the case of the triangular and hcp lattices the summary magnetic moments are equal respectively to 1/3 and 1/4 of the saturation moment. The vacancies are preferentially concentrated on a ceratin magnetic sublattice and upset the initial symmetrical density distribution in the vacancy-free crystal.

We note that our results present a rigorous variational estimate of the ground-state energy in the Hubbard model with inifinite repulsion and with nearly half-filled band for these lattices. The authors thank V. L. Pokrovskiĭ for helpful discussions.

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Onset of inhomogeneous magnetic ordering of the spins in a superconductor

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The relative role of the magnetic dipole and of the indirect exchange interaction of localized spins in the onset of magnetic ordering in a superconductor is investigated. It is assumed that in the absence of superconductivity the ordering in the spin system would be ferromagnetic. It is shown that even weak exchange exerts a substantial influence on the value of the wave vector of the magnetic structure produced in the superconductor.

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The question of coexistence of magnetism and superconductivity is presently attracting increased interest in connection with the synthesis of superconducting compounds with a regular arrangement of magnetic atoms. Many compounds of this type have been synthesized by now[1-3] and their number continues to increase rapidly. There is at present no doubt that antiferromagnetism has little effect on superconductivity, and approximately ten compounds of the $GdMo_6S_8$ type are known, in which coexistence of superconductivity and antiferromagnetism has been found to take place.[1-3] A situation of much greater interest is one in which, in the absence of superconducting pairing, the localized moments should become ferromagnetically ordered when the temperature is lowered. This case is realized in the compounds $ErRh_4B_4$ and $HoMO_6S_8$, which become first magnetically disordered superconductors at a point T_{c1}, and then go over into a ferromagnetic superconducting phase at a point $T_{c2} < T_{c1}$. This situation is the subject of our study.

The magnetic ordering of the considered compounds can be brought about by two main types of interaction: magnetic dipole interaction of localized moments, and indirect exchange via the conduction electrons. Spin ordering in a superconductor in the model with indirect exchange interaction was considered in Ref. 4 and in detail in Ref. 5. It was shown there that in the region where superconductivity and magnetism coexist there is realized a helicoidal ordering of the localized spoins with wave vector Q. and near the point of onset of magnetic ordering.

$$Q \approx (k_F^2 \xi_0^{-1})^{1/3},$$

where ξ_0 the superconducting correlation length. The model with magnetic dipole interaction was investigated in Refs. 6 and 7. In this case the magnetic ordering in the superconductor is likewise helicoidal with wave vector

$$Q \approx (k_F/\lambda_L)^{1/2},$$

where λ_L is the London penetration depth. In real systems, the magnetic-dipole and the exchange interactions act simultaneously, and we investigate in this article their relative contributions to the structure of the inhomogeneous magnetic ordering in the superconducting phase.

We note first that a weak magnetic dipole interaction or an interaction of the order of the exchange interaction does not alter the results obtained in Refs. 4 and 5 for the case of pure exchange interaction. The situation is different in the case of weak exchange interaction against the background of the magnetic dipole interac-

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tion. Although the temperature of the magnetic transition T_M is determined in this case mainly by dipole interaction, the vector $\mathbf{Q}$ of the superstructure depends on the exchange interactions, and the expressions given in Refs. 6 and 7 for $\mathbf{Q}$ are not applicable. We obtain here the value of $\mathbf{Q}$ for our case, assuming that $T_M \ll T_{c1}$.

We describe the magnetic system by the Ginzburg-Landau functional. We express that part of the functional which is quadratic in the magnetization $\mathbf{M}(\mathbf{r})$ in the form

$$F=\sum_{\mathbf{q}}\left\{\frac{1}{2}\left[\frac{T-\Theta_0}{\Theta_{em}}+\gamma q^2-\frac{\Theta_{ex}}{\Theta_{em}}\chi_s(\mathbf{q})\right]\mathbf{M}_\mathbf{q}\mathbf{M}_{-\mathbf{q}}-\mathbf{B}_\mathbf{q}\mathbf{M}_{-\mathbf{q}}+\frac{1}{8\pi}\mathbf{B}_\mathbf{q}\mathbf{B}_{-\mathbf{q}}+\frac{1}{2}Q_s(\mathbf{q})\mathbf{A}_\mathbf{q}\mathbf{A}_{-\mathbf{q}}\right\},\quad \mathbf{B}_\mathbf{q}=i[\mathbf{q}\times\mathbf{A}_\mathbf{q}], \tag{1}$$

where $\mathbf{M}_q$, $\mathbf{B}_q$, and $\mathbf{A}_q$ are the Fourier components of the magnetization, of the induction, and of the vector potential, $Q_s(\mathbf{q})$ is the electromagnetic kernel, and $\chi_s(\mathbf{q})$ is the spin susceptibility of the superconductor. The last three terms of (1) describe the long-range (electromagnetic) part of the magnetic dipole interaction, while the term with the parameter Θ_{ex} describes the indirect exchange interaction, which is determined by the susceptibility $\chi_s(\mathbf{q})$ of the electron system, with $q \ll K$, where K is the reciprocal vector of the spin lattice. Those parts of the dipole and exchange interaction which are determined by the response of the electron system with large values $q \approx K$ do not depend on the superconducting properties and are taken into account by the quantity Θ_0 (see Ref. 5). The dependence of this part of the free energy on q is described by the term with coefficient $\gamma = a^2$ and $a \approx K^{-1}$.

Allowance for the direct exchange of the spins reduces to a renormalization of the parameters Θ_0 and γ. At $q \ll \xi_0^{-1}$ we have for a pure superconductor

$$Q_s(\mathbf{q})=(4\pi\lambda_L^2)^{-1},\quad \chi_s\approx 0;$$

at $q \gg \xi_0^{-1}$ we have

$$Q_s(\mathbf{q})=3/4\xi_0 q\lambda_L^2,\quad \chi_s(\mathbf{q})=1-\pi/2\xi_0 q.$$

The characteristic temperature of the electromagnetic interaction is $\Theta_{em} \approx n\mu^2$, where μ is the magnetic of the localized magnetic moments and n is their density. We have

$$\Theta_{ex}\approx J^2(0)N(0)/n,$$

where J is the indirect-exchange parameter and $N(0)$ is the density of the electronic states on the Fermi surface. We assume that $\Theta_{ex} \ll \Theta_{em}$ and that without superconductivity the system would go over into a ferromagnetic state at the temperature

$$\Theta=\Theta_0+4\pi\Theta_{em}+\Theta_{ex}.$$

Magnetic order sets in then in the superconductor at $T_M \approx \Theta$, with a wave vector $\mathbf{Q}$ determined from the condition that (1) be a minimum with respect to q.

We have found the values of the exchange parameters at which the interaction influences significantly the magnitude of the wave vector $\mathbf{Q}$. We turn first to the case of pure superconductors with $l \gg \xi_0$. Under the condition $\xi_0^2/a\lambda_L \ll 1$ we have

$$\begin{aligned} &Q\sim\left(\frac{\Theta_{ex}}{\Theta_{em}}\frac{1}{a^2\xi_0}\right)^{1/3},\quad \text{if}\quad \frac{\Theta_{ex}}{\Theta_{em}}\gg\left(\frac{a}{\xi_0}\right)^2,\\ &Q\sim(a\lambda_L)^{-1/2},\quad \text{if}\quad \Theta_{ex}/\Theta_{em}\ll(a/\xi_0)^2.\end{aligned} \tag{2}$$

The second line in (2) correspond to the result of Blount and Varma.[6] In the opposite limiting case $\xi_0^2/a\lambda_L \gg 1$ we obtain

$$\begin{aligned} &Q\sim\left(\frac{\Theta_{ex}}{\Theta_{em}}\frac{1}{a^2\xi_0}\right)^{1/3},\quad \text{if}\quad \frac{\Theta_{ex}}{\Theta_{em}}\gg\left(\frac{a^2\xi_0}{\lambda_L^3}\right)^{2/3},\\ &Q\sim(\xi_0 a^2\lambda_L^2)^{-1/5},\quad \text{if}\quad \Theta_{ex}/\Theta_{em}\ll(a^2\xi_0/\lambda_L^3)^{2/5}.\end{aligned} \tag{3}$$

The second line of (3) corresponds to the result of Ferrell *et al.*[8]

It is seen from (2) and (3) that in real systems with $\Theta_{ex} \lesssim \Theta_{em}$, even though the magnetic-transition temperature is determined mainly by the dipole interaction, the wave vector $\mathbf{Q}$ must be obtained with account taken of the exchange interaction. The situation is similar also in the case of a dirty superconductor with $l \ll \xi_0$. Under the condition $l^5/\lambda_L^2 a^2 \xi_0 \ll 1$ we get

$$\begin{aligned} &Q\sim\left(\frac{\Theta_{ex}}{\Theta_{em}}\frac{1}{a^2\xi_0}\right)^{1/3},\quad \text{if}\quad \frac{\Theta_{ex}}{\Theta_{em}}\gg\frac{a^2\xi_0}{l^3},\\ &Q\sim\left(\frac{\Theta_{ex}}{\Theta_{em}}\frac{1}{a^2\xi_0 l}\right)^{1/4},\quad \text{if}\quad \frac{a^2\xi_0}{l}\gg\frac{\Theta_{ex}}{\Theta_{em}}\gg\frac{l^2}{\lambda_L^2},\frac{a^2 l}{\xi_0^3},\\ &Q\sim\left(\frac{l}{\xi_0 a^2\lambda_L^2}\right)^{1/4},\quad \text{if}\quad \frac{a^2 l}{\xi_0^3}\ \text{or}\ \frac{l^2}{\lambda_L^2}\gg\frac{\Theta_{ex}}{\Theta_{em}}.\end{aligned} \tag{4}$$

The case $l^5/\lambda_L^2 a^2 \xi_0 \gg 1$ corresponds to the situation with $\xi_0^2/a\lambda_L \gg 1$ for pure superconductors.

In conclusion, we can deduce that even a weak indirect exchange interaction of the localized spins in a superconductor exerts a substantial influence on the magnitude of the wave vector of the magnetic structure, and by the same token on the difference between the energy of the helicoidal superconducting phase and the energy of the normal ferromagnetic phase. The results of Refs. 6, 7, and 9, in which the indirect exchange interaction was not taken into account, can hardly be applied to real compounds. Thus, for $ErRh_4B_4$, the parameter $\Theta_{em} \approx 2K$ ($\mu \approx 5\ \mu_B$ and $n = 10^{22}$ cm^{-3}). The parameter Θ_{ex} can be estimated with the aid of the Abrikosov-Gor'kov theory for magnetic scattering from the data for $T_{c1}(x)$ of the alloy $Er_xY_{1-x}Rh_4B_4$ (Ref. 10) (the Y ion is nonmagnetic). This estimate yields $\Theta_{ex} \approx 1$ K. Thus, for $ErRh_4B_4$ the allowance for the magnetic dipole interaction reduces to a renormalization of Θ_0 and γ, and the helicoidal structure is determined mainly by the exchange interaction.

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Noise-induced phase transition and the percolation problem for fluctuating media with diffusion

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The diffusion problem is considered for a medium in which processes of disintegration and of reproduction of the diffusing substance are possible. The critical intensities of the fluctuations in the disintegration and reproduction rates, which lead to the occurrence of noise-induced explosive instability, are determined. The effects of suppression of such instability by nonlinear limitation mechanisms are analyzed. Fluctuation phenomena near the noise-induced phase-transition point are investigated.

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The action of external noise can lead to a qualitative reorganization of the behavior of a dynamical system.[1)] In those cases in which such a reorganization is observed in a distributed system and is accompanied by establishment of a new stationary mode, it is customary to speak of a noise-induced phase transition in such a system. The study of noise-induced transitions is one of the problems of the general theory of fluctuating and disordered media, in the construction of which there are today used a whole series of traditional methods of the of the theory of equilibrium phase transitions.

In the present paper, we consider the problem of diffusion through a fluctuating medium, in which processes of disintegration and reproduction of the diffusing substance are possible. In the paper, the threshold of noise-induced explosive instability in such a system is determined, and a possible mechanism for its suppression is considered. Fluctuation phenomena near the noise-induced phase-transition point are analyzed, and the spatially inhomogeneous problem is investigated.

§1. FORMULATION OF THE MODEL

Let $K_1(\mathbf{r},t)$ be the rate of disintegration of the diffusing substance at the instant t at the point $\mathbf{r}$ of the medium, and let $K_2(\mathbf{r},t)$ be its reproduction rate at this point, so that the diffusion equation has the form

$$\dot{n}=-(K_1-K_2)n+D\Delta n. \tag{1}$$

In general, the values of K_1 and K_2 depend on the local concentration $n(\mathbf{r},t)$ of the diffusing substance. Limiting ourselves to consideration of small concentrations, we shall suppose that this dependence is linear:

$$K_1-K_2=k_1-k_2+\beta n, \tag{2}$$

where the coefficient β is positive; that is, increase of the concentration n suppresses reproduction or increases the rate of disintegration of the diffusing component.

In a closed system, reproduction cannot continue without limit and must end upon exhaustion of the substrate required for this. We shall suppose, however, that the medium is open and that the mean rate of reproduction is maintained constant by influx of substrate from external sources.

We assume that the reaction rates k_1 and k_2 fluctuate in a prescribed manner in space and in time (fluctuations in the coefficient β may be disregarded, since the concentration n is small). Then we may distinguish in them the regular and the fluctuating components,

$$k_{1,2}(\mathbf{r},t)=\bar{k}_{1,2}+\delta k_{1,2}(\mathbf{r},t),\qquad \bar{k}_{1,2}\equiv\langle k_{1,2}(\mathbf{r},t)\rangle \tag{3}$$

and may finally rewrite the diffusion equation (1) in the form

$$\dot{n}=-\alpha n-\beta n^2+D\Delta n+f(\mathbf{r},t)n, \tag{4}$$

$$\alpha=\bar{k}_1-\bar{k}_2,\qquad f(\mathbf{r},t)=\delta k_2(\mathbf{r},t)-\delta k_1(\mathbf{r},t). \tag{5}$$

We shall hereafter assume that the coefficient α is positive; this corresponds to a situation in which the mean rate of disintegration exceeds the mean rate of reproduction.

The random field $f(\mathbf{r},t)$ is external with respect to the problem under consideration, since it does not depend on the concentration distribution $n(\mathbf{r},t)$. The mean value of this field is zero by definition, and we shall suppose that its pair correlators fall off exponentially in space and in time:

$$\langle f(\mathbf{r},t)f(\mathbf{r}',t')\rangle=S\exp\{-k_0|\mathbf{r}-\mathbf{r}'|-\gamma|t-t'|\}. \tag{6}$$

We shall also assume that the random field $f(\mathbf{r},t)$ is Gaussian.

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