Radiation of relativistic particles moving quasiperiodically

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The spectral and angular distributions and the polarization characteristics of the radiation are obtained in the quasiclassical approximation in the case when a relativistic particle executes a motion that is periodic in the frame moving with the mean velocity of the particle. The results are given in a form convenient for applications such as the description of radiation in undulators and the radiation produced when relativistic particles are channeled in crystals.

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1. INTRODUCTION

In recent years, the radiation of relativistic channeled particles has attracted much attention.[1-3] Investigations also continue into radiation produced in periodic structures (undulators), this being stimulated, in particular, by the development of free-electron lasers.[4-6] The development of laser technology, which makes it possible to obtain waves with a high intensity of the electromagnetic field (up to 10^9 V/cm), has stimulated extensive investigations of quantum processes in the field of a strong electromagnetic wave.[7-8] Of particular interest is the radiation of ultrarelativistic electrons and positions ($\gamma=\varepsilon/mc^2\gg 1$).

Many papers have been devoted to these problems; as a rule, the theoretical description is developed independently for each of these phenomena and by very different methods. However, it should be borne in mind that at $\gamma\gg 1$ the elementary radiation event takes place in the same way for all the above processes and, therefore, is described by the same expressions, since we are dealing with the radiation of a relativistic particle whose motion is periodic in the frame moving with the mean velocity of the particle (we shall say that such motion is quasiperiodic). The problems listed above differ because of the specific dependence on the physical parameters and also the need to average the obtained characteristics of the radiation over some of these parameters.

Let us consider the main qualitative features of the radiation in the case of quasiperiodic motion. It is convenient to do this in a comoving frame, in which the mean velocity of the particle is zero, and then make an inverse Lorentz transformation. In the comoving frame, the properties of the radiation depend strongly on the ratio of the kinetic energy to the rest mass of the particle. In the nonrelativistic limit, we have dipole radiation, which is completely determined by the Fourier components of the particle's velocity.[4] As a rule, one or a few of the first harmonics, which are multiples of the frequency of the particle's motion, are emitted. Transforming back to the laboratory system, we obtain as a result for the frequency of the emitted photon (Doppler effect)

$$\omega\approx\frac{n\omega_0}{1-\mathbf{n}_0\mathbf{V}}\approx\frac{2\gamma^2\omega_0 n}{1+\gamma^2\vartheta^2},\qquad(1.1)$$

where ω_0 is the frequency of the particle's motion in the laboratory system, ϑ is the angle of emission of the photon with respect to the mean velocity $\mathbf{V}$ of the particle's motion, n is the number of the harmonic, and $\mathbf{n}_0=\mathbf{k}/\omega$, where $\mathbf{k}$ is the wave vector of the photon. If allowance is made for the recoil on emission, the substitution $\omega\to\omega'=\omega\varepsilon/(\varepsilon-\omega)$ must be made in (1.1).

When the motion in the comoving frame becomes relativistic, the nature of the radiation changes. First, the higher harmonics become important in the radiation and, second, it becomes necessary to take into account the dependence of the longitudinal velocity of the particle (in the direction of the mean velocity) on its transverse motion. Indeed, to terms $\sim U/\varepsilon$ (U is the potential in which the particle moves) $\gamma=\text{const}$ and by definition $\gamma^2(1-v_\parallel^2-v_\perp^2)=1$, so that for the longitudinal ($v_\parallel$) and mean (V) velocities of the particle we have

$$v_\parallel\approx 1-(1+v_\perp^2\gamma^2)/2\gamma^2,\quad V=1-(1+\overline{v_\perp^2}\gamma^2)/2\gamma^2.\qquad(1.2)$$

Here and below, it is assumed that $|\mathbf{v}_\perp|\ll V$. The expression for the radiation frequency becomes

$$\omega\approx\frac{2\gamma^2\omega_0 n}{1+\gamma^2\vartheta^2+\overline{v_\perp^2}\gamma^2},\qquad(1.3)$$

where $\overline{v_\perp^2}$ is the mean square of the transverse velocity of the particle.

In the ultrarelativistic limit of the motion in the comoving frame ($v_\perp\gamma\gg 1$), the main contribution to the radiation is made by high harmonics with $n\gg 1$. The radiation spectrum is quasicontinuous and the well-known expressions that describe synchrotron radiation (see, for example,[1)] Ref. 9) hold. In this limit, the radiation is formed over a short section of the trajectory during a time $\tau\sim c/|\dot{\mathbf{v}}|\gamma$, and the characteristic frequency of the radiation, with allowance for the Doppler effect, is $\omega\sim|\dot{\mathbf{v}}|\gamma^3$. Bearing in mind that the main contribution to the radiation is made by angles $\vartheta\sim v_\perp$, we obtain the estimate

$$n\sim v_\perp^2|\dot{\mathbf{v}}|\gamma^3/\omega_0\approx(v_\perp\gamma)^3.\qquad(1.4)$$

The present paper is devoted to a systematic study of the radiation in the case of quasiperiodic motion. Since the nonrelativistic limit of transverse motion (see, for example, Ref. 4) and its ultrarelativistic limit (see, for example, Ref. 9) have been well studied, the main attention will be devoted to the general case. For each of the radiation characteristics we have tried to obtain representations convenient for both qualitative analysis and numerical calculations.

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2. BASIC EXPRESSIONS

The most adequate approach to the problem of the radiation of relativistic particles in quasiperiodic motion is a formalism that employs the operator quasiclassical method developed by two of the present authors (see Ref. 9). After the necessary commutations have been made and the exponential expressions disentangled, this method permits a transition to be made to quantities determined from the classical trajectory of the particle; the recoil resulting from the emission of the photon is also taken into account exactly. In the general case, the method is applicable if the commutators of the dynamical variables are small compared with the variables themselves (see Ref. 9). For example, in the case of motion in the field of a plane wave and in an undulator we must have $\hbar\omega_0/\varepsilon \ll 1$, and for the case of channeling $\hbar\omega_0/\varepsilon\overline{v_\perp^2} \ll 1$.

The general expression for the radiation intensity is

$$dI=e^2\frac{d^3k}{(2\pi)^2}\left|\int dt\, R(t)\, e^{ik'x(t)}\right|^2, \tag{2.1}$$

$$\begin{gathered} R(t)=\varphi_f^+[A(t)+i\sigma\mathbf{B}(t)]\varphi_i, \\ A(t)=\frac{\mathbf{e}^*\mathbf{p}(t)}{2(\varepsilon\varepsilon')^{1/2}}\left[\left(\frac{\varepsilon'+m}{\varepsilon+m}\right)^{1/2}+\left(\frac{\varepsilon+m}{\varepsilon'+m}\right)^{1/2}\right], \\ \mathbf{B}(t)=\frac{1}{2(\varepsilon\varepsilon')^{1/2}}\left\{\left(\frac{\varepsilon'+m}{\varepsilon+m}\right)^{1/2}[\mathbf{e}^*\times\mathbf{p}(t)]-\left(\frac{\varepsilon+m}{\varepsilon'+m}\right)^{1/2}[\mathbf{e}^*\times(\mathbf{p}(t)-\mathbf{k})]\right\}, \\ k'^\mu=k^\mu\varepsilon/\varepsilon', \quad \varepsilon'=\varepsilon-\omega, \quad k^\mu=(\omega,\mathbf{k}). \end{gathered} \tag{2.2}$$

The above expressions determine all the characteristics of the radiation, including the polarization characteristics. The intensity of the radiation, summed over the spin of the final electron and averaged over the spin of the initial electron, has the form [see Eqs. (10.75)–(10.76) in Ref. 9]

$$dI=e^2\frac{d^3k}{(2\pi)^2}\int dt_1\int dt_2\, L(t_1,t_2)\exp[ik'(x_1-x_2)], \tag{2.3}$$

$$L(t_1,t_2)=\frac{(\varepsilon+\varepsilon')^2}{4\varepsilon'^2}(\mathbf{e}^*\mathbf{v}_1)(\mathbf{e}\mathbf{v}_2)-\frac{\omega^2}{4\varepsilon'^2}(\mathbf{e}^*\mathbf{v}_2)(\mathbf{e}\mathbf{v}_1)+\frac{\omega^2}{4\varepsilon'^2}\left(\mathbf{v}_2\mathbf{v}_1-1+\frac{1}{\gamma^2}\right)(\mathbf{e}^*\mathbf{e}); \tag{2.4}$$

where we have introduced the notation $x_{1,2}=x(t_{1,2})$, $\mathbf{v}_{1,2}=\mathbf{v}(t_{1,2})$. Summing over the photon polarizations, we have

$$L=\sum_{\lambda=1}^{2}L(t_1,t_2)=\frac{\varepsilon^2+\varepsilon'^2}{2\varepsilon'^2}(\mathbf{v}_2\mathbf{v}_1-1)+\frac{\omega^2}{2\gamma^2\varepsilon'^2}. \tag{2.5}$$

There are two ways in which calculations with the above expressions can be made. The first is associated with direct calculation of the integral

$$\int v^\mu(t)\,e^{ik'x(t)}\,dt,$$

where $v^\mu=(1,\mathbf{v})$. This way reveals fairly clearly the physical picture of the phenomenon and the main features of the radiation. However, the further calculations may prove difficult, this being particularly so in the case of plane motion of the particle.

The other approach is based on direct use of the expressions (2.4) and (2.5). The integration over the angular variables of the emitted photon can be performed readily, which makes it possible to obtain, for example, expressions for the spectral distribution that are convenient for numerical calculations. Another advantage of this method is the comparative simplicity and clarity in the derivation of the asymptotic expressions for both nonrelativistic transverse motion and the other limiting case when the transverse motion is ultrarelativistic. In the case when $\omega \ll \varepsilon$, i.e., the radiation is classical, the integration over ω in these formulas leads to the well-known results of classical theory, expressed in terms of the Lienard–Wiechert potentials.

3. SPECTRAL AND ANGULAR DISTRIBUTIONS OF THE RADIATION

We represent the integral in (2.3) in the form

$$\int_{-\infty}^{+\infty} v^\mu(t)\,e^{ik'x(t)}\,dt=v_n^\mu\sum_{m=-\infty}^{\infty}e^{im\varphi_0}=\sum_n 2\pi v_n^\mu\delta(\varphi_0-2\pi n), \tag{3.1}$$

$$\varphi_0=\omega'T(1-\mathbf{n}_0\mathbf{V}), \quad v_n^\mu=\int_0^T v^\mu(t)\,e^{ik'x(t)}\,dt, \quad T=\frac{2\pi}{\omega_0}, \tag{3.2}$$

where T is the period of the motion. Substituting (3.1) in (2.5), we obtain for the intensity of the radiation in unit time

$$\begin{gathered} dI=e^2\frac{d^3k}{4\pi T}\sum_{n=1}^{\infty}\delta(\varphi_0-2\pi n)\left\{[1+(1+u)^2](|\mathbf{v}_n|^2-|v_n^0|^2)+\frac{u^2}{\gamma^2}|v_n^0|^2\right\}, \\ v^\mu=(v^0,\mathbf{v}), \quad v_0=1, \quad u=\omega/\varepsilon'=\omega/(\varepsilon-\omega). \end{gathered} \tag{3.3}$$

Using (1.2), introducing the notation

$$\Delta(t)=\omega_0\int_0^t(\mathbf{v}_\perp^2(t')-\overline{\mathbf{v}_\perp^2})\,dt', \tag{3.4}$$

where $\omega_0=2\pi/T$, and ignoring terms of higher order in $1/\gamma^2$ and $v_\perp^2$, we obtain the following expression for the spectral and angular distribution of the radiation:

$$dI=e^2\frac{d^3k}{(2\pi)^2}\frac{1+u}{\omega_0\gamma^2}\sum_{n=1}^{\infty}\delta(\varphi_0-2\pi n)\left[-|I_0|^2+\gamma^2\left(1+\frac{u^2}{2(1+u)}\right)(|\mathbf{I}_\perp|^2-\operatorname{Re}I_0^*I_\parallel)\right], \tag{3.5}$$

$$\begin{gathered} I_0=\int_0^{2\pi}e^{if(\psi)}\,d\psi, \quad \mathbf{I}_\perp=\int_0^{2\pi}\mathbf{v}_\perp(\psi)\,e^{if(\psi)}\,d\psi, \\ I_\parallel=\int_0^{2\pi}\mathbf{v}_\perp^2(\psi)\,e^{if(\psi)}\,d\psi, \quad \psi=\omega_0 t, \\ f(\psi)=n\psi+\omega'\Delta(\psi)/2\omega_0-\omega'\mathbf{n}_0\mathbf{x}_\perp(\psi). \end{gathered} \tag{3.6}$$

Here, we have used the circumstance that $\varphi_0=\omega'T(1-\mathbf{n}_0\mathbf{V})$, and the frequency of the emitted photons is determined by the relation [cf. Eq. (1.3)]

$$\omega'=\omega\frac{\varepsilon}{\varepsilon-\omega}=\frac{2\gamma^2\omega_0 n}{1+\gamma^2\vartheta^2+\rho/2}, \tag{3.7}$$

where ϑ is the angle of emission of the photon with respect to $\mathbf{V}$, and $\rho=1\gamma^2\overline{v_\perp^2}$. The nature of the radiation depends strongly on ρ: for $\rho \ll 1$, we have dipole radiation, for $\rho \gg 1$ radiation of synchrotron nature.

The integrals in (3.6) have a relatively simple form only for circular transverse motion, when $|\mathbf{v}_\perp|=\text{const}$. In this case $\Delta(\psi)=0$,

$$\begin{gathered} f(\psi)=n\psi-\varkappa\sin(\psi-\varphi), \quad I_\parallel=v_\perp^2 I_0, \\ |I_0|^2=(2\pi)^2 J_n^2(\varkappa), \quad \varkappa=\omega' v_\perp\vartheta/\omega_0, \\ |\mathbf{I}_\perp|^2=2\pi^2 v_\perp^2[J_{n+1}^2(\varkappa)+J_{n-1}^2(\varkappa)], \end{gathered} \tag{3.8}$$

where $J_n(\varkappa)$ is a Bessel function, and φ is the azimuthal angle of emission of the photon. Integrating over the emission angles of the photon [the integration over φ is trivial, and the integration over ϑ reduces to integration of the δ function in (3.5)], we obtain the following expression for the spectral density of the radiation in the case of circular motion of the particle (index cr):

$$dI_{cr}^{(i)}=e^2\frac{\omega\, d\omega}{\gamma^2}\sum_{n=1}^{\infty}\vartheta(n\omega_c-\omega')G_{cr}^{(i)}, \tag{3.9}$$

$$G_{cr}^{(0)}=-J_n^2(\varkappa)+{}^1/_4\rho[1+u^2/2(1+u)][J_{n+1}^2(\varkappa)+J_{n-1}^2(\varkappa)-2J_n^2(\varkappa)],\ G_{cr}^{(1)}=0,$$

$$G_{cr}^{(2)}=[1+u^2/2(1+u)]J_n(\varkappa)[J_{n-1}(\varkappa)-J_{n+1}(\varkappa)](n\rho/2\varkappa-\varkappa/2\xi),$$

$$G_{cr}^{(3)}=J_n^2(\varkappa)(1+\rho/2)-{}^1/_2\rho J_{n-1}(\varkappa)J_{n+1}(\varkappa).$$

Here and below, the index 0 is used for expressions summed over the polarizations of the photon;

$$\xi^{(i)}=dI^{(i)}/dI^{(0)}$$

are the Stokes parameters; and

$$\omega_c=\frac{2\gamma^2\omega_0}{1+\rho/2},\qquad \xi=\frac{\omega'}{2\gamma^2\omega_0}.$$

The vectors $\mathbf{e}^{(\lambda)}$, the photon polarizations, are chosen such that

$$\mathbf{e}^{(1)}=[\mathbf{n}_0\times\mathbf{V}]/|[\mathbf{n}_0\times\mathbf{V}]|,\quad \mathbf{e}^{(2)}=[\mathbf{n}_0\times\mathbf{e}_1].$$

In the case of plane motion of the particle, we can, without loss of generality, take $v_x(\psi)$ to be an even function of $\psi(v_y=0)$; then $f(\psi)$ in the expression (3.6) becomes (apart from a constant term) an odd function of ψ. Thus, apart from a common phase factor, the quantities I_0, I_x, and $I_\|$ are real. This means that for plane quasiperiodic motion the radiation can be only linearly polarized. From the physical point of view, this fact is obvious, since for such motion there is no pseudovector with which circular polarization of the radiation could be associated.

With allowance for what we have said above, we obtain for plane motion (subscript p)

$$\begin{gathered} dI_p^{(i)}=e^2\frac{\omega\, d\omega\, d\varphi}{(2\pi)^3\gamma^2}\sum_{n=1}^{\infty}\vartheta(n\omega_c-\omega')G_p^{(i)},\\ G_p^{(0)}=-I_0^2+\gamma^2[1+u^2/2(1+u)](I_x^2-I_0I_\|),\\ G_p^{(2)}=0,\quad G_p^{(1)}=2I_x\gamma^2\sin\varphi(I_x\cos\varphi-\vartheta I_0),\\ G_p^{(3)}=[I_x^2\sin^2\varphi-(I_x\cos\varphi-\vartheta I_0)^2]\gamma^2. \end{gathered}\tag{3.10}$$

After integration over the azimuthal emission angle of the photon $G_p^{(1)}=0$.

Assuming that the plane motion of the particle is determined by some potential $U(x)$ and using the quasiclassical condition of quantization of the energy in this one-dimensional well, we can represent $\overline{v_x^2}$ in the form

$$\overline{v_x^2}=\frac{1}{T\varepsilon}\oint p_x\,dx\approx\frac{\omega_0 n_x}{\varepsilon}=-2\frac{\partial\varepsilon_\perp(\varepsilon,n_x)}{\partial\varepsilon}, \tag{3.11}$$

where n_x is the number of the level, and $\varepsilon_\perp$ is the energy of the transverse motion.

In the special case of motion in an oscillator potential, the spectral density of the radiation can be written in the form

$$dI_p=e^2\frac{\omega\, d\omega\, d\varphi}{(2\pi)^3\gamma^2}\sum_{n=1}^{\infty}\vartheta(n\omega_c-\omega')\left[-A_0^2+\rho\left(1+\frac{u^2}{2(1+u)}\right)(A_1^2-A_0A_2)\right], \tag{3.12}$$

$$A_m=\int_0^{2\pi}e^{if(\psi)}\cos^m\psi\, d\psi,$$

$$f(\psi)=n\psi-\alpha_0\sin\psi+\beta_0\sin 2\psi, \tag{3.13}$$

$$\alpha_0=\frac{\omega'}{\omega_0}\left(\frac{2\varepsilon_\perp}{\varepsilon}\right)^{1/2}\vartheta\cos\varphi,\qquad \beta_0=\frac{\omega'\varepsilon_\perp}{4\omega_0\varepsilon}.$$

In this case, $\rho=2\gamma\varepsilon_\perp/m$.

The expressions (3.9) and (3.12) are identical to the expressions for the spectral density of the radiation in the field of a monochromatic plane wave of circular and linear polarization, respectively. This agreement is not fortuitous and is due to the circumstance that in the comoving frame of the electron the field in which the electron moves is, with relativistic accuracy, a wave field. The special distribution of the radiation in the field of a plane wave has been investigated in detail already[7,8] (see also the literature cited there).

As one further example of the use of the general expressions (3.5) and (3.6), we consider the case when the transverse motion takes place in an ellipse, but the time dependence of the coordinates is not harmonic:

$$x=a(\cos\xi-\delta),\quad y=a(1-\delta^2)^{1/2}\sin\xi,\quad \psi=\omega_0 t=\xi-\delta\sin\xi. \tag{3.14}$$

Such a motion can be realized if the potential $U(\mathbf{x}_\perp)$ has Coulomb form. In this case,

$$\begin{gathered} f(\psi)=n\xi-\mu\sin(\xi+\xi_0),\\ \operatorname{ctg}\xi_0=[n\delta-\delta\omega'\omega_0a^2+\omega' a(1-\delta^2)^{1/2}\vartheta\sin\varphi]/(a\omega'\vartheta\cos\varphi),\\ \mu^2=[n\delta-\delta\omega'\omega_0a^2+a\omega'(1-\delta^2)^{1/2}\vartheta\sin\varphi]^2+(a\omega'\vartheta\cos\varphi)^2,\quad \rho=2\gamma^2a^2\omega_0^2. \end{gathered}\tag{3.15}$$

Substituting (3.14) and (3.15) in Eqs. (3.5) and (3.6) and integrating over the polar angle ϑ, we obtain the intensity distribution of the radiation:

$$dI=\frac{e^2}{2\pi\gamma^2}\omega\, d\omega\, d\varphi\sum_{n=1}^{\infty}\vartheta(n\omega_c-\omega')\left\{\frac{\rho}{2}\left[1+\frac{u^2}{2(1+u)}\right]\right.$$
$$\left.\times\left[\left(\frac{n^2}{\mu^2}-1\right)J_n^2(\mu)+J_n'^2(\mu)\right]-\left(1-\frac{\delta n}{\mu}\cos\xi_0\right)^2J_n^2(\mu)-\delta^2\sin^2\xi_0J_n'^2(\mu)\right\}. \tag{3.16}$$

For $\delta=0$, the expression (3.16) goes over into the intensity distribution for circular transverse motion [cf. (3.9)], and in the limit $\rho\to 0$ we have the dipole case.

4. SPECTRAL DISTRIBUTION OF THE RADIATION

To obtain the spectral distribution of the radiation intensity, we can integrate over the photon emission angles in the expression (2.3) by means of the relation

$$\int_0^{\infty}\vartheta\, d\vartheta\int_0^{2\pi}d\varphi\exp[i(a\vartheta^2+b\vartheta\cos\varphi)]=\frac{i\pi}{a}\exp\left(-i\frac{b^2}{4a}\right). \tag{4.1}$$

Going over also to the variables

$$t={}^1/_2\omega_0(t_2-t_1),\quad \tau={}^1/_2\omega_0(t_2+t_1),$$

we find for the radiation intensity

$$dI=\frac{ie^2m^2u\, du}{(2\pi)^2(1+u)^3}\int_{-\infty}^{+\infty}\frac{dt}{t}\int_0^{2\pi}d\tau\left[\gamma^2(1-\mathbf{v}_1\mathbf{v}_2)\left(1+\frac{u^2}{2(1+u)}\right)\right.$$
$$\left.-\frac{u^2}{2(1+u)}\right]\exp\left\{-\frac{iu\varepsilon}{2\omega_0\gamma^2}\left[2t\left(1+\frac{\rho}{2}\right)+(\Delta_2-\Delta_1)\gamma^2-\frac{1}{2t}\omega_0^2\gamma^2(\mathbf{x}_1-\mathbf{x}_2)_\perp^2\right]\right\}. \tag{4.2}$$

The indices 1 and 2 denote the dependence on the variables $t_1=(\tau-t)/\omega_0$ and $t_2=(\tau+t)/\omega_0$, respectively.

Note an important circumstance. If the integral over t in (4.2) is to be correctly defined, it is sufficient to assume that the contour of integration with respect to t is displaced below the real axis. This operation corresponds to the subtractional procedure in the finding of the mass operator in an external field (see Ref. 8) and ensures, in particular, vanishing of the radiation intensity when the field is switched off.

We consider the case of elliptic transverse motion of the particle:

$$v_x(t)=a\cos\omega_0 t,\quad v_y(t)=b\sin\omega_0 t.$$

We reduce the expressions in (4.2) to the form

$$\begin{gathered}\gamma^2(1-\mathbf{v}_1\mathbf{v}_2)=1+\rho\sin^2 t(1-\alpha\cos 2\tau),\\ \gamma^2(\Delta_2-\Delta_1)={}^1/_2\alpha\rho\sin 2t\cos 2\tau,\\ \gamma^2\omega_0^2(\mathbf{x}_1-\mathbf{x}_2)_\perp{}^2=2\rho\sin^2 t(1+\alpha\cos 2\tau);\\ \rho=\gamma^2(a^2+b^2),\quad \alpha=(a^2-b^2)/(a^2+b^2).\end{gathered}\tag{4.3}$$

After this, the integration over τ can be readily performed[2]:

$$\begin{gathered}dI=\frac{ie^2m^2}{2\pi}\frac{u\,du}{(1+u)^3}\int_{-\infty}^{+\infty}\frac{dt}{t}\Big\{J_0(\alpha z_0)+(1+u^2/2(1+u))\rho\sin^2 t\\ \times[J_0(\alpha z_0)-i\alpha J_1(\alpha z_0)]+\frac{iu(2+u)}{2(1+u)}\rho\beta(\zeta V)\sin t\left(\frac{\sin t}{t}-\cos t\right)J_0(\alpha z_0)\Big\}e^{-iz_1};\\ z_0=x_0-x_1,\quad z_1=2\lambda t-x_0,\quad x_0=\rho\xi\sin^2 t/t,\\ x_1={}^1/_2\rho\xi\sin 2t,\quad \xi=u\varepsilon/2\omega_0\gamma^2,\quad \lambda=\xi(1+\rho/2),\quad \beta=2ab/(a^2+b^2).\end{gathered}\tag{4.4}$$

The expression (4.4) corresponds to the imaginary part of the expression for the mass operator of an electron in the field of a plane wave obtained earlier in Ref. 8.

Note that, using the expressions (2.3) and (2.4) and the calculation procedure adopted above, we can obtain all the polarization properties of the radiation. For example, in the case of a helical motion of the particle ($\alpha=0$), we have

$$\begin{gathered}dI_{cr}^{(i)}=\frac{e^2m^2}{2\pi}\frac{u\,du}{(1+u)^3}\int_{-\infty}^{\infty}\frac{dt}{t}A_{cr}^{(i)}(t)\exp[-2i\lambda t+ix_0(t)],\\ A_{cr}^{(0)}(t)=i\left[1+\left(1+\frac{u^2}{2(1+u)}\right)\rho\sin^2 t\right],\\ A_{cr}^{(2)}(t)=\left(1+\frac{u^2}{2(1+u)}\right)\rho\sin t\left(\cos t-\frac{\sin t}{t}\right),\\ A_{cr}^{(3)}=\frac{\rho}{2x_0(t)}(e^{-ix_0(t)}-1)-i.\end{gathered}\tag{4.5}$$

In the case of plane motion ($\alpha=1$), we obtain the following expressions for the intensity and polarization of the radiation:

$$\begin{gathered}dI_p=\frac{ie^2m^2}{2\pi}\frac{u\,du}{(1+u)^3}\int_{-\infty}^{+\infty}\frac{dt}{t}\Big[J_0(z_0)\\ +\left(1+\frac{u^2}{2(1+u)}\right)\rho\sin^2 t(J_0(z_0)-iJ_1(z_0))\Big]e^{-iz_1}.\end{gathered}\tag{4.6}$$

The degree of linear polarization $\xi^{(3)}$ is given by

$$\begin{gathered}\xi^{(3)}=dI^{(3)}/dI_p,\\ dI^{(3)}=-\frac{e^2m^2}{4\pi}\frac{u\,du}{(1+u)^3}\int_{-\infty}^{+\infty}\frac{dt}{t}e^{-iz_1}\{2iJ_0(z_0)+\rho[(1/x_0+it\sin 2t)\\ \times(J_0(z_0)-J_0(x_1)e^{-ix_0})+t\sin 2t(J_1(z_0)+J_1(x_1)e^{-ix_0})]\}.\end{gathered}\tag{4.7}$$

Let us analyze this expression for the case of plane motion. For simplicity, we shall assume that soft photons are emitted,[3] i.e., $u\ll 1$ [this is valid if $2\omega_0\gamma^2(1+\rho)^{1/2}/\varepsilon\ll 1$]. In this case, the expression (4.4) becomes

$$dI_p=\frac{2ie^2}{\pi}\omega_0^2\gamma^2\xi\,d\xi\int_{-\infty}^{+\infty}\frac{dt}{t}\{J_0(z_0)+\rho\sin^2 t[J_0(z_0)-iJ_1(z_0)]\}e^{-iz_1};\tag{4.8}$$

$$\begin{gathered}\xi=\frac{\omega}{\omega_c}=\frac{\omega}{2\omega_0\gamma^2},\quad z_0=\rho\xi\left(\frac{\sin^2 t}{t}-\frac{\sin 2t}{2}\right),\\ z_1=2\xi t\left[1+\frac{\rho}{2}\left(1-\frac{\sin^2 t}{t^2}\right)\right],\quad \rho=\gamma^2a^2.\end{gathered}\tag{4.9}$$

In general, the integral in (4.8) is a discontinuous function of the parameter $\lambda=\xi(1+\rho/2)$ at the points where $\lambda=n$ with n an integer. To separate the discontinuity, we use the relation

$$\int_{-\infty}^{+\infty}\frac{dt}{t}F(t)=\int_{-\infty}^{+\infty}\frac{dt}{t}[F(t)-F_a(t)]+\int_{-\infty}^{+\infty}\frac{dt}{t}F_a(t).\tag{4.10}$$

From (4.8), we have

$$\begin{gathered}F_a(t)=\underset{t\to\infty}{F(t)}=\{J_0(y_0)+\rho^2\sin^2 t[J_0(y_0)+iJ_1(y_0)]\}e^{-2i\lambda t},\\ y_0={}^1/_2\xi\rho\sin 2t.\end{gathered}\tag{4.11}$$

Using the Fourier expansion of the Bessel functions,

$$\begin{gathered}J_0(a\sin 2t)=J_0^2\left(\frac{a}{2}\right)+2\sum_{n=1}^{\infty}J_n^2\left(\frac{a}{2}\right)\cos 4nt,\\ J_1(a\sin 2t)=2\sum_{n=0}^{\infty}J_n\left(\frac{a}{2}\right)J_{n+1}\left(\frac{a}{2}\right)\sin 2(2n+1)t,\end{gathered}\tag{4.12}$$

we can explicitly separate the discontinuities at the points $\lambda=n$. The remaining integral is a continuous function of λ, and in the limit $t\to\infty$ the integrand behaves as $1/t^2$, which is convenient for numerical calculations. The separated part of the spectral distribution of the radiation intensity has the form

$$\begin{gathered}\frac{dI_{p(a)}}{d\xi}=4e^2\gamma^2\omega_0^2\sum_{n=0}^{\infty}\{\vartheta[2(n+1)-\lambda][2(n+1)-\lambda]\\ \times J_{n+1}^2(y)+\vartheta(2n+1-\lambda)y[J_n(y)-J_{n+1}(y)]^2\},\quad y=\rho\xi/4.\end{gathered}\tag{4.13}$$

It can be seen from the expression (4.13) that discontinuities of the integral exist only for odd n, i.e., for $\lambda=2m+1$. In the language of the radiation in the field of a plane wave, this circumstance is due to the fact that if the angle of the emitted photon is $\vartheta=0$ only an odd number of photons can be absorbed from a linearly polarized wave, this, in its turn, being due to the conservation of the projection of the angular momentum onto the momentum.

In the case of helical motion, the integral in (4.5) is a discontinuous function at $\lambda=1$. Proceeding as in the case of plane motion, we separate explicitly the terms containing this discontinuity. For simplicity, we consider the case $\omega/\varepsilon\ll 1$:

$$\begin{gathered}dI^{(i)}=2e^2\omega_0^2\gamma^2\xi d\xi\left\{\frac{1}{\pi}\int_{-\infty}^{+\infty}\mathcal{A}^{(i)}e^{-2i\lambda t}[e^{ix_0(t)}-1-ix_0(t)]\frac{dt}{t}+F_a^{(i)}\right\};\\ \mathcal{A}^{(0)}(t)=i[1+\rho\sin^2 t],\quad \mathcal{A}^{(2)}(t)=\rho\sin t\left(\cos t-\frac{\sin t}{t}\right),\\ \mathcal{A}^{(3)}(t)=-(i+t/2\xi\sin^2 t);\\ F_a^{(0)}={}^1/_2\rho\{[1-2\xi(1+\rho)(1-\lambda)]\vartheta(1-\lambda)+{}^1/_2\rho\xi(2-\lambda)\vartheta(2-\lambda)\},\\ F_a^{(2)}={}^1/_2\rho\{[1-2(1-{}^1/_2\rho\xi(1-2\lambda))(1-\lambda)]\vartheta(1-\lambda)\\ -{}^1/_2\rho\xi(1-\lambda)(2-\lambda)\vartheta(2-\lambda)\},\\ F_a^{(3)}=\rho\xi(1-\lambda)\vartheta(1-\lambda).\end{gathered}\tag{4.14}$$

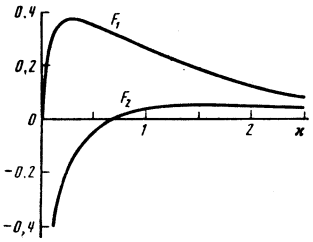

FIG. 1.

In the dipole approximation ($\rho \ll 1$), we can ignore the integral in (4.14), and in the expression for F_a we retain only the leading terms in ρ; then

$$\begin{aligned} F_a^{(0)} &\approx {}^1/_2\rho[1-2\xi(1-\xi)]\vartheta(1-\xi), \\ F_a^{(2)} &\approx {}^1/_2\rho(2\xi-1)\vartheta(1-\xi), \\ F_a^{(3)} &\approx \rho\xi(1-\xi)\vartheta(1-\xi). \end{aligned} \tag{4.15}$$

As can be seen from (4.15), $\xi^{(2)}=1$ at $\xi=1$ and $\xi^{(2)} \to -1$ as $\xi \to 0$. This is due to the helicity conservation law (from the point of view of the Compton effect), as can be seen by noting that in the electron rest frame a circularly polarized photon is scattered backward or almost forward.

To conclude this section, we give the asymptotic expressions for the spectral distribution of the radiation intensity (4.4) in the case opposite to the dipole case when $\rho \gg 1$. Using the integral representations of the Bessel functions, and also the method of stationary phase for $\xi\rho \gg 1$, we obtain for unpolarized particles

$$\begin{aligned} dI = {} & \frac{e^2 m^2 u\,du}{\pi\sqrt{3}(1+u)^3}\int_0^{2\pi}\frac{d\psi}{2\pi}\Big\{\int_\varkappa^\infty dy K_{5/3}(y) \\ & +2[f(u)-1]K_{2/3}(\varkappa)-\frac{4}{3\rho(1-\alpha\cos\psi)\varkappa}\Big[\Big(1-\frac{8}{15}g(\alpha)\Big) \\ & \times f(u)\frac{d}{d\varkappa}(\varkappa K_{1/3}(\varkappa))+\frac{2f(u)-1}{10}g(\alpha)\frac{d}{d\varkappa}(\varkappa^2 K_{2/3}(\varkappa))\Big]\Big\}, \\ & f(u)=1+\frac{u^2}{2(1+u)},\quad g(\alpha)=\frac{1-2\alpha\cos\psi}{1-\alpha\cos\psi},\quad \varkappa^2=\frac{32\xi^2}{9\rho(1-\alpha\cos\psi)}, \end{aligned} \tag{4.16}$$

where α is determined in (4.3) and ξ in (4.4). The first two terms in the curly brackets agree with the expression for the intensity of the synchrotron radiation of an ultrarelativistic particle [see Eq. (10.27) in Ref. 8], and the remainder are corrections $\sim 1/\rho$. For $\alpha=0$, $u \ll 1$, we can write (4.16) in the form

$$\frac{\gamma^2}{e^2\omega_c^2}\frac{dI}{d\varkappa}=F_1(\varkappa)+\frac{1}{\rho}F_2(\varkappa).$$

The functions F_1 and F_2 are shown in Fig. 1.

5. ANGULAR DISTRIBUTION OF THE RADIATION INTENSITY

We consider, finally, the angular distribution of the radiation integrated over all frequencies. When $\omega \sim \omega_c \ll \varepsilon$, such integration in Eqs. (2.4) and (2.5) gives a δ function of the difference between the times, and the subsequent calculation of the intensity and linear polarization of the radiation leads to results that can be obtained directly from the Lienard–Wiechart potentials (see Ref. 9).

In the case of helical motion

$$\begin{aligned} dI^{(i)} &= 4e^2\rho\omega_0^2\gamma^2\eta\,d\eta\frac{B^{(i)}(\eta)}{(\Omega^2-2\rho\eta^2)^{5/2}}, \\ B^{(0)} &= \frac{1}{2}\Omega^2+\Big(\frac{\rho}{2}-1\Big)\eta^2-\frac{5\rho\eta^4}{2(\Omega^2-2\rho\eta^2)}, \\ B^{(3)} &= \frac{1}{2}\eta^2\Big(\frac{\rho}{2}+2\Big)+\frac{5\rho\eta^4}{2(\Omega^2-2\rho\eta^2)}, \\ \Omega &= 1+{}^1/_2\rho+\eta^2,\quad \rho=2\gamma^2\overline{v_\perp^2},\quad \eta=\gamma\vartheta. \end{aligned} \tag{5.1}$$

Here, as in the preceding sections, the index 0 is used for the total intensity of the radiation, and the remaining indices are used for the quantities needed to find the Stokes parameters $\xi^i = dI^{(i)}/dI^{(0)}$.

For $dI^{(2)}$, we have

$$\begin{aligned} dI^{(2)} = {} & \frac{4}{\pi}e^2\rho\omega_0^2\gamma^2\eta\,d\eta\int_0^\infty dt \\ & \times\Big[\frac{-{}^1/_2\sin 2t(\rho\eta^2\sin^2 t+t^2\Omega^2)+3\Omega t\eta^2\sin^2 t}{(t^2\Omega^2-2\rho\eta^2\sin^2 t)^{5/2}} \\ & +\frac{(1+\rho/2)^2+\eta^2(\rho/2-1-2\eta^2)}{t^2(\Omega^2-2\rho\eta^2)^{5/2}}\Big]. \end{aligned} \tag{5.2}$$

Integrating (5.1) and (5.2) over η, we obtain for the total intensity of the radiation and the Stokes parameters

$$\begin{aligned} & I=\frac{2}{3}e^2\frac{\rho}{2}\omega_0^2\gamma^2,\quad \xi^{(3)}=\frac{1}{2}\Big(1+\frac{\rho}{4(1+\rho/2)}\Big), \\ & \xi^{(2)}=-\frac{3}{\pi}\int_0^\infty dt\frac{\sin t(t\cos t-\sin t)}{[t^2(1+\rho/2)-{}^1/_2\rho\sin^2 t]^2}. \end{aligned} \tag{5.3}$$

For $\rho \ll 1$, the expressions (5.1)–(5.3) go over into the expressions of the dipole approximation.[4] In the case $\rho \gg 1$, we obtain the corresponding results for synchrotron radiation.[9] The dependence of the degree of circular polarization $\xi^{(2)}$ determined by (5.3) on the parameter ρ is shown in Fig. 2.

In the plane case for sinusoidal transverse motion of the particle, the angular distribution of the radiation intensity is

$$\begin{aligned} \frac{dI}{d\varphi\,d\eta} = {} & \frac{2e^2}{\pi}\rho\omega_0^2\gamma^2\eta\int_0^{2\pi}\frac{d\psi}{2\pi} \\ & \times\sin^2\psi\Big[\frac{1}{F^3}-\frac{4}{F^4} \\ & +\frac{4(1+\eta^2\sin^2\varphi)}{F^5}\Big], \\ F = {} & 1+\eta^2+\rho\cos^2\psi \\ & -2\rho^{1/2}\eta\cos\psi\cos\varphi. \end{aligned} \tag{5.4}$$

The integrals over ψ in (5.4) can be calculated by means

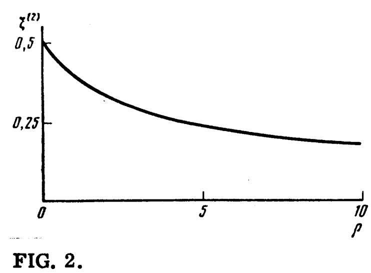

FIG. 2.

of the relations

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{\sin^2\psi}{F^l}d\psi=\frac{(-1)^{l-1}}{(l-1)!}\frac{d^{l-1}}{dx^{l-1}}f(x)|_{x=1},$$
$$f(x)=\frac{1}{\rho}\left\{\left[\frac{h_+ +(h_+^2-4\rho\eta^2\cos^2\varphi)^{1/2}}{h_- +(h_+^2-4\rho\eta^2\cos^2\varphi)^{1/2}}\right]^{1/2}-1\right\}, \tag{5.5}$$
$$h_\pm=x+\eta^2\pm\rho.$$

In the dipole case $\rho\ll 1$ and in (5.4) we can set $F\approx 1+\eta^2$; then the integration over ψ in (5.4) becomes trivial. At the same time, for both plane (5.4) and circular (5.1) motion the main contribution to the radiation is made by angles $\vartheta\sim\gamma^{-1}(\eta\sim 1)$ and arbitrary angles φ. For relativistic transverse motion $\rho\gg 1$, and the main contribution to the radiation in the case of plane motion is made by the region of angles $\vartheta\leq\rho^{1/2}/\gamma$, $|\sin\varphi|\leq\rho^{-1/2}$, i.e., the radiation is concentrated in the plane of the motion. For circular transverse motion in the limit, the radiation is concentrated on the surface of a cone with angle $\vartheta\approx\gamma^{-1}(\rho/2)^{1/2}$ and width $\Delta\vartheta\sim 1/\gamma$ of the distribution.

6. CONCLUSIONS

The results obtained above give the spectral, angular, and polarization characteristics of the radiation in the quasiclassical approximation in the case of quasiperiodic motion for arbitrary values of the parameter $\rho=2\gamma^2\overline{v_\perp^2}$, which characterizes the extent to which the transverse motion is relativistic. It is of interest to compare our results with those of other authors.

In a series of papers (see Ref. 5 and the literature cited there), Alferov, Bashmakov, and Bessonov considered radiation in undulators. In the framework of classical theory and on the basis of the Lienard–Wiechert potentials, they obtained the spectral and angular characteristics of the radiation in a form analogous to the well-known expressions for the radiation in the case of circular motion. If many harmonics contribute to the radiation, the use of such expressions is very difficult.

The region $\rho\gtrsim 1$ has been discussed by a number of authors[3,10] in connection with the problem of the radiation produced when particles are channeled in crystals. The basic formula [Eq. (14)] in the first of the papers in Ref. 3 is incorrect (the square of a sum is replaced by a sum of squares). The assertion that for $\omega\ll\varepsilon$ the coefficients in this formula satisfy $C_{ff'}\approx\delta_{ff'}$ is erroneous, since in the region of energies in which the dipole approximation is invalid the important point is that $C_{ff'}$ does not reduce to $\delta_{ff'}$. Also incorrect is the assertion that in a certain interval of emission angles around $\vartheta=0$ the radiation is of a dipole nature for all energies, whereas, in fact, in the case $\beta\gg 1$ (in the notation of Zhevago) all harmonics up to $n\sim\beta^3$ are radiated. This is the cause of the incorrect estimate of the spectral density of the radiation [Eq. (25)]. In the second paper in Ref. 3, the treatment is given for scalar particles, but in the region $\omega\sim\varepsilon$ (where quantum recoil effects are manifested) the spin terms become important, so that the conclusion drawn at the end of this paper—that the obtained expressions are valid for the radiation of electrons of arbitrarily high energies—is false. Equation (35) in the second paper in Ref. 3, which describes radiation in the potential $-\alpha/\rho$, is incorrect [cf. Eq. (3.16) of the present paper]. Contrary to the assertion of the authors, Eq. (35) does not lead to Eq. (36), which describes the radiation in the dipole approximation. In Ref. 10, the basic expression for the spectral properties of the radiation is incorrect [in the notation of Eq. (3.6) of the present paper, the term with $\Delta(\psi)$ is absent in Ref. 10]. Therefore, all the remaining expressions describing the radiation intensity are incorrect.

1) It should be borne in mind that these expressions cease to hold at sufficiently low frequencies when $n\sim 1$.

2) For greater generality, we have included in the expression (4.4) the spin term; ζ is the spin vector in the electron's rest frame.

3) For $u\gtrsim 1$, the treatment is similar.

[1] V. N. Baĭer, V. M. Katkov, and V. M. Strakhovenko, Dokl. Akad. Nauk SSSR **246**, 1347 (1979) [Sov. Phys. Dokl. **24**, 469 (1979)]; Preprint 80-03 [in Russian], Institute of Nuclear Physics, Siberian Branch, USSR Academy of Sciences (1980); V. N. Baier, V. M. Katkov, and V. M. Strakhovenko, Phys. Lett. **73A**, 414 (1979).

[2] M. A. Kumakhov, Phys. Lett. **57A**, 17 (1976); M. A. Kumakhov, Zh. Eksp. Teor. Fiz. **72**, 1489 (1977) [Sov. Phys. JETP **45**, 781 (1977)].

[3] N. K. Zhevago, Zh. Eksp. Teor. Fiz. **75**, 1389 (1978) [Sov. Phys. JETP **48**, 701 (1978)]; V. A. Bazylev, V. I. Glebov, and N. K. Zhevago, Zh. Eksp. Teor. Fiz. **78**, 62 (1980) [Sov. Phys. JETP **51**, 31 (1980)].

[4] V. N. Baĭer, V. N. Katkov, and V. M. Strakhovenko, Zh. Eksp. Teor. Fiz. **63**, 2121 (1972) [Sov. Phys. JETP **36**, 1120 (1973)].

[5] D. F. Alferov, Yu. A. Bashmakov, and E. G. Bessonov, Tr. Fiz. Inst. Akad. Nauk SSSR **80**, 100 (1975).

[6] V. N. Baier and A. I. Milstein, Phys. Lett. A **65**, 319 (1978).

[7] A. I. Nikishov and V. I. Ritus, Tr. Fiz. Inst. Akad. Nauk SSSR **111**, 16, 56 (1979).

[8] V. N. Baĭer, V. M. Katkov, A. I. Mil'shteĭn, and V. M. Strakhovenko, Zh. Eksp. Teor. Fiz. **69**, 783 (1975) [Sov. Phys. JETP **42**, 400 (1975)].

[9] V. N. Baĭer, V. M. Katkov, and V. S. Fadin, Izluchenie relyativistskikh élektronov (Radiation of Relativistic Electrons), Atomizdat, Moscow (1973), §§9, 10.

[10] M. A. Kumakhov and Kh. G. Trikalinos, Zh. Eksp. Teor. Fiz. **78**, 1623 (1980) [Sov. Phys. JETP **51**, 815 (1980)].

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