_x/k\nu_1, \quad \mu_2=\mu k_y/k\nu_2,$$

$$\mu_3=\mu k_z/k, \ \nu_1=\Omega_1/\Omega, \ \nu_2=\Omega_2/\Omega.$$

In contrast to the one-dimensional oscillator of §2, Eq. (10) describes three periodic resonance arrays. We shall be interested in the resonance at the center of the absorption line. The amplitude of this resonance is equal to the term with $m=n=\mathcal{L}=0$ in $\mathcal{L}(0)$ of Eq. (10):

$$A=\frac{1}{\pi\Gamma}\exp\left[-\frac{\mu^2}{2}\left(\frac{\sin^2\theta\cos^2\varphi}{\nu_1^2}+\frac{\sin^2\theta\sin^2\varphi}{\nu_2^2}+\cos^2\theta\right)\right]\times I_0\left(\frac{\mu^2}{2\nu_1^2}\sin^2\theta\cos^2\varphi\right)I_0\left(\frac{\mu^2}{2\nu_2^2}\sin^2\theta\sin^2\varphi\right)I_0\left(\frac{\mu^2}{2}\cos^2\theta\right).$$

We shall consider the case when $\mu\gg1$. If the wave travels along the Z axis ($\theta=0$), then

$$A=1/\pi^{3/2}\mu\Gamma.$$

We shall continue this analysis for small angles $\theta\ll1$, but we shall assume that $\mu\theta\gg1$. When $\varphi\ll1$, i.e., when the wave travels almost in the XZ plane, the formulas for two specific cases are as follows:

1. $A=\nu_1/\pi^{3/2}\Gamma\mu^2\theta$, $\mu\theta\varphi\ll1$.

2. $A=\nu_1\nu_2/\pi^{3/2}\Gamma\mu^3\theta^2\varphi$, $\mu\theta\varphi\gg1$.

These formulas demonstrate a strong dependence of the amplitude of the resonance at the center of the absorption line on the angles θ and φ.

We shall now estimate the precision with which a trap has to be oriented relative to the incident wave in order to ensure that the resonance amplitude is of the same order as that of the Doppler profile. In these estimates we shall assume that $\mu\sim10^3$, $\Gamma\sim10^{-2}$, and $\nu_1\sim1$. For the angles $\theta\sim10^{-2}$ and $\varphi\sim10^{-1}$, the resonance amplitude is of the order as the Doppler profile ($A\sim D_0$). Thus, we find that when $\Gamma\ll1$, a resonance may appear at the center of the absorption line and this resonance has a homogeneous width and fairly high amplitude.

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APPENDIX

We need the sum

$$N=\beta\sum_{n=0}^{\infty}e^{-\beta zn}B_{nk}(\lambda)$$

$$=\lambda^k e^{-\lambda\beta}\sum_{n=0}^{\infty}e^{-\beta zn}\frac{n!}{\Gamma(n+k+1)}(L_n^k(\lambda))^2,$$

where z is a complex number. We shall carry out summation using Eqs. 8.976 (1) from Ref. 6 and we shall assume that $|\beta z|\ll1$. Then,

$$N=\frac{1}{z}\exp\left[\frac{\beta zk}{2}-\frac{\mu^2}{2z}\left(1+\frac{\beta^2z^2}{12}\right)\right]I_k\left(\frac{\mu^2}{2z}\left(1-\frac{\beta^2z^2}{24}\right)\right),$$

where $\mu^2=4\lambda/\beta$ and $I_k(t)$ is a Bessel function with an imaginary argument. In our case, we have $|\mu^2/2z|\gg1$, so that we shall find the asymptotic value of $I_k(t)$ for $|t|\gg1$ and arbitrary k. We shall do this employing the integral representation of Ref. 6 [Eq. 8.431 (5)]

$$I_k(t)=\frac{1}{\pi}\int_0^{\pi}e^{t\cos\theta}\cos k\theta d\theta.$$

The main contribution to the integral in the $|t| \gg 1$ case is due to a small region $\theta \sim |t|^{-1/2}$ near zero and, therefore,

$$I_k(t) \approx \frac{1}{\pi} \int_0^\infty \exp\left[t\left(1-\frac{\theta^2}{2}\right)\right] \cos k\theta d\theta$$
$$= \frac{e^t}{(2\pi t)^{1/2}} \exp\left(-\frac{k^2}{2t}\right).$$

It follows from the above formula that

$$\beta \sum_{n=0}^{\infty} e^{-\beta \varepsilon n} B_{nk}(\lambda) = D_k / z^{1/2},$$

$$D_k = \frac{1}{\sqrt{\pi}\,\mu} \exp\left[-\frac{(k-\lambda)^2}{\mu^2} z\right].$$

We shall now assume that $z=1$ in D_k. Then, to within terms of the order of $\exp(-\pi^2\mu^2)$, we find that

$$\sum_{k=-\infty}^{\infty} D_k = 1.$$

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