Contribution to the microscopic theory of the fluctuations of a quantum system interacting with a Gaussian thermostat

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(Submitted 12 March 1980, resubmitted 8 October 1980)
Zh. Eksp. Teor. Fiz. **80**, 1071–1086 (March 1981)

Exact stochastic equations are obtained for a system interacting with a microscopic system (thermostat) whose variables fluctuate. New expressions are obtained for the fluctuation sources and their correlation functions are calculated. The general method is demonstrated with a two-level system interacting with a photon thermostat as the example. It is shown that there is no ultraviolet divergence in this method.

PACS numbers: 32.80. – t, 05.30. – d

1. Much attention is being paid of late to microscopic derivation of simplified equations (in the form of stochastic Langevin equations) that describe simultaneously the relaxation and the fluctuations in macroscopic systems.

The need for simultaneous study of the transport processes (relaxation) and of fluctuations is dictated by a number of physical phenomena that take place under nonstationary conditions at strong disequilibrium. At the same time, relaxation and fluctuations are very closely related processes, as is directly manifest, in particular, in the so-called linear and nonlinear fluctuation-dissipation relations.[1-6] Efremov and Kazakov[7,8] obtained stochastic equations for a dynamic system interacting with a thermostat, using the only physical assumption that the thermostat is little changed (since it is macroscopic) by the dynamic system. In contrast to the known studies,[9-12] the equations obtained in Ref. 8 contain the fluctuations of the parameters as well as the nonlinear terms due to the non-Gaussian character of the fluctuations of the thermostat variables.

The purpose of the present paper is, first, to treat from a single point of view the kinetics and the fluctuations in nonlinear systems that are either in thermodynamic equilibrium or in strong disequilibrium. Second, to go outside the framework of the customary approximations wherein the interaction constant is small and different relaxation times exist. The method is based on the assumption that the fluctuations of the (unperturbed) variables of the thermostat interacting with the nonlinear dynamic system have Gaussian statistics. This case corresponds to a large number of physical problems, such as electron-phonon interactions in solids (in the case of small anharmonicity of the lattice vibrations), the interaction of electrons with a quantized electromagnetic field (with an electromagnetic vacuum or with the thermal radiation field of a medium), and others. To make things simpler and more perspicuous, the method is demonstrated by using as an example the interaction of a two-level system with a quantized electromagnetic field (a photon thermostat); this example is also of independent interest. We investigate in detail the case when the initial state of the thermostat is that of thermodynamic equilibrium.

2. In a large number of physical problems it is convenient to separate in the system under consideration two subsystem, one dynamic and the other a macroscopic system interacting with the first and arbitrarily called a thermostat. An important feature of the dynamic system is that it has a small number of degrees of freedom compared with the large number of degrees of freedom of the thermostat. The subdivision of the system into a "large" and a "small" part, to separate the most significant variables, makes it possible above all to treat dynamic-system states that are in strong disequilibrium (impurity atoms, long-wave radiation field, and others), whereas the macroscopic part of the system can be in a state close to equilibrium.

Let the dynamic system, defined by the Hamiltonian $\mathscr{H}_0$, interact with a thermostat having a Hamiltonian F, and let it be acted upon by an external force $f(t)$, so that the Hamiltonian of the entire system is

$$\mathscr{H}=\mathscr{H}_0+F-\lambda Qx-xf(t), \tag{1}$$

where the term λQx describes the interaction between the dynamic system and the thermostat, while Q and x are respectively the variables of the thermostat and of the dynamic system.[1)]

We shall assume the behaviors of the dynamic system and of the thermostat to be known when they do not interact. The task of the theory is to determine the behavior of the dynamic system that interacts with the thermostat. We assume further that at a certain instant (for example, $t=-\infty$) the state of the system is specified and the interaction is turned on adiabatically. In this case the entire time evolution is contained in the operators (the Heisenberg representation). Following Ref. 8, we take first into account the change of the thermostat variables Q upon interaction with the dynamic system, and then substitute the obtained (perturbed) values $Q^H(t)$ in the equations for the variables of the dynamic system (solving by the same token the self-consistent problem). In the classical formulation of the problem, the quantity λx can be regarded as a certain specified arbitrary force acting on the thermostat. By considering the behavior of the thermostat under the influence of the perturbation $V=-\lambda Qx$ in accord with nonstationary perturbation theory, we obtain for the thermostat variable $Q^H(t)$ the expansion[8]

$$Q^H(t)=Q(t)+\lambda\int dt_1\varphi(t,t_1)x(t_1)+\lambda^2\int dt_1\int dt_2 \times\varphi(t,t_1,t_2)x(t_1)x(t_2)+\dots. \tag{2}$$

Here $Q(t)$ is determined by the unperturbed behavior of the thermostat

$$\begin{gathered}\varphi(t,t_1)=-\{Q(t),Q(t_1)\}\eta(t-t_1),\quad \eta(t-t_1)=\begin{cases}1, & t>t_1\\ 0, & t<t_1\end{cases},\\ \varphi(t;t_1,t_2)=\frac{(-1)^2}{2!}\mathscr{P}_{12}\{\{Q(t),Q(t_1)\},Q(t_2)\}\eta(t-t_1)\eta(t_1-t_2)\end{gathered} \tag{3}$$

are random functions of the reaction of the thermostat to the external force $\lambda x(t)$. In quantum theory, the classical Poisson brackets $\{Q(t),\ Q(t_1)\}$ in (3) are replaced by the corresponding quantum Poisson brackets $(i\hbar)^{-1}[Q(t), Q(t1)]_-$, and the expansion of the operator $Q^H(t)$ differs from the corresponding classical expression (2) only in that the operators x are symmetrized with the functions of the reaction (3) in a definite manner. In the general case the expansion (2) contains an infinite number of terms and is determined by an infinite number of functions φ. Therefore substitution of the expansion (2) in the Heisenberg equations of motion for the operators of the dynamic subsystem

$$\dot{x}_j(t)=\frac{1}{i\hbar}[x_j(t),\mathscr{H}]_--\frac{1}{i\hbar}[x_j(t),x(t)]_-(\lambda Q^{\Gamma}(t)+f(t)) \tag{4}$$

makes it possible to solve, without additional assumptions, the problem of the behavior of the dynamic system.

To derive simpler stochastic equations that describe the behavior of a nonlinear dynamic system we make full use, when averaging over the initial state of the thermostat, of the assumption that the unperturbed variables $Q(t)$ are Gaussian. The thermostat variables are Gaussian in a large number of physical problems and is a good approximation because the thermostat, which has an infinite number of degrees of freedom, is macroscopic. It is natural to expect in this case the statistical properties of the system to be describable by a smaller number of statistical parameters.

3. Let us dwell very briefly on the properties of Gaussian variables. A random classical process $Q(t)$ is Gaussian if all the statistical properties of $Q(t)$ (relative to a chosen statistical ensemble) are determined by the first two moment functions $\langle Q(t)\rangle$ and $\langle Q(t)Q(t1)\rangle$.[2)]

The main property of Gaussian variables $Q(t)$ at $\langle Q(t)\rangle=0$ is that the mean value of any product of Guassian variable breaks up into a sum of products of all possible paired mean values. For example

$$\langle Q(t)Q(t_1)Q(t_2)Q(t_3)\rangle=\mathscr{P}_{123}\langle Q(t)Q(t_1)\rangle\langle Q(t_2)Q(t_3)\rangle,$$

where $\mathscr{P}_{123}$ is the operator of the sum of cyclic permutation of the indices.

In quantum theory, a random function $Q(t)$ is set in correspondence with an operator $Q(t)$, and operators $Q(t)$ and $Q(t_1)$ taken at different instants of time do not commute in the general case. The noncommutativity of the operators calls for a certain refinement of the very notion of a Guassian operator, as well as for a generalization of the proofs of the main properties of Gaussian operators (see the Appendix). All the statistical properties of Gaussian operators $Q(t)$ are determined by a mean value $\langle Q(t)\rangle$, which can be set equal to zero without loss of generality, and by paired correlation functions $\langle Q(t)Q(t_1)\rangle$. In place of the complex function $\langle Q(t) Q(t_1)\rangle$ it is more convenient to specify two real functions, namely the mean value of the symmetrized operator product

$$M(t,t_1)=\langle{}^1/_2[Q(t),Q(t_1)]_+\rangle \tag{5}$$

and the Poisson bracket

$$\{Q(t),Q(t_1)\}=\left\langle\frac{1}{i\hbar}[Q(t),Q(t_1)]_-\right\rangle.$$

The main property of Gaussian operators is that the mean product of operators over the initial state of the thermostat breaks up into a sum of products of all possible paired mean values, with the sequence of the operators in the mean values preserved. This property leads, in particular, to the useful relation (see Sec. 9)

$$\begin{gathered}\langle\dots Q(t_\alpha)[Q(t),Q(t_1)]_-Q(t_\beta)\dots\rangle\\ =\langle[Q(t),Q(t_1)]_-\rangle\langle\dots Q(t_\alpha)Q(t_\beta)\dots\rangle.\end{gathered} \tag{6}$$

It is important to note here that the commutator $[Q(t)Q(t_1)]_-$ is, generally speaking not a C-number.

4. We turn now to the initial equation (4), in which the Heisenberg operator $Q^H(t)$ is represented in the form of an infinite series (2) in the operators of the dynamic subsystem $x(t)$. We imagine that an exact solution of Eqs. (4) was obtained for the operators $x_j(t)$ of the dynamic system. According to the expansion (2), the solutions for the operators $x_j(t)$ (the dynamic system) are certain functions of the operators $Q(t)$, $\varphi^{(1)}=\varphi(t,t_1)$, $\varphi^{(2)}=\varphi(t,t_1,t_2)$, ...:

$$x_j(t)=F_j\{Q,\varphi^{(1)},\varphi^{(2)},\dots\}.$$

All the statistical properties of the dynamic subsystem, for some specified initial state for it, will be determined by the mean values of the products of functionals over the initial state

$$\langle F_i\{Q,\varphi^{(1)},\varphi^{(2)},\dots\}F_j\{Q,\varphi^{(1)},\varphi^{(2)},\dots\}\rangle. \tag{7'}$$

By virtue of the important property (6) of the Gaussian operators of the thermostat, the linear responses (3) in the mean values (7′) will have a behavior "similar" to that of determined quantities. It is important here that the replacement of the commutators by their mean values in the linear response functions (3), which constitute multiple commutators (Poisson brackets), causes them to vanish.

Consequently, the mean values, over the state of the thermostat, of any product of the dynamic-system operators obtained by exact solution of Eqs. (4), do not contain nonlinear response functions and depend only on the paired correlation function (5) and on the mean value of the linear response function (3)

$$\langle\varphi(t,t_1)\rangle=\left\langle\frac{i}{\hbar}[Q(t),Q(t_1)]_-\right\rangle\eta(t-t_1). \tag{7''}$$

Returning now to expansion (2), we must thus discard the nonlinear terms and replace $\varphi(t,t_1)$ by the mean value

$$Q^H(t)=Q(t)+\lambda\int_{-\infty}^{\infty}dt_1\langle\varphi(t,t_1)\rangle x(t_1), \tag{8}$$

where $Q(t)$ is determined by the unperturbed behavior of the thermostat, while the second term takes rigorously into account [in the case of Gaussian functions $Q(t)$] the reaction of the thermostat to the action of the dynamic system. Taking expansion (8) into account, we obtain from Eqs. (4) exact stochastic equations for the operators of the dynamic systems:

$$\begin{gathered}\dot{x}_j(t)=\frac{1}{i\hbar}[x_j(t),\mathscr{H}_0(t_0)]_- -\frac{\lambda}{2}[Q(t),y_j(t)]_+ -y_j(t)f(t)\\ -\lambda^2\int dt_1\langle\varphi(t,t_1)\rangle\frac{1}{2}[y_j(t),x(t_1)]_+,\\ y_j(t)=\frac{1}{i\hbar}[x_j(t),x(t)]_-.\end{gathered} \tag{9}$$

Since the unperturbed operator $Q(t)$ does not commute with the operators $y_j(t)$ of the dynamic subsystems, the brackets $[\,]_+$ were used to symmetrize beforehand $Q^H(t)$ and $y_j(t)$.

A distinguishing feature of the obtained equation is the presence of the parametric terms $\frac{1}{2}[Q(t),y(t)]_+$, whose mean values over the initial state of the thermostat are given by

$$\left\langle\frac{1}{2}[Q(t),y_j(t)]_+\right\rangle=\lambda\int_{-\infty}^{\infty}dt_1M(t,t_1)\left\langle\frac{\delta y_j(t)}{\delta f(t_1)}\right\rangle, \tag{10}$$

where

$$\left\langle\frac{\delta y_j(t)}{\delta f(t_1)}\right\rangle=\left\langle\frac{i}{\hbar}[y_j(t),x(t_1)]_-\right\rangle\eta(t-t_1) \tag{11}$$

is the functional derivative of the mean value $\langle y_j(t)\rangle$ over the external force $f(t)$; $M(t,t_1)$ is the correlation function (5) of the thermostat variable $Q(t)$.

Let us prove Eq. (10). By virtue of the Gaussian properties, we have

$$\begin{gathered}\left\langle\frac{1}{2}[Q(t),y_j(t_1)]_+\right\rangle=\int_{-\infty}^{\infty}dt_2\left\langle\frac{1}{2}[Q(t),Q(t_2)]_+\right\rangle\\ \times\left\langle\frac{\delta y_j(t_1)}{\delta Q(t_2)}\right\rangle.\end{gathered} \tag{12}$$

Noticing now that $f(t)$ and $\lambda Q(t)$ enter additively in (9), we can write

$$\langle\delta y_j(t_1)/\delta Q(t_2)\rangle=\lambda\langle\delta y_j(t_1)/\delta f(t_2)\rangle. \tag{13}$$

By substituting (13) in (12) we arrive at expression (10). Using (10), we can rewrite (9) in the form

$$\begin{gathered}\dot{x}_j(t)=\frac{1}{i\hbar}[x_j(t),\mathscr{H}_0(t)]_- -y_j(t)f(t)-\lambda^2\int_{-\infty}^{\infty}dt_1M(t,t_1)\\ \times\frac{\delta y_j(t)}{\delta f(t_1)}-\lambda^2\int_{-\infty}^{\infty}dt_1\langle\varphi(t,t_1)\rangle\frac{1}{2}[y_j(t),x(t_1)]_+ +\xi_j(t),\end{gathered} \tag{14}$$

where

$$\xi_j(t)=-\frac{\lambda}{2}[Q(t),y_j(t)]_+ +\lambda^2\int_{-\infty}^{\infty}dt_1M(t,t_1)\frac{\delta y_j(t)}{\delta f(t_1)} \tag{15}$$

are fluctuation sources with zero mean values over the state of the thermostat. Since the thermostat variable $Q(t)$ is Gaussian, we can use (12) and (13) to calculate the correlation functions of the fluctuation sources of any order. For example, the correlations of the fluctuation sources (15) of second order take in the simpler classical case the form

$$\begin{gathered}\langle\xi_i(t)\xi_j(t_1)\rangle=\lambda^2M(t,t_1)\langle y_i(t)y_j(t_1)\rangle\\ +\lambda^4\int_{-\infty}d\tau\int_{-\infty}d\tau_1M(t,\tau)M(t_1,\tau_1)\left\langle\frac{\delta y_i(t)}{\delta f(\tau_1)}\frac{\delta y_j(t_1)}{\delta f(\tau)}\right\rangle.\end{gathered} \tag{16}$$

Thus, the behavior of a dynamic system interacting with a Gaussian thermostat is determined completely by the correlation function (5) and by the linear response of the thermostat to the external action (7″), quantities connected for the thermodynamic-equilibrium initial state of the thermostate by the fluctuation-dissipation theorem (FDT).[1,2]

For linear dynamic systems, the equations of motion are particularly simple. Thus, for example, for a one-dimensional harmonic oscillator interacting with a Gaussian thermostat, the exact equations take the form of a linear stochastic equation[8]

$$\ddot{x}(t)+\omega_0^2x(t)-\lambda^2\int_{-\infty}^{\infty}dt_1\langle\varphi(t,t_1)\rangle x(t_1)=\lambda Q(t)+f(t), \tag{17}$$

where the left-hand side of the equation determines the dynamics of the fluctuations, while $\lambda Q(t)$ is the fluctuation force. The first to obtain an equation for an oscillator of type (17) under additional insufficiently rigorous assumptions was Senitzky.[9] However attempts to extend his theory to include nonlinear systems were unsuccessful.[10] Our use of the Gaussian properties of the thermostat variables $Q(t)$ makes it possible to solve this problem.

The nonlinearity of the stochastic equations for the operators of the dynamic subsystem can be due either to the nonlinearity of the dynamic system itself or to the character of the interaction of the dynamic system with the thermostat. Separation of the "dynamics" of the fluctuations and of the fluctuation force turns out, as follows from (14) and (15), more complicated in the nonlinear case.

5. For greater clarity and simplicity, we shall use in the exposition that follows an example wherein the thermostat interacts with a very simple nonlinear model—a two-level system. At the same time, the two-level model is of independent interest for quantum radiophysics and for other fields. We write down the Hamiltonian of a two-level system in the form

$$\mathscr{H}_0=\Delta a^+a, \tag{18}$$

where the operators a and a^+ are subject to the conditions

$$[a,a^+]_+=1,\quad a^2=(a^+)^2=0.$$

Let the two-level system interact with the thermostat and be acted upon by an external force, so that the Hamiltonian of the entire system is

$$\mathscr{H}=\Delta a^+a-\lambda Qi(a^+-a)-f(t)i(a^+-a)+F. \tag{19}$$

The stochastic equations (14) for the operators of the two-level system then take the form ($\hbar=1$)

$$\ddot{x}(t)+\Delta^2x(t)+2\lambda^2\Delta\int_{-\infty}^{\infty}dt_1M(t,t_1)\frac{\delta n(t)}{\delta f(t_1)}$$

$$+2\lambda^2\Delta\int_{-\infty}^{\infty}dt_1\varphi(t-t_1)\frac{1}{2}[n(t),x(t_1)]_+=-2\Delta n(t)f(t)+\xi_x(t), \tag{20}$$

$$\dot{n}(t)=\frac{2\lambda^2}{\Delta}\int_{-\infty}dt_1M(t,t_1)\frac{\delta\dot{x}(t)}{\delta f(t_1)}$$

$$+\frac{2\lambda^2}{\Delta}\int_{-\infty}^{\infty}dt_1\varphi(t,t_1)\frac{1}{2}[\dot{x}(t),x(t_1)]_++\frac{2}{\Delta}\dot{x}(t)f(t)+\xi_n(t), \tag{21}$$

where $x=i(a^+-a)$ is the dipole moment of the two-level system; $n=a^+a-aa^+$ is the operator of the population difference between the upper and lower level. The correlation and response functions $M(t,t_1)$ and $\varphi(t,t_1)$ are given by (5) and (7'').

The equations for the fluctuation sources $\xi_x(t)$ and $\xi_n(t)$ are respectively

$$\xi_x(t)=-\lambda\Delta[Q(t),n(t)]_++2\lambda^2\Delta\int_{-\infty}^{\infty}dt_1M(t,t_1)\frac{\delta n(t)}{\delta f(t_1)}, \tag{22}$$

$$\xi_n(t)=\frac{\lambda}{\Delta}[Q(t),\dot{x}(t)]_+-\frac{2\lambda^2}{\Delta}\int_{-\infty}^{\infty}dt_1M(t,t_1)\frac{\delta\dot{x}(t)}{\delta f(t_1)}. \tag{23}$$

6. We assume further that at the initial instant of time (before the interaction is turned on) the thermostat is in thermodynamic equilibrium at a temperature T. In this case all the statistical properties of the dynamic system (at given values of Δ and λ) are determined by the thermostat response function (6) or by the susceptibility

$$\chi(\omega)=\int_{-\infty}^{+\infty}d\tau e^{i\omega\tau}\varphi(\tau). \tag{24}$$

Indeed, to calculate the spectral density of the fluctuations of the thermostat variable $Q(t)$

$$S(\omega)=\int_{-\infty}^{\infty}d\tau M(\tau)e^{i\omega\tau} \tag{25}$$

we can use the Callen-Welton FDT,[1] according to which ($\hbar=1$, $T=\beta^{-1}$) the spectrum of the fluctuations (25) is determined by the imaginary part of the susceptibility (24):

$$S(\omega)=\chi''(\omega)\ \mathrm{cth}\,(\beta\omega/2). \tag{26}$$

The terms with the functional derivatives in Eqs. (20) and (21) can be rewritten in a form more convenient for the subsequent approximations. To this end we move the unit function $\eta(t-t_1)$ from the functional derivative

$$\delta n(t)/\delta f(t_1)=i[n(t),x(t_1)]_-\eta(t-t_1)$$

to the correlation function $M(t-t_1)$

$$\int_{-\infty}^{\infty}dt_1M(t-t_1)\frac{\delta n(t)}{\delta f(t_1)}=\int_{-\infty}^{+\infty}dt_1\tilde{M}(t-t_1)i[n(t),x(t_1)]_-, \tag{27}$$

after which we write

$$\int_{-\infty}^{\infty}dt_1M(t-t_1)\frac{\delta\dot{x}(t)}{\delta f(t_1)}=\int_{-\infty}^{+\infty}dt_1\tilde{M}(t-t_1)i[\dot{x}(t),x(t_1)]_-. \tag{28}$$

We introduce next the Fourier transform of the function (27)

$$\tilde{S}(\omega)=\int_{-\infty}^{\infty}d\tau e^{i\omega\tau}\tilde{M}(\tau). \tag{29}$$

Using the spectral representation for the unit function

$$\eta(\tau)=\frac{i}{2\pi}\int\frac{d\Omega}{\Omega+i\varepsilon}e^{-i\Omega\tau}$$

together with the operator identity

$$\frac{1}{\Omega+i\varepsilon}=\frac{\mathscr{P}}{\Omega}-i\pi\delta(\Omega), \tag{30}$$

we obtain[3)] from the definitions (25) and (29)

$$\tilde{S}(\omega)=\frac{1}{2}\left[S(\omega)+\frac{i}{\pi}\int_{-\infty}^{+\infty}\frac{d\omega'}{\omega-\omega'}S(\omega')\right]. \tag{31}$$

To calculate in (31) the integral in the sense of the principal value, we use the analytic properties of the susceptibility $\chi(\omega)$ in the upper complex ω half-plane. It is easily seen that if $\chi(\omega)$ is analytic in the upper ω half-plane, then the function $\chi(\omega)/(\omega_0-\omega-i\varepsilon)$ has likewise no poles in the upper ω half-plane. Correspondingly

$$\frac{\chi(\omega)}{\omega_0-\omega-i\varepsilon}\mathrm{cth}\left(\frac{\beta\omega}{2}\right) \tag{32}$$

is an analytic function in the upper ω half-plane, with the exception of the points $z_n=4\pi ni/\beta$ on the imaginary axis, at which the function (31) has simple poles. In accord with the residue theorem we have

$$\int_{-\infty}^{+\infty}d\omega'\frac{\chi(\omega')}{\omega_0-\omega'-i\varepsilon}\mathrm{cth}\left(\frac{\beta\omega'}{2}\right)=2\pi i\frac{2}{\beta}\left[\sum_{n=1}^{\infty}\frac{\chi(z_n)}{\omega_0-z_n}+\frac{1}{2}\frac{\chi(0)}{\omega_0}\right], \tag{33}$$

whence we obtain for (31) with the aid of the identity (30)

$$\tilde{S}(\omega)=-\frac{i}{2}\chi(\omega)\mathrm{cth}\left(\frac{\beta\omega}{2}\right)+iT\frac{\chi(0)}{\omega}+i2T\,\mathrm{Re}\sum_{n=1}^{\infty}\frac{\chi(z_n)}{\omega-z_n}, \tag{34}$$

where

$$z_n=4\pi ni/\beta,\quad n=0,\pm1,\ldots. \tag{b}$$

Equation (34) takes the simplest form in the limiting cases of low and high temperatures. In the limiting classical case ($T\gg\omega$) we have

$$\tilde{S}(\omega)=\frac{iT}{\omega}[\chi(0)-\chi(\omega)]. \tag{35}$$

Since the susceptibility is real on the imaginary axis, we obtain for the limiting quantum case ($T\ll\omega$) from (34)

$$\tilde{S}(\omega)=-\frac{i}{2}\chi(\omega)\,\mathrm{sign}\,\omega. \tag{36}$$

The same formula can be obtained directly from the analytic properties of $\chi(\omega)$ signω in the upper ω half-lane, by defining the sign of the complex variable as sign(Re ω).

The meaning of the obtained equations (34)–(36) is that the properties of the dynamic system can be expressed in terms of the susceptibility of the thermostat at the characteristic frequencies of the dynamic system, thereby eliminating the ultraviolet divergence. The result is general and does not depend on the model chosen for the two-level atom.

7. We shall show that Eqs. (20) and (21) do not contradict the condition of thermodynamic equilibrium between the thermostat and the two-level system. To this end we average (20) and (21) over the initial state of the system:

$$\langle \ddot{x}(t)\rangle+\Delta^2\langle x(t)\rangle+2\lambda^2\Delta\int_{-\infty}^{\infty} dt_1 M(t,t_1)\left\langle\frac{\delta n(t)}{\delta f(t_1)}\right\rangle$$

$$+2\lambda^2\Delta\int_{-\infty}^{\infty} dt_1\varphi(t,t_1)\left\langle\frac{1}{2}[n(t),x(t_1)]_+\right\rangle=-2\Delta\langle n(t)\rangle f(t), \tag{37}$$

$$\langle \dot{n}(t)\rangle=\frac{2\lambda^2}{\Delta}\int_{-\infty}^{\infty} dt_1 M(t,t_1)\left\langle\frac{\delta \dot{x}(t)}{\delta f(t_1)}\right\rangle$$

$$+\frac{2\lambda^2}{\Delta}\int_{-\infty}^{\infty} dt_1\varphi(t,t_1)\left\langle\frac{1}{2}[\dot{x}(t),x(t_1)]_+\right\rangle+2\langle\dot{x}(t)\rangle f(t). \tag{38}$$

Assuming complete averaging over the states of the entire system in (37) and (38), we substitute in the right-hand side of (38) (at $f=0$) the expansions of the Fourier functions of the correlations and of the responses

$$M(t-t_1)=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega(t-t_1)}S(\omega), \tag{39}$$

$$\left\langle\frac{1}{2}[\dot{x}(t),x(t_1)]_+\right\rangle=\int\frac{d\omega'}{2\pi}e^{-i\omega'(t-t_1)}(-i\omega')K(\omega'), \tag{40}$$

$$\varphi(t-t_1)=\int\frac{d\omega}{2\pi}e^{-i\omega(t-t_1)}\chi(\omega), \tag{41}$$

$$\left\langle\frac{\delta\dot{x}(t)}{\delta f(t_1)}\right\rangle=\int\frac{d\omega'}{2\pi}e^{-i\omega'(t-t_1)}(-i\omega')\varkappa(\omega'). \tag{42}$$

Integrating with respect to t_1 and recognizing that $S(\omega)$ and $K(\omega)$ are even with respect to reversal of the sign of the frequency, we obtain

$$\frac{\Delta}{2\lambda^2}\langle\dot{n}(t)\rangle=-\int\frac{d\omega}{2\pi}\omega\{-S(\omega)\varkappa'(\omega)+\chi''(\omega)K(\omega)\}. \tag{43}$$

This expression vanishes if the temperature of the entire system coincides with the temperature of the thermostat. Indeed, according to the FDT we have at the temperature T ($T=\beta^{-1}$, $\hbar=1$)

$$S(\omega)=\chi''(\omega)\,\mathrm{cth}\,(\beta\omega/2),\quad K(\omega)=\varkappa''(\omega)\,\mathrm{cth}\,(\beta\omega/2). \tag{44}$$

By substituting these expressions in (31) we verify that $\langle\dot{n}(t)\rangle=0$. Thus, at an arbitrary coupling constant, statistical equilibrium is established in the entire system, with a temperature T equal to the initial temperature of the thermostat. This result is a generalization, to the case of an arbitrary coupling constant, of Bogolyubov's proof[13] that a static equilibrium is established in the dynamic subsystem at the thermostat temperature if the interaction between the dynamic subsystem and the thermostat is small enough. It can also be shown that the obtained stochastic equations agree with the linear and nonlinear FDT.[1-6] This means that the FDT for the variables of the dynamic system follow from the FDT for the variables of the thermostat.

8. We consider now the mechanism of greatest importance in atom-fluctuation processes and due to their interaction with a quantized electromagnetic field (with a photon thermostat). Choosing the Coulomb gauge for the vector potential ($\mathrm{div}\,\mathbf{A}=0$), we express the energy of the interaction between the electron and the field in the form

$$\mathcal{V}(\mathbf{r},t)=-\frac{e}{mc}\mathbf{p}\mathbf{A}(\mathbf{r},t)+eA_0(\mathbf{r},t). \tag{45}$$

If the initial state of the thermostat (prior to the turning-on of the interaction) is a state of thermodynamic equilibrium with temperature T, then all the statistical properties of the atom are determined by the retarded Green's functions for the potentials of the field ($\hbar=1$):

$$D_{jk}(\mathbf{r},t;\mathbf{r}_1,t_1)=\langle i[A_j(\mathbf{r},t),A_k(\mathbf{r}_1,t_1)]_-\rangle\eta(t-t_1);\quad k,j=0,1,2,3. \tag{46}$$

In the chosen gauge ($\mathrm{div}\,\mathbf{A}=0$) the Fourier transforms of the Green's functions for an isotropic medium are of the form[14, 15]

$$D_{ij}(\mathbf{r},\omega)=\delta_{ij}\frac{e^{-\varkappa r}}{r}-\frac{1}{\varkappa^2}\frac{\partial^2}{\partial x_i\partial x_j}\left(\frac{e^{-\varkappa r}-1}{r}\right),\quad i,j=1,2,3; \tag{47}$$

$$D_{i0}(\mathbf{r},\omega)=0,\quad D_{00}(r,\omega)=\frac{1}{\varepsilon(\omega)}\frac{1}{r}, \tag{48}$$

where

$$\varkappa^2=\omega^2\varepsilon(\omega)/c^2,\quad r=|\mathbf{r}_1-\mathbf{r}_2|. \tag{49}$$

We note that the interaction of the electrons with the scalar part of the field potential is responsible for the Coulomb interaction between the electrons, and leads at the same time [by virtue of the singularity of the Green's function $D_{00}(R,\omega)$ at the point $r=0$] to an infinite constant shift of each of the levels (infrared divergence).[4)] The interaction with the scalar potential will therefore be omitted from now on. The singularity of the vector part of the Green's function (47) at zero frequency at the point $r=0$ also leads to infrared divergence. However, infrared divergences are beyong the scope of the two-level approximation considered here.

We proceed now to the two-level-atom approximation. We assume for the sake of argument that the wave functions of the upper and lower levels $\psi_2(5)$ and $\psi_1(r)$ are real. Taking into account the transitions due to the off-diagonal matrix element of the interaction operator (45), we arrive at a Hamiltonian of the type (19) with an interaction constant $\lambda=e/mc$ in with a thermostat variable

$$Q(t)=-i\int d^3r_1\psi_2(\mathbf{r}_1)A_j(\mathbf{r}_1,t)\nabla_j\psi_1(\mathbf{r}_1). \tag{50}$$

The response function $\varphi(t-t_1)$ in Eqs. (20) and (21) is consequently determined for a two-level system by the photon retarded Green's function (47), and the expression for the susceptibility (23) takes the form

$$\chi(\omega)=\int d^3r_1d^3r_2\psi_2(\mathbf{r}_1)(\nabla_j\psi_1(\mathbf{r}_1))\psi_2(\mathbf{r}_2)(\nabla_k\psi_1(\mathbf{r}_2))D_{jk}(|\mathbf{r}_1-\mathbf{r}_2|,\omega). \tag{51}$$

All the statistical properties of a two-level atom (kinetics and fluctuations) are thus determined for an arbitrary nonequilibrium state by the susceptibility (51).

9. It is not our purpose here to carry out a complete statistical investigation of the dynamic system for an arbitrary susceptibility dispersion law (51) and for an arbitrary coupling constant. We confine ourselves only to a number of limiting cases. The general formulas readily permit additional physical approximations.

We dwell first on the dipole approximation or a weak frequency dispersion for the susceptibility (51), assuming $\varepsilon(\omega)=1$. As a result of the condition $|\omega a/c|\ll 1$ (a is the size of the atom) we can write for the Green's function (47) in (51) the expansion

$$D_{jk}(r,\omega)\sim\frac{1}{3}\sum_j D_{jj}(r,\omega)\sim\frac{2}{3}\left(\frac{1}{r}+i\frac{\omega}{c}\right)$$

after which we obtain for the susceptibility

$$\chi(\omega)=\frac{2\alpha}{3a}\left(\frac{m\Delta d_{12}}{e}\right)^2+i\omega\frac{2}{3c}\left(\frac{m\Delta d_{12}}{e}\right)^2, \tag{52}$$

where d_{12} is the matrix element of the dipole moment of the atom. The coefficient α is defined as

$$\alpha=\int d^3r_1d^3r_2\psi_2(\mathbf{r}_1)\nabla_j\psi_1(\mathbf{r}_1)\frac{a}{|\mathbf{r}_1-\mathbf{r}_2|}\psi_2(\mathbf{r}_2)\nabla_k\psi_1(\mathbf{r}_2)$$
$$\times\left\{\int d^3r_1d^3r_2\psi_2(\mathbf{r}_1)\nabla_j\psi_1(\mathbf{r}_1)\psi_2(\mathbf{r}_2)\nabla_k\psi_1(\mathbf{r}_2)\right\}^{-1}.$$

Equations (20) and (21) assume the simplest form in the limiting classical case. It follows from (35) and (52) that

$$\mathcal{S}(\omega)\sim\frac{2T}{3c}\left(\frac{m\Delta d_{12}}{e}\right)^2. \tag{53}$$

Taking the Fourier transforms of (52) and (53)

$$\varphi(t-t_1)=\frac{2\alpha}{3a}\left(\frac{m\Delta d_{12}}{e}\right)^2\delta(t-t_1)+\frac{2}{3c}\left(\frac{m\Delta d_{12}}{e}\right)^2\delta'(t-t_1),$$
$$\tilde{M}(t-t_1)=\frac{2T}{3c}\left(\frac{m\Delta d_{12}}{e}\right)^2\delta(t-t_1)$$

and substituting the resultant expressions for $\varphi(t-t_1)$ and $\tilde{M}(t-t_1)$ in (20) and (21), we obtain

$$\ddot{x}(t)+\gamma\dot{x}(t)+\Delta^2x(t)+2\Delta n(t)f(t)=\xi_x(t), \tag{54}$$

$$\dot{n}(t)-\frac{2}{\Delta}\dot{x}(t)f(t)+\gamma\left(n+\frac{\Delta}{2T}\right)=\xi_n(t), \tag{55}$$

where $\gamma=4(e/mc)^2(2T/3c)(m\Delta d_{12}/e)^2$ is the damping constant and $\langle n\rangle^0=-\Delta/2T$ is the thermodynamic-equilibrium population difference. These equations are valid at any external force $f(t)$ and for an arbitrary coupling constant.

To calculate the correlation functions x and n it suffices to know the correlators of the fluctuation sources $\xi_x(t)$ and $\xi_n(t)$. From (16) we obtain in our approximation

$$\langle\xi_x(t)\xi_x(t_1)\rangle=2\gamma\Delta^2\delta(t-t_1),$$
$$\langle\xi_n(t)\xi_n(t_1)\rangle=2\gamma\delta(t-t_1),\quad\langle\xi_x(t)\xi_n(t_1)\rangle=0. \tag{56}$$

Starting from the explicit form of the fluctuation sources, we can calculate the correlators of fluctuation forces of any order. We present the expressions for the third- and fourth-order correlators

$$\langle\xi_x(t)\xi_x(t_1)\xi_x(t_2)\rangle=2\gamma^2\Delta^2\{\langle\dot{x}(t)\rangle\delta(t_1-t_2)$$
$$+\langle\dot{x}(t_1)\rangle\delta(t-t_2)+\langle\dot{x}(t_2)\rangle\delta(t-t_1)\},$$
$$\langle\xi_x(t)\xi_x(t_1)\xi_x(t_2)\xi_x(t_3)\rangle=4\gamma^2\Delta^4\mathcal{P}_{123}\delta(t-t_1)\delta(t_2-t_3)$$
$$+2\gamma^3\Delta^2\mathcal{P}_{123}\langle\dot{x}(t)\dot{x}(t_1)\rangle\delta(t_2-t_3)+2\gamma^3\Delta^2\mathcal{P}_{123}\delta(t-t_1)\langle\dot{x}(t_2)\dot{x}(t_3)\rangle.$$

Equations (54) and (55), obtained in the classical case for a weak frequency dispersion $\chi(\omega)$ have thus a simple and physically lucid form. The left-hand sides of (54) and (55) describe the fluctuation dynamics, which coincides with the dynamics of the mean values. The right-hand sides of (54) and (55) contain fluctuation sources whose correlation functions are determined in accord with (56).

The quantum case at small frequency dispersion but arbitrary coupling constant is the most complicated. The dynamic part of the equations and the correlation functions of the fluctuation sources contain terms that are nonlocal in time and whose analysis is outside the scope of the present article.

We proceed next to the case of a small coupling constant. Let now the thermostat temperature T and the dispersion law of the thermostat variables be arbitrary, but to simplify Eqs. (20) and (21) we assume that the coupling constant λ is small. This approximation is well satisfied in an interaction between a two-level system and a photon thermostat. We note that the expansion of a commutator such as $[n(t),\ x(t-\tau)]_-$ in powers of τ is determined by coefficients

$$[n(t),\ (d/dt)^kx(t)]_-. \tag{57}$$

Since λ is small, we can assume that the operators $x(t)$ in the commutators (57) satisfy the equations of motion of a free two-level system, $\ddot{x}(t)=-\Delta^2[x(t)]$. This means that we discard in (20) and (21) terms of order λ^4 and assume the force $f(t)$ to be small.

It follows from this, in particular, that

$$i[n(t),x(t-\tau)]_-=2\left\{\frac{\dot{x}(t)}{\Delta}\cos(\Delta\tau)+x(t)\sin(\Delta\tau)\right\},$$
$$i[\dot{x}(t),x(t-\tau)]_-=-2\Delta n(t)\cos(\Delta\tau).$$

As a result, the stochastic equations at a small coupling constant and at a small force $f(t)$ take the form

$$\ddot{x}(t)+\gamma(\Delta)\dot{x}(t)+\omega^2(\Delta)x(t)=-2\Delta n(t)f(t)+\xi_x(t), \tag{58}$$

$$\dot{n}(t)+\gamma(\Delta)\{n(t)+\operatorname{th}(\beta\Delta/2)\}=\frac{2}{\Delta}\dot{x}(t)f(t)+\xi_n(t), \tag{59}$$

where

$$\gamma(\Delta)=(2\lambda)^2\mathcal{S}'(\Delta), \tag{60}$$

$$\omega^2(\Delta)=\Delta^2\left(1+\frac{(2\lambda)^2}{\Delta}\mathcal{S}''(\Delta)\right) \tag{61}$$

are determined by the real and imaginary parts of the function (34) at $\omega=\Delta$. In the limiting quantum case ($T\ll\Delta$) we obtain from (36)

$$\gamma(\Delta)=2\lambda^2\chi''(\Delta)\operatorname{sign}\Delta, \tag{62}$$

$$\omega^2(\Delta)=\Delta^2\left(1-\frac{2\lambda^2}{\Delta}\chi'(\Delta)\operatorname{sign}\Delta\right). \tag{63}$$

In this case the damping coefficient $\gamma(\Delta)$ and the shift of the line center are determined respectively by the imaginary and real parts of the thermostat susceptibility (51) at the transition frequency Δ, and these parts are connected by virtue of the causality principle by the dispersion relation

$$\chi'(\Delta)\operatorname{sign}\Delta=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{d\omega}{\omega-\Delta}\chi''(\omega)\operatorname{sign}\omega. \tag{64}$$

We note that expression (62), which determines the shift of the line center, does not contain, of course, the traditional logarithmic divergence. Substituting in (63) the expression (52) for $\chi(\omega)$ we can easily estimate the level shifts $2p$ (parameter $\alpha\approx\frac{1}{4}$) and $2s$ ($\alpha\approx1/6$) relative to the lower $1s$ level. The difference of these shifts yields for the Lamb shift a value of the same order as obtained from the experimental data.

We would obtain the well known expression (with the logarithmic divergence) for the Lamb shift of a two-level atom[16] if we were to use for the calculation of $\chi'(\Delta)$ from (64) the approximate relation $\chi''(\omega)=\omega B$, which can be used at high frequencies. To obtain a correct nondiverging result it is necessary to substitute in

(64) the total susceptibility $\chi(\omega)$, with $\chi(\omega)\to 0$ as $\omega\to\infty$. If the coupling constant is small it is easy to calculate the correlators of the fluctuation forces for the quantum case. For example, the correlator of the fluctuation forces $\xi_x(t)$ and $\xi_x(t_1)$ is given by

$$\langle {}^1/_2[\xi_x(t),\xi_x(t_1)]_+\rangle=(2\lambda\Delta)^2\{\langle {}^1/_2[Q(t),Q(t_1)]_+\rangle\langle {}^1/_2[n(t),n(t_1)]_+\rangle\}. \tag{65}$$

Since the coupling constant is small, $n(t)$ is a slow function of the time and the expression for the correlator (65) takes the simpler form (we neglect the terms $\sim\lambda^4$):

$$\langle {}^1/_2[\xi_x(t),\xi_x(t_1)]_+\rangle=(2\lambda\Delta)^2\langle {}^1/_2[Q(t),Q(t_1)]_+\rangle. \tag{66}$$

In our approximation the correlator of the fluctuation forces is proportional to the correlation functions of the unperturbed variables of the thermostat.

10. We discuss now the results obtained in the present paper.

Stochastic functional differential equations were derived for an arbitrary nonlinear dynamic system interacting with a Guassian thermostat and acted upon by an external force $f(t)$. From the properties of the system itself we obtained the fluctuation sources (15) and their correlation functions subject to the single assumption that the unperturbed variables of the thermostat are Gaussian. In contrast to the papers of Lax, Haken and others,[17,18] the fluctuation sources are not introduced in the phenomenological Langevin equations, but are obtained in natural fashion from the initial equations.

It is shown that at an arbitrary coupling constant a statistical equilibrium is established between the dynamic system and the thermostat at a temperature equal to the initial thermostat temperature; this is a generalization of the Bogolyubov's result[13] for a small coupling constant.

In the case of an initial thermostat state in thermodynamic equilibrium, it has been shown that the properties of the dynamic system are determined by the susceptibility of the thermostat at the characteristic frequencies of the dynamic system. This circumstance makes it possible to exclude the ultraviolet divergence. The result is general and does not depend on the chosen model of the two-level system. The infrared divergence is preserved in this case. Equations have been derived that describe in unified manner the dynamics and fluctuations of a two-level atom interacting with a quantized electromagnetic field. The approximation of small frequency dispersion of the susceptibility (51) (the dipole approximation) is considered. In the classical case we have an example of a simple nonlinear stochastic model in which it is easy to calculate all the statistical characteristics in an arbitrary nonequilibrium state. The third- and fourth moments, which depend on the characteristics of the dynamic system itself, are given. It is shown that by virtue of the nonlinearity of the dynamic system the fluctuation sources are non-Gaussian, even though the thermostat itself is Gaussian. The quantum case ($T=0$ at small frequency dispersion of the thermostat but at an arbitrary coupling constant) is more complicated. The dynamic part of the equations and the correlation function of the fluctuation sources contains terms that are nonlocal in time.

The behavior of a two-level system is investigated for a low coupling constant and a small external force $f(t)$. The damping and the finite shift of an emission line containing no logarithmic divergence are obtained. The reasons why these divergences were obtained in other approaches are analyzed. It is made clear in these two approximations that the relaxation as well as the frequency shift is due to the parametric action of the field on the dipole moment and on the population difference, whereas the equilibrium value of the population difference is determined by the reaction of the radiation.

APPENDIX

We consider an arbitrary set of noncommuting operators, which we number in a certain specified order:

$$A_1A_2A_3\ldots A_\alpha A_{\alpha+1}. \tag{A.1}$$

We are interested in the moment functions of the operators in the established order (A.1). We introduce the operator-ordering symbol T_A, the action of which on any product of operators arranges them in the selected order (A.1). We determine next the characteristic function for the ordered product of the operators

$$\Theta_{T_A}(u)=\langle T_A\exp(iu_jA_j)\rangle. \tag{A.2}$$

Then arbitrary ordered moments of the operators A can be obtained by differentiating (A.2) with respect the corresponding quantities u. For example,

$$\langle A_2A_3A_5\rangle=\frac{1}{i^3}\frac{\partial^3\Theta_{T_A}(u)}{\partial u_2\,\partial u_3\,\partial u_5}\bigg|_{u=0}.$$

In an analogy with the cumulants of classical theory,[19] we define the cumulants of the ordered operators K_α, $K_{\alpha\beta}$, $K_{\alpha\beta\gamma}$, ... in accord with

$$\Theta_{T_A}(u)=\exp\left\{\sum_{s=1}\frac{i^s}{s!}K_{\alpha_1\alpha_2\ldots\alpha_s}u_{\alpha_1}u_{\alpha_2}\ldots u_{\alpha_s}\right\}. \tag{A.3}$$

From the definition (A.3) follow the relations between the moments and the cumulants; these relations differ from the classical ones[19] only in the ordering of the operators under the averaging signs. For example,

$$K_\alpha=\langle A_\alpha\rangle,\quad K_{\alpha\beta}=\langle T_AA_\alpha A_\beta\rangle-\langle A_\alpha\rangle\langle A_\beta\rangle. \tag{A.4}$$

In analogy with the classical case we shall call the operators A_α Gaussian if all the ordered cumulants of order above the second are equal to zero. The characteristic function (A.2) in this case is written at $K_\alpha=0$ in the form

$$\langle T_A\exp(iA_\alpha u_\alpha)\rangle=\exp\left\{\frac{i^2}{2}K_{\alpha_1\alpha_2}u_{\alpha_1}u_{\alpha_2}\right\}. \tag{A.5}$$

For Gaussian operators it follows from the relations between the moments and the cumulants that the mean value of the product of the operators in a certain chosen sequence (A.1) breaks up into a sum of products of all possible paired averages, and the sequence of the operators and of the averages remains the same. The possible pairings in the mean value of the product $\langle A_1A_2\ldots A_{\alpha-1}A_\alpha A_{\alpha+1}\ldots\rangle$ can be carried out stage by stage.

One first carries out all possible pairings of a certain chosen operator A_1:

$$\langle A_1A_2\ldots A_{\alpha-1}A_{\alpha+1}\ldots\rangle=\sum_\alpha\langle A_1A_\alpha\rangle\langle A_2\ldots A_{\alpha-1}A_{\alpha+1}\ldots\rangle. \tag{A.6}$$

This operation is then repeated in the product $\langle A_2 \ldots A_{\alpha-1} A_{\alpha+1} \ldots \rangle$ etc. Introducing the notation

$$F[A]=A_2A_3\ldots A_{\alpha-1}A_\alpha A_{\alpha+1},$$

$$\partial F[A]/\partial A_\alpha=A_2A_3\ldots A_{\alpha-1}A_{\alpha+1},$$

we can rewrite (A.6) in the form

$$\langle A_1 F[A]\rangle=\sum_\alpha \langle A_1A_\alpha\rangle\langle \partial F[A]/\partial A_\alpha\rangle. \tag{A.7}$$

This formula is valid for any function F of Gaussian operators A_α. It is easy to generalize (A.7) to include the case when the subscript α runs through a continuous set of values. Then $F[A]$ is a certain functional of A_α and (A.7) takes the form

$$\langle A_1 F[A]\rangle=\int d\alpha \langle A_1A_\alpha\rangle\langle \delta F[A]/\delta A_\alpha\rangle, \tag{A.8}$$

where $\delta F[A]/\delta A_\alpha$ is a functional derivative. The formula obtained generalizes the well known Furuzi-Novikov formula to include the quantum case.

In conclusion we obtain one more useful formula for Gaussian operator. We consider the mean value of a product of the form

$$\langle [A_1, A_2]_- F[A]\rangle.$$

Using (A.6), we find

$$\langle A_1A_2F[A]\rangle=\langle A_1A_2\rangle\langle F[A]\rangle+\sum_\alpha \langle A_1A_\alpha\rangle \left\langle A_2\frac{\partial F[A]}{\partial A_\alpha}\right\rangle,$$

$$\langle A_2A_1F[A]\rangle=\langle A_2A_1\rangle\langle F[A]\rangle+\sum_\alpha \langle A_1A_\alpha\rangle \left\langle A_2\frac{\partial F[A]}{\partial A_\alpha}\right\rangle,$$

whence we obtain by subtraction the second expression from the first

$$\langle [A_1, A_2]_- F[A]\rangle=\langle [A_1, A_2]_-\rangle\langle F[A]\rangle. \tag{A.9}$$

[1)] In the general case Q and x should be taken to mean the sets of variables of the thermostat and of the dynamic system.

[2)] In other words, the random process $Q(t)$ is Gaussian if all the cumulant functions of third and higher order are equal to zero.

[3)] We assume here that $\lim_{\tau\to\infty} M(\tau)=0$, so that a convolution of the product $M(\tau)\eta(\tau)$ exists.

[4)] The divergence due to the singularity of the photon Green's function at low frequencies is called the infrared divergence.

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Translated by J. G. Adashko