Interface between a hot and cold gas and the evolution of cold clouds in intergalactic space

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A study is made of the structure and evolution of the interface between hot and cold clouds due to influence of bulk energy losses and nonlinear heat conduction. It is shown that in the absence of heat sources the general decrease in the temperature of the hot phase is accompanied by the appearance of a cooling wave, in which hot gas goes over into the cold phase. Various similarity solutions to the heat conduction equation that describe both thermal waves and cooling waves are studied. The evolution of a cold gas cloud moving through hot gas is discussed. Some astrophysical applications of the problems are considered.

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In astrophysical problems, it is frequently necessary to consider the interface between very hot and comparatively cold gases. Such problems arise in the analysis of the nonlinear stage in the development of thermal instability in a hot gas, in the analysis of the structure and evolution of condensations in the nonlinear theory of gravitational instability, in the analysis of the motion of a cold gas cloud (isolated or in a galaxy) through the hot gas of a cluster of galaxies, and so forth. In cosmological problems of this type, a decisive role in the evolution of the structure of the interface is played by bulk radiative energy losses (for example, bremsstrahlung and recombination radiation) and nonlinear heat conduction (for example, electron heat conduction).

All the thermal processes in these problems take place much more slowly than the hydrodynamic processes (for estimates, see Sec. 3). One is therefore dealing with a situation similar to the slow propagation of a flame through a burning mixture, namely, the pressure in the hot and cold phases is the same, and it is only near the front of the thermal wave (or cooling wave) that there is a slight excess (or decrease) in the pressure due to the transition of the matter from one phase to the other. The amplitude of the pressure discontinuity, determined by the ratio of the wave velocity to the velocity of sound in the hot phase, is usually small, and we shall assume that the pressure is constant. To take into account the displacement of the matter accompanying the change in the volume, it is convenient to use Lagrangian coordinates to analyze the heat conduction equation.

It is well known that in problems with nonlinear heat conduction, in contrast to linear problems of the propagation of heat, there is a sharp boundary between the heated region and the cold region; this is the front of the thermal wave.[1,2] The same is true for problems with nonlinear bulk losses in which the influence of heat conduction can be ignored (in this case, the sharp boundary is the front of a cooling wave). As is shown below, in the general case when allowance is made for both bulk radiation of energy and nonlinear heat conduction the wave nature of the solution is preserved and one can have not only cooling waves but also, under certain conditions, thermal waves.

These questions are intimately related to problems in a well-studied field—the theory of combustion and explosion and the theory of propagation of a flame through a mixture that reacts at an initial temperature,[3,4] nonlinear heat release problems,[5,6] cooling waves associated with strong temperature dependence of the transparency of the medium,[2,7,8] and other such problems. Unexpectedly, an equivalent problem has arisen in biology in the theory of the displacement of an old and less well adapted population by a new and better adapted population produced by a mutation. This problem has been considered by Kolmogorov, Petrovskiĭ, and Piskunov,[9] and, independently by Fisher.[10]

The important part played by heat conduction in the evolution of a two-phase system to a stationary state was noted in Ref. 11 by one of the present authors and Pikel'ner. In the problem of evolution of interstellar gas considered in Ref. 11 there were not only bulk heat losses but also bulk energy sources—x rays, cosmic rays, etc. In such a system, there is a critical pressure at which the bulk sources and bulk heat losses balance on the average. The heat conduction at the interface of the two phases regulates the processes of evaporation and condensation in such a way that the system as a whole is displaced toward the critical pressure. For the intergalactic medium, bulk energy sources are not typical, the evolution is determined by the bulk losses, and the critical state is the cold phase. In the intergalactic medium it is therefore natural to expect a predominance of cooling waves and condensation processes.

Problems relating to the interface between hot and cold gases were also discussed in Refs. 12–15, and it was concluded on the basis of an analysis of quasistationary problems[14] and problems without bulk losses[13] that there is effective evaporation of the cold phase if the cold cloud has a smaller than critical size and that there is condensation of the hot phase into a cloud with larger than critical size. The analysis below shows that when allowance is made for both heat conduction and volume energy losses the situation is more complicated. Among the asymptotically stable regimes under cosmological conditions, cooling waves, leading to condensation of matter on the surface of clouds of the cold

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phase, must be predominant because of the absence of effective energy sources.

Thermal waves are possible either in the initial phase of contact (when the initial temperature discontinuities begin to disappear) or in the cases when there are bulk energy sources or reserves of heat (high temperature in some region) sufficient to sustain thermal waves.

In §1, we consider in detail the problem of contact between hot and cold phases when the temperature of the hot phase is bounded. We show that bulk heat losses lead to the appearance of cooling waves as an intermediate asymptotic behavior. In §2, we consider similarity solutions to the heat balance equation that describe cooling waves, thermal waves, and the evolution of inhomogeneities of the temperature in the nonlinear regime. In §3, we discuss the conditions of applicability of the results, and in §4 we take into account the influence of the motion of cold clouds on their evolution. In §5, we consider possible astrophysical applications of the results.

§1. COOLING WAVES

We consider the plane one-dimensional problem of heat propagation described by the heat conduction equation

$$\frac{5}{2}\frac{p}{T}\frac{dT}{dt}+\varepsilon(p,T)=\frac{\partial}{\partial x}\lambda(p,T)\frac{\partial T}{\partial x}, \tag{1}$$

where $T=T(t,x)$ is the temperature of the medium, $p=nkT$ is the pressure, $n(x,t)$ is the density of the particles, $\varepsilon(p,T)$ are the bulk energy sources and sinks, and $\lambda(p,T)$ is the thermal conductivity of the medium. We assume that the propagation velocity of the thermal waves and cooling waves is small compared with the velocity of sound in the hot phase, and therefore the pressure can be assumed to be constant. To take into account the displacement of matter in Eq. (1), it is convenient to go over to Lagrangian coordinates q, which are determined by the condition $dq=n(t,x)dx$. In the Lagrangian coordinates, Eq. (1) is transformed into

$$\frac{5}{2}\frac{dT}{dt}+\varepsilon^*(T)=\frac{\partial}{\partial q}\lambda^*(T)\frac{\partial T}{\partial q}, \tag{2}$$

where $\varepsilon^*=T\varepsilon/p$, $\lambda^*=n^2T\lambda/p$. Since $p=\mathrm{const}$, the dependence of ε^* and λ^* on p can be ignored.

We assume that at the initial time $t=0$ the half-space $x>0$ is filled with hot gas at temperature T_h, and the half-space $x<0$ with cold gas at temperature $T_c \ll T_h$. Bearing in mind that $\varepsilon^*(T_c)=0$ and $\lambda^*(T_c)=0$, we assume in what follows that $T_c=0$.

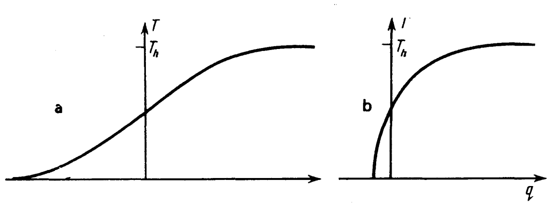

FIG. 1. Temperature profile for $\lambda^*=$ const (a) and temperature profile for $\lambda \propto T^3$ (b).

We consider successively a number of problems, which differ in the assumptions made about the form of the function $\varepsilon^*(T)$.

1. First of all, we recall the situation when $\varepsilon^*(t)=0$. Then (2) describes an ordinary problem of nonlinear heat conduction. In this case (see, for example, Ref. 2), the conservation of energy and the dimensions of the coefficient of thermal conductivity λ^* (at fixed temperature T_h) completely determine the nature of the solution:

$$T=T_hf(\xi)=T_hf\left[\frac{\sqrt{5}}{2}q\lambda^{*-1/2}t^{-1/2}\right], \tag{3}$$

where $f\to 1$ as $\xi\to\infty$ and $f\to 0$ as $\xi\to-\infty$. In the special case of the linear problem ($\lambda^*=\lambda^*(T_h)=\mathrm{const}$), we obtain the well-known solution [Fig. 1(a)]

$$f=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\xi}e^{-\alpha^2/2}d\alpha. \tag{4}$$

In the case of the power-law dependence $\lambda=\lambda_0T^\beta$, $\lambda^*=\lambda_0^*T^{\beta-1}$, a thermal wave propagating into the region $q<0$ arises, and at some $\xi=\xi_0$ the temperature vanishes in accordance with the law

$$\begin{gathered}\xi=\xi_0(1-\eta)<0,\quad 0<\eta\ll 1,\quad f^{\beta-1}=\xi_0^2(\beta-1)^2\eta,\\ T=T_h[\xi_0^2(\beta-1)^2(1-q/q_f)]^{1/(\beta-1)},\quad 0\geqslant q>q_f,\\ T=0,\quad q<q_f,\quad q_f=\xi_0\frac{2}{\sqrt{5}}(\lambda^*(T_h)t)^{1/2}\end{gathered} \tag{5}$$

[Fig. 1(b)]. This situation is also well known.[1,2] Note that because of the obvious relation $f(0)=\mathrm{const}$, $T(t,0)=T_hf(0)=\mathrm{const}$ the solution to the problem with the initial condition chosen above gives simultaneously the solution to the problem with boundary condition at a fixed point $T(t=0)=\mathrm{const}$. At high temperatures ($f\to 1$), the nonlinear nature of the heat conduction is not manifested, and in this region the temperature profile coincides with (4).

2. We now consider a more complicated problem by taking into account heat release and heat loss, and we assume that the heat release compensates the heat loss for both $T=T_h$ and $T=0$, while in the region $0<T<T_h$ the heat losses exceed the heat release. Thus, we arrive at Eq. (2) with $\varepsilon=0$ for $T=0$ and $T=T_h$ and $\varepsilon>0$ for $0<T<T_h$. In such a formulation, the problem is a special case of the problem considered earlier in Ref. 11 of the motion of the interface between two regions each of which are themselves stationary. Since $\varepsilon>0$ and ε does not change sign in the intermediate interval $T_h>T>0$, there is no doubt with regard to the direction of propagation of the front—the amount of hot gas decreases and the amount of the cold gas increases with the time. The solution has the form (Fig. 2)

$$\begin{gathered}T=T_hf(x-ut)=T_hf(\xi),\quad u>0,\\ f(-\infty)=0,\quad f(\infty)=1.\end{gathered}$$

It is important that in this problem the distance traversed by the wave is proportional to the time and the structure of the front does not change with the time. The width of the front and the velocity with which it moves are determined by the temperature T_h of the hot phase and will be obtained below. By comparing the properties of the first and the second variant, we can establish the fate of an arbitrary initial temperature distribution.

If a step in the temperature distribution is specified at $t=0$, then initially a thermal wave arises; because of the steep temperature profile, heat conduction has a greater influence than the bulk losses, $(\partial/\partial q)\lambda^*\partial T/\partial q$ is greater than ε^*, $[\xi_0\lambda^*(T_h)]^{1/2}$ is greater than ut, and the influence of ε^* can be ignored. With the passage of time, however, the gradients decrease, the width of the front of the thermal wave increases, approaching the width of the front of the cooling wave (second variant), and a smooth transition to the second asymptotic behavior takes place.

Knowing the law of motion of the front of the thermal wave and the cooling wave, we can find an interpolation formula describing the motion of the temperature front. It is obvious that when t is small $[\xi_0\lambda^*(T_h)t]^{1/2} > ut$ and the front is displaced to the left, whereas at large t the sign of the inequality is reversed and the front moves to the right. For the motion of the front (or, more precisely, a point with given temperature, since the profiles of the front of the thermal wave and the cooling wave differ somewhat) one would be tempted to write down an expression of the form

$$q(T_1,t)=ut-[\xi_0\lambda^* t]^{1/2}.$$

However, such an expression would lead to an unbounded growth of the front width,

$$q(T_1,t)-q(T_2,t)=(\lambda^* t)^{1/2}(\xi_2^{1/2}-\xi_1^{1/2}),$$

in the period when the cooling wave has already been established, which is incorrect. The width of the cooling wave remains constant. The correct interpolation formula is

$$q(T_1,t)=+u(T_h)t-a(T)t^{1/2}e^{-t/\tau}+b(T)(1-e^{-t/\tau}), \tag{6}$$

where τ is the characteristic time of transition from the first to the second regime, $a(T)$ describes the profile of the front of the thermal wave, and $b(T)$ the profile of the front of the cooling wave.

The problem of determining the temperature profile and the velocity of the cooling front is completely analogous to the same problems in the theory of propagation of a flame in a medium with nonlinear heat release law. For the existence of a solution, it is necessary that the condition $\varepsilon^*(T_h)=0$ hold. For

$$d\varepsilon^*/dT|_{T=T_h}\neq 0$$

and under some restrictions on the rate of growth of the function $\psi(T)=\lambda^*(T)\varepsilon^*(T)$ for $T<T_h$, Kolmogorov–Petrovskiĭ–Piskunov theory[9] holds and the wave velocity is determined by

$$u=u^*=\frac{4}{5}\left[\lambda^*(T_h)\left(\frac{d\varepsilon^*}{dT}\right)_{T=T_h}\right]^{1/2}. \tag{7}$$

But if $\psi(T)=\lambda^*(T)\varepsilon^*(T)$ increases sufficiently rapidly, then, as was shown by one of the present authors and Frank-Kamenetskiĭ,[16]

$$u^2=\tilde{u}^2\approx 2T_h^{-2}\int_{T_c}^{T_h}\varepsilon^*(T)\lambda^*(T)\,dT, \tag{8}$$

and in this case the velocity $u=\tilde{u}$ may depend on not only T_h but also on the minimal temperature T_c and on the form of the function $\psi(T)=\lambda^*(T)\varepsilon^*(T)$. Numerical calculations for a combustion wave[17] showed that in practice the largest of the values of $\tilde{u}$ and u is realized.

3. We now consider a real problem—the cooling of gas with bulk heat losses but without heat sources. Such a situation is typical for the extragalactic medium. The initial stage in the evolution of the temperature distribution (small t) is the same as in the first problem. However, the asymptotic behavior in the late stages is quite different from the second variant. For $\varepsilon^*(T_h)\neq 0$, the hot phase cools everywhere, over the complete region of high temperature, independently of the arrival of the cooling wave. The law of cooling of the hot phase is determined by Eq. (2) with $\partial T/\partial q=0$, where

$$dT_a/dt=-{}^2/_3\varepsilon^*(T_a) \tag{9}$$

provided the initial conditions $T_a|_{t=0}=T_h$ are satisfied. The general cooling of the gas does not prevent propagation of the cooling waves, but the wave now propagates through a gas of variable (decreasing with the time) temperature.

The solution to the problem in the analogous situation in the theory of combustion was recently found in Ref. 3, and this method can be transferred fully to the case of cooling waves. We seek a solution in the form

$$T=T\left(x-\int u(t)\,dt,\,t\right)=T(\xi,t).$$

Such a form of solution takes into account explicitly the variation of the temperature ahead of the cooling front, and the solution (Fig. 3) satisfies the boundary conditions

$$T(\xi,t)|_{\xi\to\infty}=T_a(t),\quad T(\xi,t)|_{\xi\to-\infty}=0. \tag{10}$$

Such a solution obviously exists and is meaningful only for a limited time $t<t_a$, since during the time t_a the hot phase cools through the bulk losses. However, in real problems the time t_a may be fairly long.

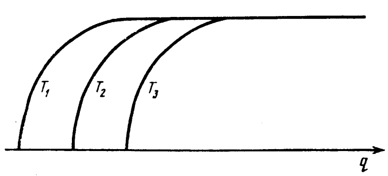

FIG. 2. Cooling wave in a medium with T_h = const. The temperature profile at three times, $t_3=t_2+\Delta t=t_1+2\Delta t$.

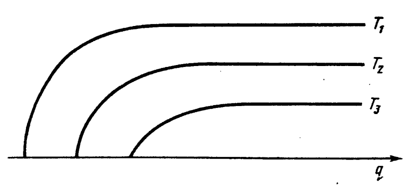

FIG. 3. Cooling wave in a medium with decreasing temperature. The temperature profile at three times $t_3>t_2>t_1$.

Thus, in the considered problem solutions of a cooling wave type exist only for an interval of time that is bounded below (period of establishment) and above (cooling time). This is an important difference between the problem (3) and the problem (2), in which a solution of the cooling wave type exists for an arbitrarily long time. Therefore, cooling waves in problems with $\varepsilon^*(T_h) > 0$ describe an intermediate asymptotic behavior of the solution.

We note finally that solutions of this type also arise naturally in the case of initial temperature distributions of the most general form. In particular, the development of a perturbation associated with thermal instability of a medium leads to cooling waves. However, one can then encounter "kinematic" cooling waves, in which neither the temperature profile nor the wave velocity is related to the heat transfer but both are completely determined by the initial (sufficiently smooth) temperature profile.

We find an approximate solution to Eq. (2) in the considered situation. We seek the solution in the form of a cooling wave:

$$T=T(t,\xi)=T_a(t)f(t,\xi),\quad \xi=q-\int u(t)\,dt,$$
$$\frac{5}{2}\left(\frac{\partial T}{\partial t}\right)_\xi-\frac{5}{2}u\frac{\partial T}{\partial \xi}+\varepsilon^\cdot(T)=\frac{\partial}{\partial \xi}\lambda^\cdot\frac{\partial T}{\partial \xi}, \tag{11}$$

where $T_a(t)$ is determined in (9). In the region $\xi\to\infty$, far from the cooling front, $T=T_a(t)$, $f=1$, $\dot T=\dot T_a=-\frac{2}{5}\varepsilon^*(T_a)$, whereas at the other end, $\xi\to-\infty$, we have $T=0$, $\dot T=0$. Following Ref. 3 and regarding $\dot T$ in (11) as a correction and assuming for it a linear interpolation satisfying both the limiting conditions, we obtain

$$\frac{5}{2}\left(\frac{\partial T}{\partial t}\right)_\xi=-\frac{T}{T_a}\varepsilon^\cdot(T_a),$$
$$-\frac{5}{2}u\frac{\partial T}{\partial \xi}+\varepsilon^\cdot(T)-\frac{T}{T_a}\varepsilon^\cdot(T_a)=\frac{\partial}{\partial \xi}\lambda^\cdot(T)\frac{\partial T}{\partial \xi}. \tag{12}$$

In this approximation, the problem reduces to the second problem considered above.

We consider in more detail the case of a power-law dependence, which is of interest for practical application: $\varepsilon=\varepsilon_0T^{-1}(\gamma>0)$ and $\lambda=\lambda_0T^\beta(\beta>1)$ for $T\geqslant T_c$, and $\varepsilon=0$ for $T<T_c$. (For $\beta+1-\gamma>0$, the value of T_c is unimportant and we shall assume in what follows that $T_c=0$.) For example, $\gamma=3/2$ corresponds to bremsstrahlung losses, $\gamma=5/2$ to recombination losses, and $\beta=5/2$ to electron heat conduction. We go over to dimensionless variables ζ and v related to ξ and u by

$$\xi=\zeta n_a(\lambda_aT_a/\varepsilon_a)^{1/2},\quad u=\frac{2}{5}v\frac{n_a}{p_a}(\lambda_aT_a\varepsilon_a)^{1/2}, \tag{13}$$

where the subscript a is appended to quantities taken for $T=T_a$, $f=1$. Then Eq. (12) reduces to the form

$$-vf'+f(f^{-\gamma}-1)=\beta^{-1}(f^\beta)''. \tag{14}$$

This equation satisfies the requirements of Kolmogorov–Petrovskiĭ–Piskunov theory,[9] and for the eigenvalue (the front velocity v) we obtain $(\beta+1-\gamma>0)$

$$v=\pm2\gamma^{1/2},\quad u=\frac{4}{5}\frac{n_a}{p_a}(\gamma\lambda_aT_a\varepsilon_a)^{1/2}.$$

For the temperature profile near the front, we obtain

$$0<\zeta\ll1,\quad f=f_0\zeta^\mu,\quad v=2\sqrt{\gamma}>0,$$
$$1)\ \beta>\gamma+1,\quad \mu=1/\gamma,\quad f_0^{-\gamma}=v/\gamma=2\gamma^{-1/2}, \tag{15}$$
$$2)\ \gamma+1>\beta>\gamma-1,\quad \mu=2(\gamma+\beta-1)^{-1},\quad f_0^{\beta+\gamma-1}=0.5(\beta+\gamma-1)^2(\beta+1-\gamma)^{-1},$$

which corresponds to a cooling wave moving in the direction of large q. Asymptotically, at large ζ, we have $f\to1$ in accordance with the law $(v=2\gamma^{1/2})$

$$f=1-c(\zeta-\zeta_0)e^{-2\gamma^{1/2}\zeta}. \tag{16}$$

But if $\gamma-1\geqslant\beta$, then the Kolmogorov–Petrovskiĭ–Piskunov theory[9] is invalid. In this case, the wave velocity depends explicitly on the temperature of the cold phase $T=T_c>0$, and the wave velocity is determined by the expression (8).

§2. SIMILARITY SOLUTIONS OF THE HEAT BALANCE EQUATION

Similarity solutions are important in the analysis of steady regimes of evolution of the structure of the interface between hot and cold gases. It is well known that in a large class of hydrodynamic and thermal problems in which a steady regime exists the initial conditions are forgotten and the solution becomes a similarity solution. Even if this is not correct for the complete studied region, steady solutions can be locally close to similarity solutions, in the most dynamic region. In this case, similarity solutions can be regarded as describing intermediate asymptotic behaviors[2,18-20] that are valid in a bounded interval and in a restricted part of the investigated region.

For the heat balance equation (2), similarity solutions can be obtained in the case of the power-law dependence $\varepsilon=\varepsilon_0T^{-\gamma}$ $(\gamma>0)$, $\lambda=\lambda_0T^\beta$ $(\beta>1)$ for $T>T_c$, and $\varepsilon=0$ for $T<T_c$. As was noted above, we can set $T_c=0$ when $\beta+1-\gamma>0$. The coefficients ε_0 and λ_0 may depend on the pressure, but if $p=\text{const}$ this dependence is unimportant. Equation (2) reduces to the form

$$\frac{5}{2}\frac{dT}{dt}+\varepsilon_0^*T^{1-\gamma}=\lambda_0^*\beta^{-1}\frac{\partial^2T^\beta}{\partial q^2}, \tag{17}$$

where

$$\varepsilon_0^*=\varepsilon_0/p,\quad \lambda_0^*=\lambda_0(nT)^2/p.$$

Equation (17) can be reduced to a self-similar form in two ways, which describe somewhat different physical situations. In the first,

$$T=(\varepsilon_0^*t)^{1/\gamma}f(\xi),\quad 0<t<\infty,\quad \varkappa=(\beta+\gamma-1)/2\gamma,$$
$$\xi=qt^{-\varkappa}\lambda_0^{*-1/2}\varepsilon^{*(1-\beta)/2\gamma}=\xi_0q/q_f. \tag{18}$$

In this solution, the temperature at a point with given value of $\xi=\xi_T$, i.e., at a given point of the profile $f(\xi_T)$, increases with the time, $q_f\sim t^\varkappa$ also increases, and the point $\xi=\xi_T$ moves in the direction of large q. This solution describes, for example, a cooling wave of the type considered in Sec. 1.

The second method differs by the substitution $t\to-t$, $-\infty<t<0$ in (18). For this choice of the domain of definition of t, the temperature at the point $\xi=\xi_T$ decreases with time, $q_f\sim(-t)^\varkappa$ also decreases, and the point $\xi=\xi_T$ moves in the direction of small q. This solution describes, for example, a thermal wave of the type considered in Sec. 1. It is convenient to combine the two cases, replacing the time t by the new variable $\tau=\nu t>0$, $\nu=\pm1$, respectively. For $\nu=1$, τ increases; for $\nu=-1$, τ decreases.

The similarity solution is constructed in the whole of space at once, and there is no similarity solution with constant temperature as $q\to\infty$. In the region of large ξ, similarity solutions of the type (18) always increase unboundedly. Therefore, solutions of the form (18) are helpful primarily as an intermediate asymptotic behavior valid in a bounded region of space near the wave front. Solutions of the form (18) together with a thermal wave and a cooling wave describe the late nonlinear stages in the evolution of temperature inhomogeneities.

In the similarity variables, Eq. (17) reduces to the form

$$\begin{gathered}{}^{5}/_{2}\nu(\sigma f-\varkappa\xi f')+f^{1-\gamma}=\beta^{-1}(f^{\beta})'':\\ \sigma=1/\gamma,\quad \varkappa=(\gamma+\beta-1)/2\gamma,\quad \nu=\pm1.\end{gathered}\tag{19}$$

The general solution of (19) can be obtained for $\gamma=\beta=1$. However, this solution has a mainly methodological significance.

The solution of (19) that is most interesting from the point of view of applications has at low temperatures the characteristic wave form

$$\begin{aligned}&\xi=\xi_0(1+\eta),\quad 0<\eta\ll1,\quad f=f_0\eta^{\mu},\\ &T=(\varepsilon_0^*\tau)^{1/\gamma}f_0(q/q_f-1)^{\mu},\quad q>q_f,\\ &T=0,\qquad q<q_f.\end{aligned}\tag{20}$$

For $\nu=1$, q_f increases, and a cooling wave (see Fig. 2) moves in the direction of the hot phase; for $\nu=-1$, q_f decreases and a thermal wave [see Fig. 1(b)] moves in the direction of the cold phase. Depending on the values of β and γ, the following regimes are possible:

a) for a cooling wave ($\nu=1$):

$$\begin{gathered}\beta>\gamma+1,\quad \mu=1/\gamma,\quad f_0^{-\gamma}=2.5\varkappa/\gamma,\\ \gamma+1>\beta>\gamma-1,\quad \mu=2(\gamma+\beta-1)^{-1},\quad f_0^{\beta+\gamma-1}=0.5(\beta+\gamma-1)^2(\beta-\gamma+1)^{-1}\xi_0^2;\end{gathered}\tag{21}$$

b) for a thermal wave ($\nu=-1$):

$$\begin{gathered}\beta>\gamma+1,\quad \mu=(\beta-1)^{-1},\quad f_0^{\beta-1}=2.5\varkappa\xi_0^2(\beta-1),\\ \gamma+1>\beta>\gamma-1,\quad \mu=2(\gamma+\beta-1)^{-1},\\ f_0^{\beta+\gamma-1}=0.5(\beta+\gamma-1)^2(\beta-\gamma+1)^{-1}\xi_0^2.\end{gathered}\tag{22}$$

In the case $\beta>\gamma+1$, the profile of the cooling wave (21) is determined by the bulk losses, and the profile of thermal wave (22) by the heat conduction. If $\gamma+1>\beta>\gamma-1$, both factors are important. In all cases, the profile corresponding to the thermal wave is steeper than that of the cooling wave. The profile of the cooling wave is identical to (15), which confirms the validity of the approximation (12), (15). However, the velocity of the similarity waves depends on the temperature, and the front width changes. This is due to the change in the temperature in the region $\xi>\xi_0$—there are no similarity solutions with $T=\text{const}$ as $q\to\infty$.

In the special case $\beta=\gamma+1$, Eq. (19) has the exact solution

$$f^{\gamma}=0.5\gamma[(16+25\xi_0^2)^{1/2}-5\nu\xi_0](\xi-\xi_0)\tag{23}$$

with one arbitrary constant ξ_0. The situation described by (21) and (22) corresponds to

$$\begin{aligned}T^{\gamma}&=0.5\varepsilon_0^*\tau\gamma[(16+25\xi_0^2)^{1/2}-5\nu\xi_0](\xi-\xi_0)\\ &=0.5\varepsilon_0^{*1/2}\gamma\lambda_0^{*-1/2}[(16+25\xi_0^2)^{1/2}-5\nu\xi_0]q(1-q_f/q).\end{aligned}\tag{24}$$

For $\nu=1$, (24) describes a cooling wave, and for $\nu=-1$ a thermal wave. The cold phase is situated at $q\leqslant q_f$. Asymptotically far from the front in the region $\xi\gg\xi_0$ ($q\gg q_f$), the solution (24) tends to a stationary solution:

$$T_{st}^{\gamma}=0.5\gamma\varepsilon_0^{*1/2}\lambda_0^{*-1/2}[(16+25\xi_0^2)^{1/2}-5\nu\xi_0]q.\tag{25}$$

For $\xi_0=0$, $q_f=0$, the expression (24) goes over into an exact stationary solution equivalent to the one found in Ref. 14. In the asymptotic region $q\gg q_f$, the stationary solution with $\xi_0=0$ separates the set of solutions describing a thermal wave ($\nu<0$) from the set describing a cooling wave ($\nu>0$). However, in accordance with the similarily solution (24) the temperature becomes zero for $q=q_f>0$, whereas in the stationary solution this occurs only for $q=0$ (the stationary solution in the neighborhood of $q=q_f$ always passes higher than both the thermal wave and the cooling wave and cannot serve as boundary between them).

The solution (23)–(25) is realized in a situation that is particularly interesting from the point of view of applications—bremsstrahlung energy losses ($\gamma=3/2$) and electron heat conduction ($\beta=5/2$).

As the expressions (23) and (24) show, similarity thermal waves can exist only through heat reserves at the distant periphery, in the region of large q, and they are accompanied by a general decrease in the temperature in the hot phase in accordance with energy conservation.

Similarity solutions of wave type do not exist for $\beta\leqslant\gamma-1$. In this case, there are only similarity solutions having the form of a minimum of the temperature. Similarity solutions with an extremum of the temperature are also possible at other values of β and γ together with wave solutions. These solutions can, for example, describe the late nonlinear stages of thermal instability. In the neighborhood of the extremum in the region $|\xi|\ll1$, these solutions have the form

$$\begin{gathered}f=f_0(1+a\xi^2+\ldots)=f_0(1+q^2/q_*^2+\ldots),\\ T=(\varepsilon_0^*\tau)^{1/\gamma}f_0(1+a\xi^2+\ldots)=T_*[1+0.5q^2/q_*^2+\ldots],\\ a=0.5f_0^{1-\beta-\gamma}[1+2.5\nu\sigma f_0^{\gamma}],\end{gathered}\tag{26}$$

and if $\nu=1$ then $a>0$ and only a minimum of the temperature is possible, whereas if $\nu=-1$ it is possible to have both a minimum ($1>2.5\sigma f_0^{\gamma}$) and a maximum ($1<2.5\sigma f_0^{\gamma}$; $a<0$) of the temperature. For $\nu=1$, the temperature at the minimum increases, and the curvature of the profile decreases, i.e., the inhomogeneity is decreased. In contrast, for $\nu=-1$ the temperature at the center decreases and the curvature of the profile increases, i.e., the thermal inhomogeneity is localized more strongly. This is true both in the case of a minimum of the temperature (when the decrease in the temperature leads to an increase in the inhomogeneity) as well as in the case of a maximum of the temperature (when the spreading of the inhomogeneity is accompanied by an increase in the curvature of the temperature profile).

It is to be expected that in real problems the cooling ceases when a certain minimal temperature has been reached (for example, 10^4 °K), and the solution (26) is rearranged into a solution of the type (20), (21) with formation of a cooling wave.

Expressing q_e^2, which determines the curvature of the temperature inhomogeneity, in terms of the temperature T_e at the extremum, we obtain

$$q_e^2 = \lambda_e T_e n_e^2 \varepsilon_e^{-1} |1 + 2{,}5\sigma p_e/\varepsilon_e\tau|^{-1}, \tag{27}$$

where the subscript e is appended to the symbols of all quantities at $T = T_e$. Obviously, there exist two physically different situations. The first is

$$\text{a)}\ |2{,}5\sigma p_e/\varepsilon_e\tau| \ll 1, \quad q_e^2 \approx \lambda_e T_e n_e^2 \varepsilon_e^{-1} \tag{27a}$$

in which the curvature of the temperature profile depends implicitly on the time, through $T_e = T_e(t)$. This regime is realized for sufficiently large values of ε_e, i.e., for sufficiently low temperature at the minimum, in the late stages of development of thermal instability. The q_e scale in this case is equal to the critical scale for thermal instability in accordance with Ref. 21. In the second case,

$$\text{b)}\ |2{,}5\sigma p_e/\varepsilon_e\tau| \gg 1, \quad q_e^2 = \lambda_e T_e n_e^2 \tau/p_e \tag{27b}$$

and the curvature of the temperature profile depends explicitly on the time and does not depend on the (low) cooling rate. This regime is realized at high temperatures, at maxima of the temperature, and, possibly, in late stages in the disappearance of a temperature minimum.

Besides the solutions (20), (23), and (26), Eq. (19) in the region of small f has a solution of the form

$$\begin{gathered} 0 < \xi \ll 1, \quad f = f_0 \xi^{1/\beta}[1 + b\xi^\mu + \ldots]; \\ 1)\ \beta > \gamma; \quad \mu = 1/\beta; \quad b = 2.5\varkappa\beta(1+\beta)^{-1} f_0^{1-\beta}; \\ 2)\ \gamma > \beta > \gamma - 1; \quad \mu = (\beta - \gamma + 1)/\beta; \quad b = \beta^2(\beta + 1 - \gamma)^{-1}(2\beta + 1 - \gamma)^{-1} f_0^{1-\beta-\gamma}. \end{gathered} \tag{28}$$

In this solution at the point $\xi = 0$, $f = 0$ the heat flux $Q = f^{\beta-1} f'$ is not zero, and therefore (28) is valid only for describing nonstandard situations with localized energy sink (boundary condition at the point $\xi = 0$).

In addition, in Eq. (17) there is a singular stationary solution (unstable) corresponding to the one obtained in Ref. 14:

$$\begin{gathered} f = f_0 \xi^{2/(\beta+\gamma-1)}, \quad f_0^{\beta+\gamma-1} = 1{,}25\gamma^{-1}(\beta + \gamma - 1)^2(\beta + 1 - \gamma)^{-1}, \\ \beta > \gamma - 1, \quad T = T(q) \propto q^{2/(\beta+\gamma-1)}. \end{gathered} \tag{29}$$

Among the stationary solutions of (17) there is, besides (29), a solution with a minimum of the temperature and a solution of the type (28) with energy sink at the point $\xi = 0$.

In the region of large ξ and $f \gg 1$, Eq. (19) has a solution of the form

$$\begin{gathered} 1)\ f = f_0 \xi^{2/(\beta-1)}; \quad f_0^{\beta-1} = 1{,}25(\beta - 1)(\beta + 1)^{-1}; \quad \nu = -1, \\ 2)\ f = f_0 \xi^{2/(\beta+\gamma-1)}(1 + c\xi^{-1/\varkappa}), \quad c = f_0^{-\gamma} - f_0^{\beta-1}. \end{gathered} \tag{30}$$

The similarity solutions can be generalized to the case of a power-law dependence of the pressure of the form $p \sim t^\alpha$.

§3. EVAPORATION AND CONDENSATION OF CLOUDS

The solutions obtained above solve not only the problem of the structure of the interface between hot and cold gases but also permit the calculation of the evolution of inhomogeneities under conditions of high temperature of the hot phase, a strong influence of electron heat conduction, and bulk energy losses—of bremsstrahlung and recombination type—for given external pressure.

We assume for the gas with cosmological chemical composition (0.7H + 0.3He) the following values of ε and λ ($T = 10^6 T_6$ °K, particle density n measured in cm^{-3}):

$$\lambda T = 0{,}9 \cdot 10^{15} T_6^{7/2} \quad \mathrm{erg \cdot cm^{-1} \cdot sec^{-1}}$$

$$\varepsilon_{ff} = 2 \cdot 10^{-24} T_6^{1/2} n^2 \quad \mathrm{erg \cdot cm^{-3} \cdot sec^{-1}}$$

$$\varepsilon_{fb} = 1{,}5 \cdot 10^{-24} T_6^{-1/2} n^2 \quad \mathrm{erg \cdot cm^{-3} \cdot sec^{-1}}.$$

Then for the scale and velocity of the wave we obtain in accordance with (13) (for bremsstrahlung losses)

$$\begin{gathered} q_c = n_a[\lambda_a T_a/\varepsilon_{ff}(T_a)]^{1/2} \approx 2 \cdot 10^{19} T_6^{3/2} \ \mathrm{cm}^{-2}, \\ u = {}^4/_5[\gamma\lambda_a T_a \varepsilon_{ff}(T_a)]^{1/2}/kT_a \approx 3 \cdot 10^5 \ T_6 n \mathrm{cm}^{-2} \cdot \mathrm{sec}^{-1} 2 p_{15} \ \mathrm{cm}^{-2} \cdot \mathrm{sec}^{1}, \end{gathered} \tag{31}$$

where $p = 10^{-15} p_{15}$ erg/cm^3. The value of q_c is equal to the critical scale of the region of thermal instability obtained in Ref. 21. The temperature profile is determined by (15) at low temperatures and by (16) at high temperatures:

$$\begin{gathered} f = 0{,}65\xi^{2/3}, \quad \xi < 1, \\ f = 1 - c(\xi - \xi_0)e^{-\xi\sqrt{3/2}} \quad (1 - f \ll 1). \end{gathered}$$

Allowance for recombination radiation makes the profile steeper in the region of low temperatures, changing the exponent to $\frac{1}{2}$ from $\frac{2}{3}$.

As is shown by the estimates (31), the velocity of the cooling front is small compared with the isothermal velocity of sound in the hot phase up to temperatures $T_6 \approx 10^2$:

$$a_s \approx 10^7 T_6^{1/2} \ \mathrm{cm/sec} \gg u/n \approx 3 \cdot 10^5 T_6 \mathrm{cm/sec}.$$

The cooling front is strongly spread out and at low densities of the hot phase may be wider than a kiloparsec. Comparing the dimension of the cooling front with the characteristic electron mean free path in ionized (q_i) and neutral (q_0) gases,

$$\begin{gathered} q_i = (kT)^2/12\pi e^4\Lambda \approx 2{,}35 \cdot 10^{14} T_6^2 \ \mathrm{cm}^{-2}, \\ q_0 \approx 10^{18} \ T_6 \ \mathrm{cm}^{-2}, \end{gathered}$$

where Λ is the Coulomb logarithm ($\Lambda \approx 40$), we find that the diffusion approximation is satisfied with a good margin and direct heating and ionization of the cold gas by electrons of the hot phase are ruled out. The possible photoionization by the self-radiation of the hot gas is negligibly small. Photoionization by external sources may play some part. However, if this radiation cannot heat the neutral gas of the cold phase, its part merely reduces to changing the degree of ionization of both the cold gas and the gas in the region of the front. Then the position of the ionization front need not coincide with that of the thermal front.

The most important aspect of the problem discussed here is the question of the conditions under which evaporation or condensation of a cloud occurs. This question has been discussed in a series of papers.[12-14] The principle conclusion of these studies is that small clouds evaporate and large clouds condense and that there exists a critical cloud radius separating evaporating and condensing clouds. The rate of evaporation of clouds obtained in the quoted papers is widely used in astrophysical applications.[12,15] However, both the analysis of the problem made in the quoted studies and the conclusions reached in them come up against a number of serious objections that can be seen clearly by a comparison with the results obtained above.

We note first that in Refs. 13 and 14 only stationary equations are considered, and the evaporation rates are calculated[13] with neglect of bulk energy losses. The approximate stationary solution in Ref. 14, obtained for a complicated cooling law, is actually identical to (29) when allowance is made for the change in the geometry (plane instead of spherical) and the expression for the bulk losses. It can be seen from the results of Secs. 1 and 2 that this solution contains very little information about the predominance of evaporation or condensation processes. In a steady regime, as is shown by the results of Secs. 1 and 2 (see also Ref. 11), thermal waves and evaporation are always associated either with bulk heat sources (for example, in the Galaxy) or with heat reserves, as, for example, in the similarity solution (24). But if bulk heat sources are absent, $\varepsilon > 0$ in the range of temperatures in which we are interested, and the temperature of the hot phase is bounded ($T \leqslant T_h$ as $q \to \infty$), then a steady regime always corresponds to a cooling wave and condensation of the matter of the hot phase. Heat conduction, which leads to a spreading of the interface between the hot and cold phases, helps to cool the hot phase.

Under real conditions, the motions of the gas of the hot phase may distort locally the temperature profiles. However, this applies mainly to the outer, hotter regions of the region of contact. In the low-temperature region, the width of the interface is small, and the temperature profile is established rapidly. The chosen value of the coefficient of electron thermal conductivity may be distorted in the presence of magnetic fields, and the question of their influence remains open. It is clear however that the main conclusions of the paper do not depend on the particular choice and value of ε and λ.

§4. MOTION OF COLD CLOUDS IN HOT GAS

The fate of clouds of cold gas is determined by not only the processes of evaporation and condensation but also to a considerable extent by their motion in the hot gas, which can result in appreciable deformations of a cloud. A cloud of cold gas at rest in a hot gas acquires a spherical shape under the influence of gravitation. However, if the mass of the cloud is small, so that the gravitational energy of the cloud is small compared with the thermal energy, the spherical shape is readily distorted by fluctuations of the external pressure, and the cloud of cold gas may be flattened and break up.

If a spherical cloud of cold gas moves through hot gas, then the pressure on its surface depends on the polar angle θ, measured from the direction of the velocity[22,23] (we ignore the acceleration):

$$p(\theta)=p_0+1.125\rho_h u^2(\cos^2\theta-{}^5/_9), \tag{32}$$

where $p(\theta)$ and p_0 are the pressure of the hot phase on the surface of the sphere, ρ_h is the density of the hot phase, and u is the velocity. The pressure difference between the points $\theta=0$ and $\theta=\pi/2$ is $\Delta p=1.25\rho_h u^2$, and this Δp tends to flatten the sphere, transforming it into an ellipsoid. However, this deformation is associated with work done against the gravitational forces. Therefore, strong contraction is possible only in the case of a sufficiently high velocity u (or low mass M) of the sphere.

We estimate roughly the connection between the critical velocity u leading the flattening of a sphere and the mass M of the sphere. A particle is maintained on the surface of the sphere by the gravitational force $F_g = GM\rho_c/R^2$, whereas the pressure gradient displacing it is of order $\rho_h u^2/R$; we therefore find that deformation of the sphere requires

$$\rho_h u^2 > \rho_c GM/R=\rho_c a_c^2 (M/M_J)^{2/3} \approx \rho_h a_h^2 (M/M_J)^{2/3}, \tag{33}$$

where G is the gravitational constant, R is the radius of the sphere, ρ_c and a_c are the density and velocity of sound in the cold gas, M_J is the Jeans mass corresponding to approximate equality of the thermal and gravitational energies in the sphere, and a_h is the velocity of sound in the hot gas. The condition (approximate, without allowance for gravitation) of constancy of the pressure in the cold and hot phases leads to the condition $\rho_c a_c^2 = \rho_h a_h^2$. Rewriting (33) in the form

$$u^2/a_h^2=\mu^2 \gtrsim (M/M_J)^{2/3}, \tag{33a}$$

we obtain a convenient formula for estimating the degree of deformation of the cold gas cloud when it moves in the hot gas.

If the velocity of the cloud exceeds the value (33), then the cloud is flattened, its gravitational energy decreases, and it can be dissipated during the hydrodynamic time, i.e., before thermal processes become important. In contrast, slight flattening of the cloud may accelerate the process of condensation of hot gas due to the increased surface of the cold cloud.

§5. EVOLUTION OF EXTRAGALACTIC INHOMOGENEITIES

We consider as a first example the evolution of a cloud of cold ($T=10^4$ °K) gas in the hot gas of a rich cluster of galaxies. If such a cloud enters such a cluster from without, then in the process of establishment of the cooling wave regimes considered above the mass of such a cloud may decrease appreciably and, in particular, it is possible that it will evaporate entirely. However, if the mass of the original cloud is sufficiently large and the cooling wave regime has become established, the mass of the cloud increases rapidly, mass being accumulated in proportion to the surface area of the cloud.

We assume that the surface area of a cloud containing N particles at temperature $T=10^4$ °K is

$$S=600T_4^{2/3}N^{2/3}p_{15}^{-2/3}\sigma, \tag{34}$$

where σ is a factor which takes into account the possible influence of a departure of the surface area of the cloud from the surface area of a homogeneous sphere, and $p=10^{-15}p_{15}$ erg/cm^3. Using (31), we obtain

$$\begin{aligned} dN/dt=uS, \quad N^{1/3}&=N_0^{1/3}+200T_4^{2/3}\int \sigma p_{15}^{1/3}\,dt \\ &=N_0^{1/3}+200T_4^{2/3}p_{15}^{1/3}\sigma^* t \end{aligned} \tag{35}$$

(assuming $p_{15}=\text{const}$ and $\int\sigma dt=\sigma^* t$) and in the most interesting case $N\gg N_0$

$$N=10^7\ T_4^2p_{15}\sigma^{*3}t^3\approx 300T_4^2p_{15}\sigma^{*3}h^{-3}N_\odot \tag{36}$$

for $t=H_0^{-1}=3\times10^{17}h^{-1}$, where $H_0=100\text{h km}\cdot\text{sec}^{-1}\cdot\text{Mpc}^{-1}$ is the Hubble constant. Taking the Coma cluster as a guide, we assume that in the hot phase $T_6=100$, $n_p=10^{-3}$ cm^{-3}, $p_{15}=3\times10^4$. Comparing the obtained mass with the critical mass of an isothermal sphere at temperature $T=10^4T_4$ °K,

$$M_T=4\cdot10^8T_4^2p_{15}^{-1/2}M_\odot, \tag{37}$$

we find that

$$M/M_T\approx0.75\cdot10^{-6}p_{15}^{3/2}\ \sigma^{*3}h^{-3}\approx4\sigma^{*3}h^{-3}>1. \tag{38}$$

Thus, during a time several times shorter than the cosmological time gas clouds under the conditions of clusters of galaxies reach the critical mass and collapse, being transformed into stars. Therefore, under the conditions of rich clusters of galaxies it is hard to expect the presence of fossil cold gas clouds. On the other hand, the formation of new cold clouds in the hot gas of clusters is also difficult. It is possible that the formation of *SO* galaxies in rich clusters is due to these effects.

In poor clusters or in superclusters such as the Virgo cluster or A1367–A1656 the pressure in the hot phase is lower by one or two orders of magnitude, and therefore the effects of growth of the mass of cold clouds are expressed much more weakly and H1 clouds may exist both within galaxies and outside them.

Problems associated with the cooling and condensation of hot gas also arise in connection with the analysis of the structure of the outer zone of the gas disk of galaxies, in problems of the evolution of inhomogeneities in the nonlinear theory of the gravitational instability, in connection with thermal inhomogeneities in the hot gas of clusters of galaxies, etc. These questions will be considered separately.

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