Anisotropic spectra of weak sound turbulence

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The weak turbulence of waves with a weakly damped dispersion law is considered. Small anisotropic increments to the isotropic spectrum of the Kolmogorov type are found in the stationary case. It is shown that weak anisotropy of a source located in the region of small **k** should lead to an appreciably anisotropic spectrum in the region of large **k**. The strongly anisotropic stationary spectra are determined.

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INTRODUCTION

The theory of wave turbulence has been intensively developed in recent years in connection with problems of plasma physics, nonlinear optics, and hydrodynamics. In the presence of dispersion of the propagation velocity and a low level of nonlinearity, the interaction between waves is weak, which allows us to use the kinetic equation for the description of the turbulence. An important achievement of the theory has been the determination of the spectra of weak wave turbulence as exact solutions of the kinetic equation.[1-3] An isotropic medium is usually considered. In this case, for a decay law of dispersion and for a scale-invariant situation

$$\omega_{\lambda k}=\lambda^{\alpha}\omega_{k}, \qquad V_{\lambda \mathbf{k}_1\lambda \mathbf{k}_2\lambda \mathbf{k}_3}=\lambda^{m}V_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3},$$

where $V_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}$ is the matrix element of three-wave interaction, the spectrum of the turbulence is

$$n_k^0\sim k^{-m-d}. \tag{1}$$

Here d is the dimensionality of the **k** space. The spectrum (1) is realized in the inertial scale interval $k_0 \ll k \ll k_m$, where the source is located in the region $k=k_0$, and at $k=k_m$ is the drain of the energy of the waves; this corresponds to a constant energy flux over the spectrum.

In addition to the isotropic spectra, small anisotropic increments to them have also been found:[4]

$$\delta n_{\mathbf{k}}\sim n_k^0 k^{\alpha-1}\cos\theta_{\mathbf{nk}}, \tag{2}$$

where **n** specifies the source. We turn our attention to the fact that, since $\alpha > 1$ (the decay situation), the anisotropic increment falls off with increase in k more slowly than the isotropic part of the spectrum. This indicates an unusual instability of the isotropic spectrum—even a small anisotropy of the source leads to an essentially anistropic spectrum deep in the inertial interval.

In the present work, we find the strongly anisotropic spectra of weak turbulence for the physically interesting case of a weak-decay dispersion law. Since the dispersion is small, waves propagating in a narrow cone of angles take part in the three-wave interaction (in the case of linear dispersion law $\omega_k = ck$, only waves with collinear wave vectors interact). Assuming that the angular dependence of the spectrum n_k changes little over the dimensions of the interaction angle, we use a differential approximation in the angle variables. In Sec. 1, we find the small stationary increments to the spectrum (1), of the form

$$\frac{\delta n_{\mathbf{k}}}{n_k^0}\propto\sum_{l=0}^{\infty} a_l P_l(\cos\theta_{\mathbf{nk}})\,k^{\varepsilon l(l+1)/2}, \quad l=0,1,2,\dots, \tag{3}$$

for the dispersion law $\omega_k \sim k^{1+\varepsilon}(\varepsilon \ll 1)$.

Formula (2) is a special case of formula (3) when all the a_l except a_1 are equal to zero. It is easy to see from (3) that the higher the number of the harmonic l the more rapidly does the corresponding increment to the spectrum increase with increase in k. Inasmuch as an anisotropic source is always present in real physical situations, the finding of strongly anisotropic spectra is of interest. This problem is solved in Sec. 2 for a model, scale-invariant dispersion law:

$$\omega_k\sim k^{1+\varepsilon} \tag{4a}$$

and for the dispersion law

$$\omega_k=ck(1+a^2k^2), \tag{4b}$$

which is more interesting from the physical point of view. In this latter case, z is some characteristic length, $ak_m \ll 1$. The spectra of n_k (10) and (12) that are obtained possess the property that the smaller their characteristic angular size, the more slowly does the spectrum decrease with increase in k. Each such spectrum is unstable in the same sense as the isotropic spectrum—in a real system there is always a small anisotropy with a characteristic angle much smaller than the angular dimension of the spectrum, so that the results of Sec. 1 will be applicable. This allows us to draw the conclusion that in the presence of even a small anistropy of the source the spectrum is sharply chopped up in the region of large k up to angles of the order of the interaction angle.

1. SMALL INCREMENTS TO THE ISOTROPIC SPECTRUM

Three wave processes are allowed in the case of the decay dispersion laws (4), and the kinetic equation has the form

$$\begin{aligned} dn_{\mathbf{k}}/dt=4\pi\int\{&|V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}|^2\delta(\omega_k-\omega_1-\omega_2)\,\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\,[n_1n_2-n_1n_k-n_2n_k] \\ &-2|V_{\mathbf{k}_1\mathbf{k}\mathbf{k}_2}|^2\delta(\omega_1-\omega_k-\omega_2)\,\delta(\mathbf{k}_1-\mathbf{k}-\mathbf{k}_2)\,[n_2n_k-n_1n_k-n_1n_2]\}\,d\mathbf{k}_1d\mathbf{k}_2. \end{aligned} \tag{5}$$

Here the matrix element of the three-wave interaction for waves with almost linear dispersion laws can be written as

$$V_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}=B(k_1k_2k_3)^{1/2}\left[q+\frac{(\mathbf{k}_1\mathbf{k}_2)}{k_1k_2}+\frac{(\mathbf{k}_1\mathbf{k}_3)}{k_1k_3}+\frac{(\mathbf{k}_2\mathbf{k}_3)}{k_2k_3}\right],$$

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where q is a constant of the order of unity.

Let a source that is different from zero at small k be weakly non-isotropic (n is the vector specifying the preferred direction). Considering the axially-symmetric case for simplicity, we seek a solution of (5) in the form

$$n_k = A[k^{-9/2} + \varepsilon f(\cos\theta_{nk}) k^\alpha]. \qquad (6)$$

Since we are considering the stationary case, $dn_k/dt = 0$. We integrate (5) over dk_2 with the help of $\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)$. In the obtained expression, we can set $V_{\mathbf{kk}_1\mathbf{k}_2} \sim (kk_1|\mathbf{k}-\mathbf{k}_1|)^{1/2}$ without risking contradiction. Then, integrating over $d\cos\theta_{nk}$ with the help of a δ-function of ω, we get the equation ($x = k_1/k$)

$$\int_0^1 (1-x^{1+\varepsilon})^{(2-\varepsilon)/(1+\varepsilon)} x^2 dx (n_1 n_2 - n_1 n_k - n_2 n_k)$$
$$-2\int_1^\infty (x^{1+\varepsilon}-1)^{(2-\varepsilon)/(1+\varepsilon)} x^2 dx (n_2 n_k - n_1 n_k - n_1 n_2) = 0. \qquad (7)$$

Here $n_1 = n_{k1}(\theta_{\mathbf{kk}_1})$ and $n_2 = n_{k2}(\theta_{\mathbf{kk}_2})$ are taken at

$$\frac{\theta_{kk_1}^2}{2} = \frac{\theta_1^2}{2} = -\varepsilon\frac{1-x}{x}[(1-x)\ln(1-x) + x\ln x]$$

in the first integral,

$$\frac{\theta_{kk_1}^2}{2} = \frac{\theta_2^2}{2} = \varepsilon\frac{x-1}{x}[x\ln x - (x-1)\ln(x-1)]$$

in the second integral, while $\theta_{kk_2}^2 = \theta_{kk_1}^2 \cdot x^2/(1-x^2)$. The quantities θ_1, θ_2 are the angles of interaction in the processes $\mathbf{k}_1 + \mathbf{k}_2 \to k$ and $\mathbf{k} + \mathbf{k}_2 \to k_1$. As is easy to see, they are determined by the deviation of the dispersion from linearity.

We now substitute (6) in (7), linearize with respect to ε and, dividing the second integral into two identical terms, we carry out in one of which the transformation $x \to 1/x$ and in the other $x \to x(1-x)^{-1/(1+\varepsilon)}$ (these are in fact the conformal transformations proposed by Zakharov[3]). Using the smallness of the angle of interaction, we expand

$$f(\cos\theta_{nk_1}) \approx f(\cos\theta_{nk}) - (f'\cos\theta_{nk} - f''\sin^2\theta_{nk}\cos^2\varphi_{k_1})\theta_{kk_1}^2/2$$

and for $f(\cos\theta_{nk}) = f(y)$ we obtain the equation (the factor of the converging integral in the kinetic equation)

$$\varepsilon\left(y\frac{\partial f}{\partial y} - \frac{1-y^2}{2}\frac{\partial^2 f}{\partial y^2}\right) = \left(\frac{9}{2}+\alpha\right)f.$$

This is a Legendre equation. The analytic solution of the equation over the interval $y \in [-1, 1]$ has the form

$$f = \sum_{n=0}^\infty a_n D_n(y),$$

where $D_n(y)$ are the Legendre polynomials. In this case, $\alpha_n = -9/2 + \varepsilon n(n+1)/2$ and we arrive at Eq. (3).

2. ANISOTROPIC SPECTRA

For the scale-invariant dispersion law (4a) we seek anisotropic solutions of the kinetic equation in the scale-invariant form

$$n_k \sim f(y) k^{-9/2+\varepsilon\alpha}. \qquad (8)$$

Following the same procedures as in Sec. 1 (integration over dk_2, over the angle, conformal mapping), we obtain in first order in ε the following equation:

$$[-2\alpha f^2 + (1-y^2)(f'^2 - ff'') + 2yff'] \int_0^1 x^2(1-x)^2[x\ln x + (1-x)\ln(1-x)]$$
$$\times[x^{-9/2} + (1-x)^{-9/2} - x^{-9/2}(1-x)^{-9/2}]dx = 0. \qquad (9)$$

As is easy to see, the integral converges, while the solution of the equation for $f(y)$, which is finite at $y \in [-1, 1]$, has the form

$$f(y) = (1-y)^{\alpha - C/2}(1+y)^{\alpha + C/2}, \quad |C/2| \leqslant \alpha.$$

In the derivation of Eq. (9), we used a differential approximation, discarding the higher derivatives of f with respect to y. This can be done only if $f(y)$ is an analytic function. Requiring that f be an analytic function of y, we obtain

$$f(y) = (1-y)^n(1+y)^m, \quad n, m = 0, 1, 2, \ldots.$$

Thus, the anisotropic spectra have the form

$$n_k^{(ml)} \sim (1-\cos\theta_{nk})^m (1+\cos\theta_{nk})^l k^{-9/2+\varepsilon(m+l)/2}. \qquad (10)$$

We note that since the integrals in the kinetic equation converge, the spectra (3) and (10) are local.

We now consider the case of the dispersion law (4b). The spectrum $n_k^0 = Ak^{-9/2}$ now satisfies the kinetic equation only in zeroth order in a^2k^2. We seek the anisotropic solution in the form

$$n_k \sim f(\cos\theta_{nk}) k^{-9/2}(1 + \xi a^2 k^2).$$

Such a spectrum is not local, and the integrals in the kinetic equation diverge. Therefore it is necessary to introduce the cutoff $k \leqslant k_m$. The equation for $f(\cos\theta_{nk})$ is obtained in first order in $a^2k_m^2$ by equating to zero the coefficient of the most diverging integral (which behaves as $k_m^{3/2}$). To obtain this equation, we must expand $|V_{\mathbf{kk}_1\mathbf{k}_2}|^2(n_2 n_k - n_1 n_k - n_1 n_2)$ up to terms proportional to a^2, expand the argument of the δ-functions of the frequencies up to terms proportional to a^4, and then integrate (5) first with respect to dk_3 and then with respect to $d\cos\theta_{nk}$. As a result, we obtain

$$f\left[f\left(\frac{5\xi}{2} - \frac{27}{2}\right) + \frac{39}{2}\left(\frac{1-y^2}{2}\frac{\partial^2 f}{\partial y^2} - y\frac{\partial f}{\partial y}\right)\right] = 0. \qquad (11)$$

In the square brackets we again have a Legendre equation. Since, by definition, $f(y) \geqslant 0$, the solution of (11) is

$$f_n(y) = \begin{cases} P_n(y) & \text{at } P_n(y) \geqslant 0, \\ 0 & \text{at } P_n(y) < 0 \end{cases} \quad n = 0, 1, 2\ldots$$

and the turbulence spectrum has the form

$$n_k^{(l)} \sim k^{-9/2} f_l(\cos\theta_{nk})\left[1 + \left(\frac{27}{5} + \frac{39}{10}(l+1)l\right)a^2k^2\right]. \qquad (12)$$

The spectrum falls off more slowly for larger l.

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