Limiting mobility of charged particles localized near a liquid-helium surface

Yu. A. Matveev

Research, Design-Planning, and Technological Institute of Current Sources
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The resistance force acting on ions near a surface and due to excitation of surface waves is calculated. The cases of a surface anion, a vortex ring, and ions beneath the surface, vibrating and moving parallel to the surface, are considered. The obtained resistance force has a nonlinear dependence on the velocity of the body, on the immersion depth, and on the geometric dimensions. It is shown that in the case of ions beneath the surface stationary motion in sufficiently strong clamping fields ($E_\perp > 1$ cgs esu) is possible at not all velocities.

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Calculations of the mobility of charged particles near the surface of liquid helium were made in a number of papers (see the reviews of Shikin and Monarkha[1,2]). In these papers were calculated the electron mobility induced over a liquid-helium surface by scattering from surface oscillations, and also the surface-anion mobility due to energy dissipation in the volume of the liquid helium. For an ion beneath the surface, calculations were made of the friction force exerted on the ion by the quasiparticles in the volume. The change produced in negative-ion mobility by the decrease of the number of phonons near a free surface was accounted for in a paper by Shikin.[3]

The energy-dissipation mechanisms described above become ineffective for a surface anion and for ions beneath the surface, since the number of quasiparticles that take part in the process decreases. An important role is assumed in this case by the emission of surface waves, "ripplons," an effect known in classical hydrodynamics as wave resistance.[4] The present paper is devoted to the calculation of the wave resistance of objects moving parallel to a helium surface, as well as those vibrating near the surface.

§1. FORMULATION OF PROBLEM

Questions connected with the motion of solids under a liquid surface have been intensively studied at the beginning of the century (see Ref. 4 and the references therein, as well as the papers of Lamb[5] and Havelock[6]). As a rule, account was taken in these investigation of only the influence of the force of gravity on the deformation of the surface (see, however, §62 of Ref. 4). This is understandable, for a number of conditions must be satisfied to take surface tension forces into account. First, it turns out that the body immersion depth h should be less than the capillary constant a ($a^2=\alpha/\rho g$, α is the surface-tension coefficient, ρ is the liquid density, and g is the acceleration due to gravity). Second, to be able to use a linear approximation the liquid velocity excited by the motion should be less than the velocity of the body. This leads, as will be shown below, to the condition

$$R \ll h \ll a.$$

Since the objects considered in classical hydrodynamics have characteristic dimensions ~1 m and the capillary constant for all liquids is ~0.1 cm, allowance for the surface tension under these conditions in only of academic interest. In addition, when the dimensions of the body are decreased account must be taken of the viscosity forces, which are small if the Reynolds number $\mathrm{Re}=cR/\eta$ is large (here c is the velocity of the body and η is the kinematic viscosity). The body velocity, in turn, must be less than that of sound, since ordinary surface waves are solutions of the equations of incompressible liquids. Estimates show that such a situation is practically impossible to simulate in an ordinary liquid.

It is known (Ref. 7, §129) that the problem of flow of incompressible liquid helium around a body breaks up into two problems in ordinary hydrodynamics for an ideal and a viscous liquid. The contribution of the normal part of the helium, which gives rise to the viscosity, is small to the extent that $\rho_n \ll \rho_s$ (ρ_n and ρ_s are the normal and superfluid densities). Thus, the motion of a solid in helium at low temperatures is described by the equations of an ideal liquid. In addition, under real experimental conditions the ions beneath the surface are fixed at depth $h\sim 10^{-5}$–10^{-6} and this, taking the foregoing into account, provides an exceptional possibility of considering phenomena connected only with surface-tension forces. We proceed now to the formulation of the problem.

Let the helium occupy the half-space $z \leqslant \xi(x, y)$, where $\xi(x, y)$ is the shape of the free surface. The velocity potential φ satisfies the equation $\nabla^2\varphi=0$. On the boundary of the body there must be satisfied the nonleakage condition

$$\left.\frac{\partial\varphi}{\partial n}\right|_\Sigma = c\cos\theta,$$

where Σ is the surface of the body and θ is the angle between the directions of the body-velocity vector and the normal to the surface.

For mathematical convenience, we introduce into the nondissipative-hydrodynamics equations a fictitious Rayleigh force proportional to the velocity[4]:

$$\bar{R} = -\mu u, \tag{1}$$

where $\mu > 0$ is a small dissipation coefficient. We let $\mu \to 0$ in the final expressions. It is important that the dissipation introduced in this manner does not violate the potential character of the motion.[4]

When formulating the boundary condition on the free surface, account must also be taken of the polarization interaction of the ion with the surface, i.e., the fact that the electrostatic forces repel the ion from the

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boundary while the external field clamps it; the joint action of these forces deforms the surface and causes it likewise to contribute to the wave resistance.

The electric field in the liquid consists of the field of the ion, the ion image field, and the clamping field. Since the polarizability of helium is very small, the image-charge field is smaller by a factor $\varepsilon - 1$ than the ion field (ε is the permittivity of the helium), and will therefore be neglected. The constant clamping field produces a constant displacement of the interface and is equivalent simply to a change of the external pressure. The surface $z=0$ will hereafter be taken to mean just the displaced surface. Thus, the electrostatic pressure in the helium is due only to the ion and is of the form

$$P = \frac{\varepsilon-1}{8\pi}\frac{e^2}{[r^2+(z+h)^2]^2} \tag{2}$$

(e is the ion charge and r is the distances in the x and y planes). We then obtain on the free surface $z=\xi$

$$P_{\text{liq}}+P_e=0, \tag{3}$$

where P_{liq} is the pressure of the liquid:

$$P_{\text{liq}}=-\rho\partial\varphi/\partial t-\rho g\xi+\alpha\Delta_\perp\xi-\mu\rho\varphi,$$
$$\Delta_\perp=\partial^2/\partial x^2+\partial^2/\partial y^2.$$

Differentiating (3) with respect to t and recognizing that $\partial\xi/\partial t=\partial\varphi/\partial z$, we obtain the sought boundary condition in the form

$$\frac{\partial^2\varphi}{\partial t^2}+g\frac{\partial\varphi}{\partial z}+\frac{\alpha}{\rho}\frac{\partial^3\varphi}{\partial z^3}+\mu\frac{\partial\varphi}{\partial t}=\frac{(\varepsilon-1)e^2}{8\pi}\frac{\partial}{\partial t}\frac{1}{[r^2+(z+h)^2]^2}. \tag{4}$$

$\varphi \to 0$ as $z \to -\infty$.

We note that allowance for the polarization pressure leads to a distortion of the equilibrium shape of the free surface. The boundary condition (4) should therefore be specified at $z=\xi_0(r)$, where $\xi_0(r)$ is the equilibrium shape; it will be shown below, however, that this leads to inclusion of terms $\sim k\xi_0$, with $\xi_0/h \ll 1$. In other words, if the deformation $\xi_0(r)$ is small, then the contributions from the polarization mechanism and from the hydrodynamic mechanisms must be calculated independently. When the problem is so formulated, it is necessary to solve a complicated integral equation, if the boundary condition on the liquid surface and the streamlining condition for the body are to be exactly satisfied. Such an integral equation was first obtained and investigated by Kochin.[8]

A good approximate solution of the problem can be obtained by using Lamb's method, which satisfies the streamlining condition only partially and which works better the deeper the body. Namely, we seek the potential φ in the form[4]

$$\varphi=\varphi_0-\bar{\varphi}_0+\varphi_1,$$

where φ_0 is the potential of a body in an unbounded liquid. $\bar{\varphi}_0$ is its "reflection," i.e., the potential of a body located at the point $z=h$, and φ_1 is the potential of the wave motions. Lamb's method is equivalent to the requirement

$$\bar{\varphi}_0,\varphi_1|_{z=-h}\ll\varphi_0|_{z=-h},$$

i.e., we seek the shape of the surface for a given body potential, neglecting the redistribution of the velocity on the body surface.

To calculate the wave resistance in stationary motion we can use the energy conservation law. Let F be the resistance force. Then the work performed per unit time is $W=Fc$. This is equal to the rate of energy dissipation due to the Rayleigh forces (1). It is easy to show that the dissipative function for such forces is of the form

$$S=\mu\rho\int\varphi\frac{\partial\varphi}{\partial z}ds,$$

where the integral is taken over the free surface of the unperturbed liquid. From this we get an expression for the wave-resistance force[6]

$$F=\rho\mu c^{-1}\int\varphi\frac{\partial\varphi}{\partial z}ds;\quad \mu\to 0,\ z=0. \tag{5}$$

§2. WAVE RESISTANCE OF ION BENEATH A HELIUM SURFACE

We note, by way of introduction, the following. It is known that the dispersion law for surface waves in a liquid is given by

$$\omega^2=gk+\alpha k^3/\rho.$$

The phase and group velocities $v_{\text{ph}}=\omega/k$ and $v_{\text{gr}}=d\omega/dk$ of the wave have then the same shape, shown in Fig. 1. The minimum of the phase velocity corresponds to $v_m=(2ga)^{1/2}$ and $k_m=a^{-1}$. From the momentum and energy conservation laws it follows that a body moving with velocity c excites surface waves only if the condition $c \geq v_m$ is satisfied. It is then obvious from Fig. 1 that there are always two solutions for the excited wave, k_- and k_+, and they turn out to be related, $k_-k_+=a^{-2}$. The solution k_- corresponds to a gravitational wave, i.e., a wave in which the principal role is played by the force of gravity. Correspondingly, k_+ is a capillary wave. The gravitational wave is always produced behind the moving body, and the capillary ahead of it.

Let the liquid move with velocity c in the direction of the x axis. Then in the case of stationary motion $\partial/\partial t = c\,\partial/\partial x$ and the boundary condition (4) is written in the form

$$\frac{\partial^2\varphi}{\partial x^2}+\nu\frac{\partial\varphi}{\partial z}+\beta\frac{\partial^3\varphi}{\partial z^3}+\frac{\mu}{c}\frac{\partial\varphi}{\partial x}=\frac{Z}{\rho c}\frac{\partial}{\partial x}\frac{1}{[r^2+(z+h)^2]^2}, \tag{6}$$
$$\nu=\frac{g}{c^2},\quad \beta=\frac{\alpha}{\rho c^2},\quad Z=\frac{\varepsilon-1}{8\pi}e^2.$$

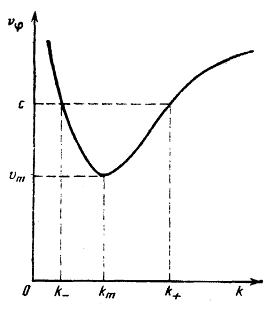

FIG. 1. Schematic representation of the solutions k_- and k_+ for a body velocity $c > v_m = (2ga)^{1/2}$.

Since the ions in the helium are geometrically spheres, we choose φ_0 and φ_1 in the form

$$\varphi_0=-M\frac{\partial}{\partial x}\frac{1}{[r^2+(z+h)^2]^{1/2}},\quad \varphi_0=-M\frac{\partial}{\partial x}\frac{1}{[r^2+(z-h)^2]^{1/2}},$$

$$M=cR^3/2.$$

Recognizing that

$$\frac{\partial}{\partial z}\left[\frac{1}{[r^2+(z+h)^2]^{1/2}}-\frac{1}{[r^2+(z-h)^2]^{1/2}}\right]_{z=0}=2\frac{\partial}{\partial h}\frac{1}{(r^2+h^2)^{1/2}},$$

we obtain from (6) a boundary condition for φ_1:

$$\frac{\partial^2\varphi_1}{\partial x^2}+\nu\frac{\partial\varphi_1}{\partial z}+\beta\frac{\partial^3\varphi_1}{\partial z^3}+\frac{\mu}{c}\frac{\partial\varphi_1}{\partial x}=\frac{Z}{\rho c}\frac{\partial}{\partial x}\frac{1}{(r^2+h^2)^2}$$
$$+2M\frac{\partial}{\partial x}\left(\nu\frac{\partial}{\partial h}+\beta\frac{\partial^3}{\partial h^3}\right)\frac{1}{(r^2+h^2)^{1/2}}. \quad (7)$$

Taking the symmetry of the right-hand side of (7) into account, we seek a solution for φ_1 in the form

$$\varphi_1=\frac{\partial}{\partial x}\int_0^\infty\varphi_{1k}e^{kz}J_0(kr)k\,dk,$$

where $J_0(kr)$ is a Bessel function. Expanding both sides of (7) in a Fourier-Bessel series, we obtain

$$\varphi_{1k}=\frac{g_k(-k\cos^2\theta+\nu+\beta k^2)}{k[(-k\cos^2\theta+\nu+\beta k^2)^2+\delta^2]},\quad \delta=\frac{\mu}{c}\cos\theta,$$

$$g_k=\frac{Z}{\rho c}\frac{k}{2h}K_1(kh)-2M(\nu+\beta k^2)e^{-kh}, \quad (8)$$

$$K_1(u)=\int_0^\infty \operatorname{ch} t e^{-u\operatorname{ch} t}\,dt.$$

We note an interesting figure of the function g_k: the contributions from the polarization and from the hydrodynamic mechanisms have opposite signs, therefore at certain values of the parameters of g_k the amplitude of the surface wave vanishes. The meaning of this phenomenon is the following. The electric field in the helium tends to push out the free surface and to form a "hump," while the hydrodynamic forces produce a sag in the surface. This can be seen already from (7), where the first term on the right is negative and the second positive. Substituting the obtained expression for φ_1 in (5), we obtain after calculations

$$F=F_++F_-,$$

where

$$F_\pm=4\pi\rho\int_0^{\theta_0}k_\pm^2\frac{g_{k_\pm}^2\cos\theta}{(\cos^4\theta-\varkappa^4)^{1/2}}d\theta, \quad (9)$$

$k_\pm$ are the roots of the equation $k\cos^2\theta+\nu+\beta k^2=0$;

$$k_\pm=(2\beta)^{-1}[\cos^2\theta\pm(\cos^4\theta-\varkappa^4)^{1/2}],$$
$$\varkappa=v_m/c,\quad \theta_0=\arccos\varkappa.$$

It is seen from the expression for $k_\pm$ that $k_+\sim k_-\sim a^{-1}$ if $\cos\theta\sim\varkappa$, therefore the contributions from F_- and F_+ become equal in a small angle region $\Delta\theta$ near θ_0. But at $\varkappa\ll 1$ it can be shown that the contribution from the region $\Delta\theta\sim\varkappa$ near θ_0 amounts to $F_+\sim F_-\sim F_0\varkappa^5$, where F_0 is the contribution from the angles $\theta\sim 1$. Therefore at $\varkappa\ll 1$ we can replace θ_0 by $\pi/2$.

If $h\gg a$, then $F_-\gg F_+$, except for velocities satisfying the condition $\varkappa\sim 1$. Neglecting the polarization term in the expression for g_k, we obtain an equation for the resistance for of an uncharged small sphere[4]:

$$F_-=4\pi\rho c^2R^6\nu^4\int_0^{\pi/2}\exp(-2\nu h\sec^2\theta)\sec^5\theta\,d\theta. \quad (10)$$

For the case $h\ll a$, $\varkappa\ll 1$ of interest to us we obtain

$$F_+=4\pi\rho\beta^{-4}\int_0^{\pi/2}g_{k_+}^2\cos^7\theta\,d\theta,\quad k_+=\beta^{-1}\cos^2\theta, \quad (11)$$
$$g_{k_+}=Z(2\rho ch)^{-1}K_1(k_+h)-2M\cos^2\theta\exp(-k_+h).$$

It is convenient to rewrite this expression in the form

$$F_+=4\pi\alpha R(R/h)^5\varphi(x),$$

$$\varphi(x)=x^5\int_0^1\left\{\frac{Q}{x}K_1[x(1-t^2)]-(1-t^2)\exp[-x(1-t^2)]\right\}^2(1-t^2)^3dt, \quad (12)$$
$$x=\frac{c^2\rho h}{\alpha},\quad Q=\frac{Z}{2R^3\alpha},\quad h=2^{-1}\left[\frac{e}{E_\perp}\frac{(\varepsilon-1)}{\varepsilon(\varepsilon+1)}\right]^{1/2}.$$

The asymptotic expressions for $\varphi(x)$ are

$$\varphi(x)=2Q^2x/3,\quad x\ll 1;\quad \varphi(x)=15/16x,\quad x\gg 1.$$

In the region $x\sim 1$ the function $\varphi(x)$ has a minimum due to the cancellation indicated above. At $c<c_{1\max}$ the resistance is determined by the polarization mechanism and is equal to

$$F_+=\frac{2}{3\pi}\left(\frac{eE_\perp}{\alpha}\right)^2\rho c^2. \quad (13)$$

At $c\gg c_{2\max}$ the decisive factor is the hydrodynamic mechanism. In this case

$$F_+=\frac{15\pi}{4}\frac{\alpha^2}{\rho c^2}\left(\frac{R}{h}\right)^6. \quad (14)$$

Plots of F_+ at $E_\perp=1$ and 2 cgs esu are shown in Figs. 2 and 3. R was assumed to be 10 Å for the positive ions and 17 Å for the negative. The upper curves on each plot are the sums of the contributions of the wave resistance and the scattering by impurities. The contribution of the residual impurities from the experiment of Schwartz and Stark is $e/\mu=5\times10^{-17}$ (Ref. 9).

It should be noted that the descending parts ($F_c'<0$) of the $F(c)$ curve are unstable in the sense that if we apply a force F_0 to a body having an initial velocity c_0 on this segment then, depending on whether this force is larger or smaller than $F_+(c_0)$, the body will respectively go off to infinity or to a stable part of the curve ($F_c'>0$), where it will continue a stationary motion. The foregoing pertains only to bodies whose energy increases with increasing velocity. For vortex rings, which will be discussed below, the energy decreases with increasing velocity, so that for them it is the curve segment with $F_c'<0$ which is stable. It is also obvious that if the applied force $F_0>F_+(c_{2\max})$, then no stationary motion is possible at all.

It follows from Figs. 2 and 3 that at $E_\parallel$ corresponding to extremal points F_+ ($E_\parallel\lesssim 10^{-2}$–$10^{-3}$ cgs esu) the mobility $\mu\sim F_+^{-1}$ depends on c. In strong clamping fields, $E_\perp>1$ cgs esu, stationary motion is likewise impossible also on the section between the first maximum and minimum of $F_+(c)$. Comparison with the results of earlier calculations of the mobility[2,3] shows that these effects become noticeable at $T\leqslant 0.2$ K.

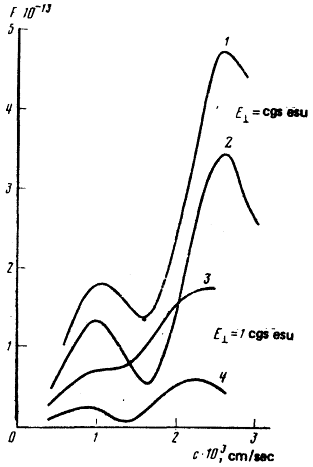

FIG. 2. Resistance force $F^+(c)$ for positive ions ($R_+ = 10$ Å), calculated from formula (12). Curves 1, 2) $E_\perp = 2$ cgs esu; 3, 4) $E_\perp = 2$ cgs esu;. Curves 3 and 1 were plotted with allowance for the residual resistance of the impurities $e/\mu = 5\times10^{-17}$ (Ref. 9).

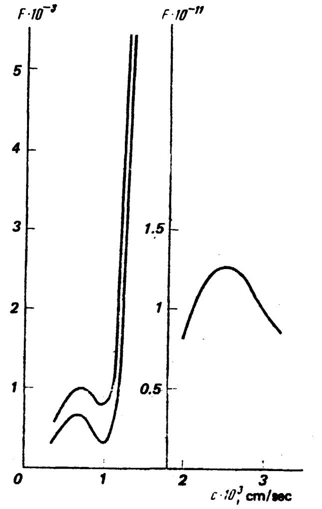

FIG. 3. Resistance force for negative ions ($R_- = 17$ Å) with the same impurity density as in Fig. 2. The segments $c \gtrsim c_{\min}$ and $c \gtrsim c_{2\max}$ were taken, $E_\perp = 2$ cgs esu. The effect of the impurities is small near $c \sim c_{2\max}$.

We consider now the velocity region $\varkappa \sim 1$. We note first that it is legitimate to introduce the fictitious dissipation coefficient only at velocities not too close to the threshold, $\varkappa \sim 1$, or more accurately speaking, it is required that $1-\varkappa^4 \gg \mu'$, with a corresponding angle region $\Delta\theta^2 \leqslant 1-\varkappa^4$ (μ' is the dimensionless dissipation coefficient at $k=a^{-1}$). The region $1-\varkappa^4 \ll \mu'$ calls for a consistent solution of the problem with finite viscosity, and will not be dealt with here. Recognizing that $F_+ = F_-$ as $\varkappa \to 1$, we easily obtain from (12)

$$F|_{\varkappa=1} = \frac{1}{\sqrt{2}}\frac{(eE_\perp)^2}{\alpha a}, \quad T \ll T_a = \omega_a. \tag{15}$$

Owing to the very strong dependence of the parameter $1-\varkappa^4$ on the particle velocity, the transition from the region $\varkappa > 1$, where $F=0$, to the region $\varkappa < 1$ will take place almost jumpwise.

We note also that the deformation of the surface on account of the polarization is $\xi \sim eE_\perp/\alpha \sim 10^{-9}-10^{-8}$ cm for $E_\perp \sim 1$ to 10 cgs esu. The characteristic wave vectors k are of the order $k \leqslant 10^6$, therefore the imposition of the boundary condition on $z=\xi_0(r)$ and not $z=0$ (referred to above) leads to the allowance for small terms $k\xi \sim 10^{-2}-10^{-3}$. In the derivation of (4) we have assumed that the shape of the polarization "hump" does not change in the course of motion, i.e., the pressure in the moving liquid is less than the polarization pressure. This is ture at velocities $c^2 \ll \alpha/\rho h$.

When an ion reaches a certain critical velocity, an annular vortex is formed behind it.[10] At distances exceeding the vortex radius d, the vortex velocity field coincides with the field of a moving sphere. Therefore if the vortex is located at a depth $h \gg d$, its wave resistance is calculated in the same manner as in the case of a sphere. Comparing the velocity field at large distances from the ring with the field of a sphere, we obtain the connection

$$M = {}^1/_4\Gamma d^2,$$

where Γ is the circulation of the ring. The polarization forces are insignificant at the velocities of interest to us, and the angular momentum of the ring exceeds that of a sphere in a ratio $(\alpha/R)^3 \gg 1$. Therefore the resistance of the "ion + ring" complex is determined entirely by the resistance of the ring, which can be represented in the form

$$F_k = \pi\rho\Gamma^6(\rho c/\alpha)^4 \int_0^{\pi/2} \cos^{11}\theta \exp(-2k_+h)\,d\theta. \tag{16}$$

§3. WAVE RESISTANCE OF SURFACE ANION

The electron pressure of a surface anion on the surface of liquid helium is given by[1]

$$P(r) = eE_\perp\varphi^2(r), \tag{17}$$

where $\varphi^2(r) = (\pi L^2)^{-1}\exp(-r^2/L^2)$ is the electron wave function, and $L^2 = 2\pi\alpha h^2/m(eE)^2$ is the size of the anion. An expression for the force F is obtained from (7), (8), and (9) with account taken of (17) and of $M=0$. We obtain

$$g_{k_\pm} = \frac{eE_\perp}{2\pi\rho c}\exp\left(-\frac{k_\pm^2L^2}{4}\right) \tag{18}$$

and for $\varkappa \ll 1$

$$F_+ = \pi^{-1}\rho c^2(eE_\perp/\alpha)^2 \int_0^{\pi/2} \cos^3\theta \exp\left[-\frac{1}{2}\left(\frac{\rho c^2 L}{\alpha}\right)^2\cos^4\theta\right] d\theta. \tag{19}$$

We have recognized that $J_+ \gg J_-$, since $L \ll a$. Just as in the case of ions beneath the surface, motion with velocities $c > c_+ \sim (\alpha/\rho L)^{1/2}$ is not stationary. Therefore, confining ourselves to the case $c < c_+$, we obtain formulas (13) and (15). We note also that in the case of a surface anion a lucid interpretation can be offered for the inequality $c < c_+$. It is obvious that the fields $E_\parallel$ must be weak enough so as not to "roll out" the electron from the deformation well,[1)] i.e., we must have $eE_\parallel L \ll w$ (w is the binding energy) $\sim (eE_\perp)^2/\alpha$. But F_+ at the maximum is of the order of $(eE_\perp)^2/\alpha^2 L$, so that the stationarity condition for the surface anion is equivalent to the requirement that the complex be indestructible. Just as in the case of an ion beneath the surface, allowance for the distortion of the shape of the anion becomes important at $c \sim c_+$.

The limits of applicability of (19) are obtained from the condition $k\xi \ll 1$; since $\xi \sim kP_k/\rho c^2 \lesssim eE/\alpha$, this condition reduces to $eE_\perp/\alpha L \ll 1$, which holds true for fields $E \lesssim 10^2$ cgs esu.

We estimate now the temperature region where the described effect becomes substantial. Since at $T \lesssim 0.1$ K, where the surface anion is produced, the phonon mean free path $l \gg a$ (see, e.g., Ref. 11), where a is the characteristic dimension of the deformation, it follows that the force of the phonon pressure on the anion must be obtained by solving the kinetic equations. We confine ourselves to estimates of this force.

At not too low temperatures, $T \gg T_a \sim \omega_a$, the wavelength of the thermal phonon is $\lambda_t \ll a$, therefore the cross section scattering by the surface anion is equal to the geometric cross section, $\sigma \sim a^2$. The projection of the force on the x axis is then $F \sim \Delta p \sin\alpha/\tau$, where $\Delta p \sim p_t \sim T/s$ is the momentum transferred to the anion, s is the speed of sound, and τ is the interaction time: $\tau^{-1} \sim \sigma c N_{\rm ph}$, $\tan\alpha = \partial\xi/\partial x$, and $N_{\rm ph}$ is the number of phonons. To observe the wave resistance we must have $F \ll F_+$, and F_+ is defined by expression (13). It is convenient to rewrite this inequality in the form $\rho_n as/\varphi\xi c \ll 1$ or $10^9 T^4/cE_\perp \ll 1$. For fields $E_\perp \sim 10$ cgs esu and $c \sim 10^2$ cm/sec, the wave resistance becomes noticeable starting with $T \lesssim 0.03$ K.

§4. WAVE RESISTANCE OF OSCILLATING ION

For an ion oscillating along the z axis[12] we have

$$\varphi = M\left\{\frac{z+h}{[r^2+(z+h)^2]^{3/2}} + \frac{z-h}{[r^2+(z-h)^2]^{3/2}}\right\} + \varphi_1,$$

$M = \frac{1}{2}A\omega\cos\omega t R^3$ and A is the oscillation amplitude. The boundary condition (4) takes than the form

$$\frac{\partial^2\varphi_1}{\partial t^2} + g\frac{\partial\varphi_1}{\partial z} + \frac{\alpha}{\rho}\frac{\partial^3\varphi_1}{\partial z^3} + \mu\frac{\partial\varphi_1}{\partial t} = \frac{Z}{\rho}\frac{\partial}{\partial t}\frac{1}{[r^2+(z+h-A\sin\omega t)^2]^2} + 2M\left\{g\frac{\partial^2}{\partial h^2}\frac{1}{(r^2+h^2)^{1/2}} + \frac{\alpha}{\rho}\frac{\partial^4}{\partial h^4}\frac{1}{(r^2+h^2)^{1/2}}\right\}. \tag{20}$$

Bearing in mind the use of the formula for the dissipative function S, we seek that part of the potential φ_1 which does not vanish as $\mu \to 0$. It is of the form

$$\varphi_1 = \cos\omega t\int_0^\infty \varphi_{1k} e^{kz} J_0(kr)\,k\,dk.$$

We then obtain from (20)[2)]:

$$\varphi_{1k} = \frac{g_k}{(-\omega^2+\omega_k^2)^2+(\mu\omega)^2},$$
$$g_k = -\frac{\partial}{\partial h}\left\{\frac{Zck}{2\rho h}K_1(kh) + 2M\left(g+\frac{\alpha}{\rho}k^2\right)e^{-kh}\right\}, \tag{21}$$
$$\omega_k^2 = gk + \alpha k^3/\rho.$$

Substituting (21) in the formula for S, we obtain after going to the limit $\mu = 0$ and averaging over the period of the oscillation

$$S = \frac{\pi^2}{2}\rho\frac{k_0^3}{\omega^2}\left(\frac{d\omega}{dk_0}\right)^{-1} g_{k_0}^2, \tag{22}$$

where k_0 is the solution of the equation $\omega^2 - \omega_k^0 = 0$. If $\omega^2 \ll g/a$ then, omitting the first term in the expression for g_k, we obtain

$$S = \pi^2\rho A^2(R^3\omega^3/g)^2\exp(-2\omega^2/hg), \tag{23}$$

which agrees with the result of Ref. 4.

For $\omega^2 \gg g/a$ we obtain

$$\nu = \frac{S}{M_\pm\omega^2 A^2} = \frac{\pi}{2}\omega\frac{m_\pm}{M_\pm}\left(\frac{\omega}{\omega_R}\right)^2\varphi(z), \quad T \ll \omega, \omega_h,$$
$$\varphi(z) = \{Qz^{-1}[K_1(z)(z^{-1}+2^{-1}) - 2^{-1}K_0(z)] + e^{-z}\}^2. \tag{24}$$

Here $m_\pm = \frac{2}{3}\pi\rho R^3$, $M_\pm$ are the ion masses, $z = k_0 h$, and $\omega_R^2 = \alpha/\rho R^2$. We note an interesting feature of $\varphi(z)$; the quantity $k_0 h$ does not depend on the clamping field $E_\perp$. Therefore S [Eq. (24)] is an increasing function of $E_\perp$, $S \sim E_\perp^{9/4}$, in contrast to (23), where S is nonmonotonic.

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1) This circumstance was pointed out to me by V. Shikin.

2) Of course, in this case there is no compensation, since the polarization and hydrodynamic forces have the same direction.

1. V. B. Shikin and Yu. P. Monarkha, Fiz. Nizk. Temp. **1**, 957 (1975) [Sov. J. Low Temp. Phys. **1**, 459 (1975)].
2. V. B. Shikin, Usp. Fiz. Nauk **121**, 457 (1977) [Sov. Phys. Usp. **20**, 226 (1977).
3. V. B. Shikin, Zh. Eksp. Teor. Fiz. **58**, 1748 (1970) [Sov. Phys. JETP **31**, 936 (1970)].
4. L. N. Sretenskiĭ, Teoriya volnovykh dvizheniĭ zhidkosti (Theory of Wave Motion of a Liquid), Nauka, 1977.
5. H. Lamb. Proc. Roy. Soc. London **A93**, 244 (1926).
6. T. Havelock, Proc. Roy. Soc. London **A118**, 24 (1928).
7. L. D. Landau and E. M. Lifshitz, Mekhanika sploshnykh sred (Fluid Mechanics), Gostekhizdat, 1953 [Pergamon, 1968].
8. N. E. Kochin, Sobran. soch. (Collected Works) Vol. 2, Gostekhizdat, 1949, pp. 86-104.
9. K. W. Schwarz and P. W. Stark, Phys. Rev. Lett. **22**, 20 (1969).
10. G. W. Reifild and F. Reif, *ibid.* **11**, 305 (1963).
11. D. Benin, Phys. B11, 145 (1975).
12. J. Poitrenaud and F. Williams, Phys. Rev. Lett. **29**, 1230 (1972); **32**, 1213 (1974).

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