Metastable states and single-particle pinning in type-II superconductors near the critical field H_{c2}

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Near the critical field H_{c2} there exist regions of magnetic fields in which a single defect leads to formation of metastable states, and by the same token to single-particle pinning. As H approaches H_{c2}, an alternation of regions of single-particle and collective pinning takes place and leads to an oscillatory dependence of the critical current on the magnetic field.

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INTRODUCTION

It is well known that flow of volume current through an ideal type-II superconductor gives rise to motion of the vortex lattice, accompanied by energy dissipation. In this sense, an ideal type II superconductor does not differ from a normal metal. The crystal-lattice defects, however, lead to a dependence of the free energy on the position of the vortex lattice relative to the defects. As a result, a superfluid current of finite density, not accompanied by energy dissipation, can flow through the superconductor.

Depending on the parameters that characterize the defects and the superconducting properties of the material, two different types of metastable states can set in and lead to the onset of collective or single-particle pinning. In collective pinning, the vortex lattice is weakly deformed and can be described with the aid of the elastic model.[1-4] On going over to single-particle pinning, however, this model goes outside the region of its validity, a strong relative displacement of the vortices takes place and leads to formation of metastable states. From the elastic model it is possible to determine the parameter range in which realization of single-particle pinning is possible. The final answer can be obtained, however, only by considering the region of short distances (on the order of the lattice period), in which the elastic model is not valid.[1]

We consider below the region of fields close to H_{c2} and show that in the case of strongly elongated defects metastable states are produced on a single defect, and the case of single-particle pinning is realized by the same token. If the individual pinning force is large enough, then the pinning center is capable of capturing a single vortex, and the result is plastic flow of the vortex lattice. As a result of this process, the effective force of pinning of the vortex lattice on one defect saturates, and the dependence of the critical current density on the proximity of the magnetic field to the critical field H_{c2} becomes quadratic.

1. VORTEX LATTICE IN THE PRESENCE OF A SMALL-RADIUS CYLINDRICAL PORE

We confine ourselves below to a defect in the form of a small-radius cylindrical pore with axis directed along the magnetic field. The generalization to the case of a spherical defect will be made later on. We confine ourselves also to temperatures close to critical. The last limitation is not principal, and the generalization to the case of arbitrary temperatures is trivial.

The expression for the free energy near the transition temperature can be represented in the form

$$F=\nu\int d^3\mathbf{r}\left\{-\tau|\Delta|^2+\frac{7\zeta(3)}{16\pi^2T^2}|\Delta|^4+\frac{\pi D}{8T}|\partial_-\Delta|^2\right\}+\frac{1}{8\pi}\int d^3\mathbf{r}\,(\mathbf{H}^2-2\mathbf{H}_0\mathbf{H}), \tag{1}$$

where $\nu=mp_0/2\pi^2$ is the state density on the Fermi surface, $D=vl_{tr}/3$ is the diffusion coefficient, ζ is the Riemann zeta function, $\partial_-=\partial/\partial\mathbf{r}-2ie\mathbf{A}$, $\mathbf{A}$ is the vector potential, and H_0 is the external magnetic field. We use a system of units in which $\hbar=c=1$.

We consider a pore with a small radius R:

$$a^2=eH_{c2}R^2/2\ll 1. \tag{2}$$

When condition (2) is satisfied, the eigenvalues of only two eigenfunctions of the operator ∂_-^2 are shifted by a value proportional to a^2. For a solution in the form

$$\rho\,\exp(i\varphi-\rho^2/2), \tag{3}$$

where $\rho=r(eH)^{1/2}$ is the dimensionless length in a plane perpendicular to the magnetic field, φ is the azimuthal angle in this plane, and the magnetic field $H_{c2}^{(1)}$ at which a nucleus of the form type appears increases and becomes equal to

$$H_{c2}^{(1)}=H_{c2}(1+4a^2),\quad eH_{c2}=4T\tau/\pi D. \tag{4}$$

For a nucleus of the type

$$e^{-\rho^2/2} \tag{5}$$

the corresponding critical field H_{c2}^{0} is decreased:

$$H_{c2}^{(0)}=H_{c2}(1-2a^2). \tag{6}$$

The corrections to the remaining eigenvalues of the operator ∂_-^2 are small.

When account is taken of Eqs. (4) and (6), the expression (1) for the free energy near the critical field H_{c2} can be represented in the form

$$F=\nu\int d^3\mathbf{r}\left\{\frac{7\zeta(3)}{16\pi^2T^2}|\Delta|^4\left(1-\frac{1}{2\varkappa^2}\right)-\frac{e\pi D}{4T}(H_{c2}-H_0)|\Delta|^2 -4a^2\tau\left(|\Delta_1|^2-\frac{1}{2}|\Delta_0|^2\right)\right\}. \tag{7}$$

The order parameter Δ is the sum over all the eigen-

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functions of the operator ∂_-^2; Δ_1 and Δ_2 are proportional to the eigenfunctions (3) and (5), while $\varkappa^2=63\zeta(3)/2\pi^3e^2p^2v^3\tau_{tr}^2$.

We choose a special gauge for the vector potential $\mathbf{A}_0$:

$$\mathbf{A}_0=H_0(0,\ x,\ 0). \tag{8}$$

In this gauge, the order parameter Δ can be represented in the form

$$\Delta=\tilde{\Delta}_0+\sum_{m=0}^{\infty}C_m\rho^m\exp\left[ixy+im\varphi-\frac{\rho^2}{2}\right],$$

$$\tilde{\Delta}_0=C\sum_{n=-\infty}^{\infty}\exp\left\{\frac{i\pi}{2}n^2+i\pi n\beta+iy\left(\frac{8\pi}{3^{1/2}}\right)^{1/2}\left(\frac{n}{2}+\alpha-\frac{1}{2}\right)-\left[x-\left(\frac{2\pi}{3^{1/2}}\right)^{1/2}\left(\frac{n}{2}+\alpha-\frac{1}{2}\right)\right]^2\right\}, \tag{9}$$

where

$$|C|^2=\frac{1-H_0/H_{c2}}{1-1/2\varkappa^2}\frac{8\pi^2T^2\tau}{7\zeta(3)\cdot3^{1/4}\beta_A},$$

$$\beta_A=\sum_{N,M=-\infty}^{\infty}\exp\{-2\pi(N^2+M^2-NM)/3^{1/2}\}=1.1596. \tag{10}$$

In (9), Δ_0 is the unperturbed solution corresponding to an ideal triangular vortex lattice. The parameters α and β specify the position of the vortex lattice relative to the pore. The equations for the coefficients C_m are obtained from the condition that the free energy (7) be an extremum.

We proceed now to consider the case of fields close enough to H_{c2} to satisfy the condition

$$a^2\gg1-H_0/H_{c2}. \tag{11}$$

When this condition is satisfied, the coefficient C_1 is large: $|C_1|\gg|C|$. The pore captures one vortex, and two regimes are possible: smooth flow of the remaining vortices around the pore with the captured vortex, and a regime in which a metastable state takes place in the flow. It will be shown below that both cases can be realized when the quantity $1-H_0/H_{c2}$ is varied.

The correction to the free energy and the equations for the coefficients C_m contain the quantities

$$M(m)=\int dx\,dy\,\tilde{\Delta}_0(x-iy)^m\exp[-ixy-(x^2+y^2)/2]. \tag{12}$$

In the principal approximation in terms of the parameter $|C/C_1|^2$, the correction to the free energy per unit length does not depend on the position of the vortex lattice relative to the pore, and is equal to

$$\frac{\delta F}{\nu/eH}=\frac{\delta F^{(0)}}{\nu/eH}=-\frac{16\pi(a^2\tau)^2}{7\zeta(3)}\frac{16\pi^2T^2}{1-1/2\varkappa^2}\left(1+\frac{1-H_0/H_{c2}}{2a^2}\right). \tag{13}$$

In the same approximation, the phase of the coefficient C_1 is arbitrary, and the modulus is given by the expression

$$|C_1|^2=\frac{8a^2\tau}{7\zeta(3)}\frac{16\pi^2T^2}{1-1/2\varkappa^2}\left(1+\frac{1-H_0/H_{c2}}{4a^2}\right). \tag{14}$$

To calculate the next order, it is necessary to find the coefficients C_m at $m\neq1$. In the principal approximation in $|C/C_1|^2$, this system of equations takes the form

$$\left\{\frac{m+1}{2^m}|C_1|^2-\frac{16\pi^2T^2\tau(1-H_0/H_{c2})}{7\zeta(3)(1-1/2\varkappa^2)}\right\}M(m)(1-D_m)+\frac{1}{\pi^2}\sum_{m_1,m_2,m_3\neq1}\delta(m_1+m_3,\ m+m_2)\frac{\Gamma(m_1+m_3+1)}{2^{m_1+m_3}}\cdot\frac{M(m_1)M^*(m_2)M(m_3)}{\Gamma(m_1+1)\Gamma(m_2+1)\Gamma(m_3+1)}(1-D_{m_1})(1-D_{m_2}^*)(1-D_{m_3})=0, \tag{15}$$

where $\Gamma(m)$ is the gamma function. The sum in (15) is over all the indices that are not equal to unity. We have replaced the coefficients Cm by new variables D_m, the relation between them being

$$C_m=-M(m)D_m/\pi\Gamma(m+1),\quad m\neq1. \tag{16}$$

The correction to the free energy in the first approximation in $|C/C_1|^2$ is of the form

$$\frac{\delta F}{\nu/eH}=\frac{\delta F^{(0)}}{\nu/eH}+\frac{2a^2\tau}{\pi}\sum_{m\neq1}\frac{(m+1)|M(m)|^2}{2^m\Gamma(m+1)}\times[2(1-D_m)(1-D_m^*)+(1-D_m)+(1-D_m^*)]. \tag{17}$$

We proceed now to the solution of the system (15).

2. CALCULATION OF THE COEFFICIENTS D_m

We define m_0 as the value of m at which the expression in the curly brackets of (15) vanishes. Using expression (14) for the coefficient $|C_1|^2$, we obtain

$$\frac{2^{m_0}}{m_0+1}=\frac{8a^2}{1-H_0/H_{c2}}. \tag{18}$$

In our approximation, $m_0\gg1$.

We determine first the behavior of the coefficient $M(m)$ at $m\gg1$. It is convenient for this purpose to represent the order parameter Δ_0 in the form

$$\tilde{\Delta}_0=C_{00}e^{ixy}\sum_{N,M}\exp\left\{-\frac{1}{2}\left[\left(x-\left(\frac{2\pi}{3^{1/2}}\right)^{1/2}\left(N+\frac{M}{2}+\alpha\right)\right)^2+\left(y-\left(\frac{2\pi}{3^{1/2}}\right)^{1/2}\frac{3^{1/2}}{2}(M-\beta)\right)^2\right]+i\left(\frac{2\pi}{3^{1/2}}\right)^{1/2}\left[y\left(N+\frac{M}{2}+\alpha\right)-\frac{3^{1/2}}{2}x(M-\beta)\right]+i\pi NM+i\pi(N\beta+M(\alpha+\beta/2))\right\}, \tag{19}$$

where the constant C_{00} is connected with C by the relation

$$|C_{00}|^2=|C|^2\cdot3^{1/4}/\beta_R^2,$$

$$\beta_R^2=\sum_{N,M}\exp\{-i\pi NM-\pi(N^2+M^2+NM)/3^{1/2}\}=1.339. \tag{20}$$

At large values of the parameter m, the quantity $M(m,\alpha,\beta)$ has sharp maxima that lie on straight lines in the (α,β) plane. The position of each such line is given by the angle φ_0. These maxima arise when the vortex lines are tangent to a circle of radius $m^{1/2}$.

Substituting the expression (19) for the parameter Δ_0 in formula (12), we obtain

$$M(m)=\exp\left(\frac{m}{2}\ln m-\frac{m}{2}\right)\tilde{P}(\alpha,\beta,m),\qquad \tilde{P}=\sum_{\varphi_0}P(\alpha,\beta,m,\varphi_0), \tag{21}$$

where

$$P(\alpha,\beta,m,\varphi_0)=\pi C_{00}\sum_{N,M}\exp\left\{-im\varphi_0-\frac{2\pi i}{3^{1/2}}R_0A_2+\frac{2\pi i}{3^{1/2}}\frac{A_2^3}{R_0}-\frac{2\pi}{3^{1/2}}(A_1-R_0)^2+\frac{2\pi i}{3^{1/2}}A_2(A_1-R_0)+i\pi NM+i\pi\left(N\beta+M\left(\alpha+\frac{\beta}{2}\right)\right)\right\}. \tag{22}$$

In formula (21), the sum over φ_0 is taken over all the external angles in formula (21), the sum over φ_0 is the sum over all the extremal angles

$$R_0=\left(\frac{3^{1/2}m}{2\pi}\right)^{1/2},\quad A_1=\left(N+\frac{M}{2}+\alpha\right)\cos\varphi_0+\frac{3^{1/2}}{2}(M-\beta)\sin\varphi_0,$$
$$A_2=\frac{3^{1/2}}{2}(M-\beta)\cos\varphi_0-\left(N+\frac{M}{2}+\alpha\right)\sin\varphi_0. \tag{23}$$

In the solution chosen to us, the principal extrema are located at the angles

$$\varphi_0=\{\pi n/3;\ \pi/6+\pi n/3\},\quad n=0,\ldots,5. \tag{24}$$

It will be shown below that the contribution of the "lateral" extrema to the critical current is small, and they will be disregarded.

From (21) and (23) we obtain for $\varphi_0=0$

$$P(\alpha,\beta,m,0)=\pi C_{00}\cdot 2^{1/2}\cdot 3^{-1/4}\beta_2 m^{1/4}e^{i\pi\alpha\beta}$$
$$\times\sum_K\sum_{\gamma=0,1}\exp\left[2\pi i\left(K+\frac{\gamma}{2}\right)\beta+\frac{i\pi\gamma^2}{2}\right]\Phi\left[-\left(\frac{8\pi}{3^{1/2}}\right)^{1/2}m^{1/4}\left(R_0-\alpha-K-\frac{\gamma}{2}\right)\right], \tag{25}$$

where K are integers, γ takes on the two values 0 and 1, and Φ is the Airy function[5]:

$$\Phi(z)=\frac{1}{\pi^{1/2}}\int_0^\infty dt\cos\left(\frac{t^3}{3}+tz\right). \tag{26}$$

β_2 is given by

$$\beta_2=\sum_{M=-\infty}^{\infty}\exp\left(\frac{i\pi M^2}{2}-\frac{\pi M^2}{2\cdot 3^{1/2}}\right),\quad |\beta_2|=1.3275. \tag{27}$$

The quantities β_2 and β_R are related by

$$|\beta_2|=3^{1/4}\beta_R. \tag{28}$$

We obtain similarly for $\varphi_0=\pi/2$

$$P\left(\alpha,\beta,m,\frac{\pi}{2}\right)=\pi C_{00}\cdot 2^{1/2}\cdot 3^{1/4}m^{1/4}\beta_3\exp\left(-\frac{i\pi m}{2}-i\pi\alpha\beta+2i\pi K\alpha\right)$$
$$\times\sum_K\exp\left(-\frac{i\pi K^2}{2}\right)\Phi\left[-\left(\frac{8\pi}{3^{1/2}}\right)^{1/2}m^{1/4}\left(R_0+\frac{3^{1/2}}{2}(\beta-K)\right)\right], \tag{29}$$

where

$$\beta_3=\sum_{-\infty}^{\infty}\exp\left(-\frac{i\pi M^2}{2}-\frac{\pi\cdot 3^{1/2}M^2}{2}\right),\quad |\beta_3|=1.0087. \tag{30}$$

Substituting the expression (21) for the coefficient $M(m)$ in (15), we reduce the system of equations for D_m to the form

$$\frac{m_0+1}{2^{m_0}}|C_1|^2\left(\frac{m+1}{m_0+1}2^{m_0-m}-1\right)\bar P(m)(1-D_m)+\frac{1}{\pi^3\cdot 2^{1/2}m}$$
$$\times\sum_{n_1,n_2=-\infty}^{\infty}\exp\left(-\frac{n_1^2+n_2^2}{4m}\right)\bar P(m+n_1)\bar P(m+n_2)\bar P^*(m+n_1+n_2)$$
$$\times(1-D_{m+n_1})(1-D_{m+n_2})(1-D^*_{m+n_1+n_2})=0. \tag{32}$$

This leads to two relations:

$$1=\frac{3^{1/4}}{6^{1/2}\beta_A}[S_1^2(0)+2S_1(0)S_1(1)-S_1^2(1)],$$
$$1=\frac{3^{1/4}|\beta_3|^2}{2^{1/2}\beta_A\beta_R^2}[S_2^2(0)+2S_2(0)S_2(1)-S_2^2(1)], \tag{32}$$

where

$$S_1(y)=\sum_{-\infty}^{\infty}\exp\left(-\frac{2\pi}{3^{1/2}}\left(N+\frac{y}{2}\right)^2\right),\ S_2(y)=\sum_{-\infty}^{\infty}\exp\left(-2\pi\cdot 3^{1/2}\left(N+\frac{y}{2}\right)^2\right).$$

From the form of the system (31) we can draw two conclusions: at $m<m_0$ the quantity $1-D_m$ tends exponentially to zero; at $m>m_0$ the quantity $1-D_m$ breaks up into two terms, one a smooth function that varies over distances of the order of $m_0^{1/2}$, and the other decreasing exponentially with increasing m. These arguments allow us to separate two stages in the solution of the system (31). We obtain first, at $m>m_0$, the smooth part of $1-D_m$, following which we obtain the exponentially decreasing term.

The smooth term satisfies the equation

$$\bar P(m)(1-D_m)=\frac{1}{2^{1/2}\cdot 3^{1/4}\pi^3\beta_A m|C|^2}\sum_{n_1,n_2=-\infty}^{\infty}\exp\left(-\frac{n_1^2+n_2^2}{4m}\right)$$
$$\times\bar P(m+n_1)\bar P(m+n_2)\bar P^*(m+n_1+n_2)(1-D_{m+n_1})(1-D_{m+n_2})$$
$$\times(1-D^*_{m+n_1+n_2})\theta(m+n_1-m_0)\theta(m+n_2-m_0)\theta(m+n_1+n_2-m_0). \tag{33}$$

It follows from (25) and (29) that at fixed values of the parameters α and β that characterize the position of the vortex lattice relative to the pore, the quantity $\bar P$ as a function of m has steep maxima of width $m^{1/3}$, separated by distances on the order of $m^{1/2}$. When the parameters α and β are varied, these maxima are shifted and at definite values of the parameters pass through $m=m_0$. It is near these points that the metastable states that lead to the single-particle pinning that are formed.

The absence of the fast factor $\exp(-im\varphi_0)$ leads to breakup of the system (33) into independent equations, in each of which $\tilde P$ must be replaced by $P(\varphi)$.

As already mentioned above, metastable states are produced if one of the maxima of the function $\tilde P$ lies near the point m_0. The equation systems obtained in this case for different values of φ_0 turn out to be similar. We consider therefore in detail, for the sake of argument, the case $\varphi_0=0$. For $\varphi_0=\pi/2$ we present only the final results. We put

$$\left(\frac{3^{1/2}m_0}{2\pi}\right)^{1/2}-\alpha=K_0+\delta_0,\quad z=\left(\frac{8\pi}{3^{1/2}}\right)^{1/2}m_0^{1/4}\delta_0,\quad |\delta_0|\ll 1, \tag{34}$$

where K_0 is an integer. Using the slow variation of $1-D_m$ in the region where the function $P(m,0)$ changes, we can integrate with respect to m the system of equations (33), in which $\tilde P$ is replaced by $P(m,0)$. The system (33) then changes over into a system for the quantities $1-D_m$ at the extremal points of the function $P(m,0)$.

At $\gamma=0$, the extrema are located at the points

$$m(N)=m_0+(8\pi/3^{1/2})^{1/2}m_0^{1/2}N,\quad N=0,1,2\ldots. \tag{35}$$

We designate the quantity $1-D_m$ at these points by

$$R(N)=1-D_{m(N)}. \tag{36}$$

The extrema at $\gamma=1$ are located at the points

$$m(N)=m_0+(8\pi/3^{1/2})^{1/2}m_0^{1/2}(N+1/2),\quad N=0,1,2\ldots.$$

At these points we put for $1-D_m$

$$Y(N)=1-D_{m(N)}. \tag{37}$$

The system of equations for the quantities $R(N)$ and $Y(N)$ is unwieldy, and we present only the one equation that is of greatest importance for what follows:

$$0=-\gamma_1(z)R(0)+\frac{3^{1/4}}{6^{1/2}\beta_A}\Big\{R(0)|R(0)|^2(\gamma_3(z)-\gamma_1(z))$$

$$+\gamma_1(z)\sum_{N_1,N_2=0}R(N_1)R(N_2)R^*(N_1+N_2)\exp\left(-\frac{2\pi}{3^{1/2}}(N_1^2+N_2^2)\right)$$

$$-\gamma_1(z)\sum_{N_1,N_2=0}^{\infty}Y(N_1)Y(N_2)R^*(N_1+N_2+1)\exp\left[-\frac{2\pi}{3^{1/2}}\left(\left(N_1+\frac{1}{2}\right)^2\right.\right.$$

$$\left.\left.+\left(N_2+\frac{1}{2}\right)^2\right)\right]+2\gamma_1(z)\sum_{N_1,N_2=0}^{\infty}R(N_1)Y(N_2)Y^*(N_1+N_2)$$

$$\times\exp\left[-\frac{2\pi}{3^{1/2}}\left(N_1^2+\left(N_2+\frac{1}{2}\right)^2\right)\right]\Big\}, \tag{38}$$

where

$$\gamma_1(z)=\frac{1}{\pi^{1/2}}\int_0^{\infty}dt\,\Phi(-z-t),$$

$$\gamma_3(z)=\frac{1}{\pi^{1/2}}\int_{-\infty}^{\infty}dx_1\,dx_2\,dx_3\,\Phi(-z-x_1)\Phi(-z-x_2)\Phi(-z-(x_1+x_2-x_3))$$
$$\times\theta(x_1)\theta(x_2)\theta(x_1+x_2-x_3). \tag{39}$$

We note that to calculate the free energy we shall need $R(0)$ as a function of the parameter z.

The system of equations for $R(N)$ and $Y(N)$ was calculated with a computer. The table lists the values of $R(0, z)$ as functions of the parameter z. At the point

$$z=z_0=1.384 \tag{40}$$

the function $\gamma_1(z_0)=0$. In the vicinity of this point

$$R(0,z)\equiv R(z)=-0.575(z-z_0)^{1/3}. \tag{47}$$

We linearize the system of equations for the functions $R(N)$ and $Y(N)$ near the point z_0. It turns out that the matrix of the linear increments B of this system has one eigenvalue λ_1 that vanishes at the point z_0:

$$\lambda_1=0.153(z-z_0)^{2/3}\ . \tag{42}$$

Therefore small perturbations of the system (31), (33) at this point lead to substantial changes. Only the coefficient $R(0)$ changes greatly in this case. We put

$$R(0)=x. \tag{43}$$

Substituting in the right-hand side of (38) the values of $R(N\neq 0)$ and $Y(N)$ taken at the point $z=z_0$, we obtain a third-degree form of $G(x)$:

$$G(x)=0.154x^3+0.029(z-z_0). \tag{44}$$

In the zeroth approximation, the value of $R(0, z)=x$ is determined from the equation

$$G(x)=0. \tag{45}$$

In the next approximation in m_0, Eq. (45) acquires a new term which we now proceed to calculate.

Let K be the integer closest to m_0:

$$|K-m_0|\leqslant 1/2. \tag{46}$$

From (33) we obtain

$$1-D_m=x/(1-2^{m_0-m}),\quad m\neq K,\quad |m-m_0|\ll m_0^{1/3}. \tag{47}$$

If m_0 tends to an integer, then $1-D_K$ increases, and the nonlinear term must therefore be retained in its calculation. From (33) we obtain

$$(1-D_K)\left\{(2^{m_0-K}-1)+\frac{0,0813}{m_0^{1/6}}+\frac{0,1137}{m_0^{2/3}}|1-D_K|^2\right\}+x=0. \tag{48}$$

Substituting expression (47) for $1-D_m$ in (33), and calculating the sum over m, we obtain, taking (43) and (44) into account,

$$G(x)+\frac{0.1803}{m_0^{1/3}}\left\{(1-D_K)+x\left(m_0-K-\frac{1}{2}\right)+xS(m_0)\right\}=0, \tag{49}$$

where

$$S(m_0)=\sum_{m\neq K}[1/(1-2^{m_0-m})-\theta(m-m_0)]. \tag{50}$$

The function $S(m_0)$ is an odd periodic function with period 1. Its numerical values are given in the table.

If the quantity $R_0-\alpha$ is close to a half-integer

$$\left(\frac{3^{1/2}m_0}{2\pi}\right)^{1/2}-\alpha=K_0+\frac{1}{2}+\delta_0,\quad |\delta_0|\ll 1, \tag{51}$$

then, accurate to the permutation $R\rightleftarrows Y$, the system (33) goes over into the system (38).

It follows from (34) and (51) that the period of the vortex lattice spans 12 lines connected with the angles $\varphi_0=\pi n/3$, $n=0, 1, \ldots, 5$, near which metastable states can be formed. A metastable state is formed if

$$S(m_0)+(m_0-K-1/2)+(1-D_K)/x<0. \tag{52}$$

In the opposite case, no metastable states are formed. Inasmuch as the sign of the expression in (52) is periodically reversed when the magnetic field is varied, an interesting pattern of alternation of the single-particle and collective pinnings is produced. Since the critical current increases sharply on going to the single-particle pinning, an oscillatory dependence of the critical current on the magnetic field should be observed in experiment.

If m_0 is not too close to an integer, then $1-D_K\sim x$ and its value is

$$1-D_K=x/(1-2^{m_0-K}). \tag{53}$$

Substituting this value of $1-D_K$ in (49), we get

$$G(x)+Ax=0, \tag{54}$$

where

$$A=\frac{0.1803}{m_0^{1/3}}\left\{S(m_0)+m_0-K-\frac{1}{2}+\frac{1}{1-2^{m_0-K}}\right\}. \tag{55}$$

As already noted, for strong magnetic fields, where $A<0$, there exists a range of parameters $z-z_0$ in which (49) has three physical solutions. The function

TABLE I.

z	$R(0, z)$	z	$R(0, z)$	$S(x)$	x
$-\infty$	1,316	1.1	0,717	0	0
−2	1,33	1,2	0,567	0,191	0.05
−1,6	1,346	1,25	0,482	0,384	0,1
−1,2	1,373	1,3	0.386	0,581	0,15
−0,8	1.413	1,35	0.258	0,787	0,2
−0,4	1.46	1.38	0.11	1.003	0,25
0	1.482	1,4	−0,103	1,233	0,3
0.4	1,401	1,5	−0.149	1,483	0.35
0,6	1,285	1,7	−0.153	1,757	0,4
0.8	1,096	2	−0,157	2,064	0,45
1	0.854	$+\infty$	0	2,414	0,5

$$x = x(z - z_0) \tag{56}$$

has in this case a characteristic S-shape. The branch between the turning points is absolutely unstable. The two other solutions are stable to small perturbations. From (54) and (55) we obtain the jump of δx on going over to the point of absolute instability from one branch to the other

$$\delta x = 4.41(-A)^{1/2}. \tag{57}$$

We note also that at $A < 0$ at the point $z = z_0$ the one singular solution (41) splits into three:

$$x = 0, \quad x = \pm 2.545(-A)^{1/2}. \tag{58}$$

A similar picture appears for the directions $\varphi_0 = \pi/6 + \pi n/3$ with $n = 0, 1, \ldots, 5$. We present the analogs of (44) and (49) for $\varphi_0 = \pi/2$.

$$G(x) = 0.267x^3 + 0.002(z - z_0), \tag{59}$$

$$G(x) + \frac{0.0289}{m_0^{1/3}}\left[(1 - D_K) + xS(m_0) + x\left(m_0 - K - \frac{1}{2}\right)\right] = 0;$$

$$z = m_0^{1/6}\left(\frac{8\pi}{3^{1/2}}\right)^{1/2}\left[\left(\frac{3^{1/2}m_0}{2\pi}\right)^{1/2} - \frac{3^{1/2}}{2}(K_0 - \beta)\right], \tag{60}$$

where K_0 is an integer.

For $\varphi_0 = \pi/2$ formula (48) takes the form

$$(1 - D_K)\left\{(2^{m_0 - K} - 1) + \frac{0.197}{m_0^{2/3}}|1 - D_K|^2 + \frac{0.013}{m_0^{1/6}}\right\} + x = 0. \tag{61}$$

Just as before, if m_0 is not too close to an integer, the quantity $1 - D_K$ is determined by formula (53) and in this case the jump at the point of absolute instability is

$$\delta x = 1.342(-A)^{1/2}, \tag{62}$$

where A is determined as before by (55).

The period of the vortex lattice spans six lines connected with the angles $\varphi_0 = \pi/6 + \pi n/3$, near which metastable states are produced.

We turn now to the calculation of the free energy. From (17), (21), (25), and (29) we obtain

$$\delta F = \delta F^{(0)} + \delta F^{(1)}, \tag{63}$$

where

$$\frac{\delta F^{(1)}}{\nu/eH} = \frac{(\pi T\tau)^2(1 - H_0/H_{c2})^2}{m_0^{1/6}(1 - 1/2\varkappa^2)} 0.781 \sum_{\varphi_0} \mathscr{F}(\varphi_0)\Big\{\Phi^2(-z) \times\left[R(0, z)\left(S(m_0) + \left(m_0 - K + \frac{1}{2}\right)\right) + \frac{1 - D_K}{2^{K - m_0}} + \frac{(1 - D_K)^2}{2^{K - m_0}} + R^2(0, z)\sum_{m \neq K}\frac{1}{2^{m - m_0}(1 - 2^{m_0 - m})^2}\right] - R(0, z)\, m_0^{1/3}\int_0^\infty dt\, \Phi^2(-z + t)\Big\}_{\varphi_0}. \tag{64}$$

In formula (64), $z = z(\varphi_0)$ and is determined by formulas (34) and (60) for the angles $\varphi_0 = 0$ and $\pi/2$. For other angles from the same families, the values of $z(\varphi_0)$ are obtained by simple rotation of the coordinates (α, β). $F(\varphi_0)$ is given by

$$\mathscr{F}(\pi n/3) = 1, \quad \mathscr{F}(\pi/6 + \pi n/3) = 3|\beta_3/\beta_2|^2 = 1.732, \quad n = 0, 1, \ldots, 5. \tag{65}$$

In the vicinity of the point $z = z_0$ we have

$$R(0, z) = x \tag{66}$$

and was determined by us above

$$\Phi(-z_0) = 0.878, \quad \int_0^\infty dt\, \Phi^2(-z_0 + t) = 1.232. \tag{67}$$

From (57), (62), and (64) we obtain the jump $\delta F^{(1)}$ of the free energy on going from one branch to another:

$$\delta F^{(1)} = \sum_{\varphi_0} \delta F^{(1)}(\varphi_0), \tag{68}$$

$$\frac{\delta F^{(1)}(\varphi_0)}{\nu/eH} = 0.96\frac{(\pi T\tau)^2(1 - H_0/H_{c2})^2}{m_0^{1/6}(1 - 1/2\varkappa^2)} \times\left\{-m_0^{1/3} + 0.63\left(S(m_0) + \left(m_0 - K - \frac{1}{2}\right) + \frac{1}{1 - 2^{m_0 - K}}\right)\right\}\mathscr{F}(\varphi_0)\,\delta x(\varphi_0). \tag{69}$$

Formula (62) is valid if m_0 is not too close to an integer. Otherwise, the jump is determined from formulas (48) (61), and (64).

The density of the critical current can be determined from energy considerations[6]: the density of the average force is equal to the dissipated energy density divided by the displacement. In our case the average force is weakly anisotropic, inasmuch as the number of metastable states produced when the lattice is displaced depends on the direction of motion relative to the unit-cell vectors.

Taking into account the number of metastable states corresponding to the angles φ_0, we obtain for the critical current density j_c the expression

$$j_c B = 4n\langle l\rangle(eH)^{1/2}(3^{1/2}/2\pi)^{1/2}\{\delta F^{(1)}(0)[|\cos\varphi| + |\cos(\varphi + \pi/3)| + |\cos(\varphi - \pi/3)|] + 0{,}5\delta F^{(1)}(\pi/2)[|\cos(\pi/6 - \varphi)| + |\cos(\pi/6 + \varphi)| + |\cos(\pi/2 - \varphi)|]\} \tag{70}$$

where n is the volume density of the defects, $\langle l\rangle$ is the average length of the defect, and δF^1 is the free-energy jump determined by formulas (64), (65), and (69). In the solution (19) chosen by us, the unit-cell vector lies along the x axis. Therefore the angle φ in (70) is the angle between the direction of motion of the vortex lattice and the x axis. For a real sample it is necessary to average (70) with a weight function that determines the distribution of the pore radii.

We note that in the plastic-flow region formula (70) for the critical current is universal in form. All that depends on the pore radius a is a numerical coefficient in (70). For a sufficiently wide distribution of the pore radii (for a distribution width larger than the average radius), the critical current is given by

$$j_c B \propto \tau^{1.5}(1 - H/H_{c2})^2 \tag{71}$$

and does not depend on the average pore radius.

An expression similar to (7) is obtained for the free energy not only in the case of cylindrical pores of small radius, but also for any filamentary small-radius defect that leads to a shift of H_{c2} (for example, a dislocation). Then the coefficient $4a^2$ in (7) is replaced by the magnetic-field shift $\delta H_{c2}/H_{c2}$ due to this effect. By making the corresponding substitution we arrive again at formula (70) for the critical current.

The number of experimental investigations of the dependence of the critical pinning current on the magnetic field and on the temperature is by now quite large. Near H_{c2}, this dependence is well described by the scaling law[7]

$$j_c B \propto \tau^n (1-H/H_{c2})^m.$$

In neutron-bombarded V_3Si samples we have $m=2$, and the value of the parameter n lies between 2.3 and 2.8.8 In NiTi alloys, $m=2$ and the parameter n ranges from 2 to 2.33, depending on the Ti density.[9]

The force of interaction of the vortex lattice with an individual defect is usually proportional to $|\Delta|^2$ (Refs. 1, 6, 10) and is by the same token linearly dependent on the proximity of the magnetic field to the critical value H_{c2}. Therefore, at least near H_{c2}, such strong pinning centers will capture the vortex filaments, and this leads, according to the result of this work, to plastic flow and to effective weakening of the securing of the vortex lattice.

The considered model presents the physical picture of the saturation of the forces of interaction between a vortex lattice and pinning centers, and of the ensuing scaling law for the quantity $j_c B$. In real samples, the pinning centers[8,9] are not lines but have small dimensions in all directions. For a detailed comparison with the experimental data it is therefore necessary to generalize the considered model to include the case of point defects.

CONCLUSION

In a magnetic field close to H_{c2}, a single vortex is captured by a defect. Then, in the case considered by us, a large region from which the vortices are expelled is produced around the defect. The size of this region is $R=m_0^{1/2}/(eH)^{1/2}$. In the principal approximation, the free energy is independent of the lattice position relative to the defect. The terms of next order, however, have a nonmonotonic dependence on the proximity of H to H_{c2}, and at certain lattice positions relative to the defect these terms lead to formation of metastable states. Even near H_{c2}, the metastable states are produced at not all values of the field. In the principal approximation, metastable states are produced if $0<m_0-K<\frac{1}{2}$ but not if $-\frac{1}{2}<m_0-K<0$. Since m_0 is the number of vortices expelled by the captured vortex, this condition means that the formation of a metastable state depends on whether free space exists in the expulsion region or, on the contrary, if there is not enough space. Therefore the dependence of the critical current on the magnetic field is nonmonotonic and is determined essentially by the distribution of the defects in size.

We note also that the formation of metastable states is connected with the behavior of the vortex lattice near the boundary of the region occupied by the single capture vortex.

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