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On the connection between the size of the Universe and its curvature

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Closed three-dimensional Riemannian spaces with curvature that is constant in all directions are considered. It is shown that the topological structure of any such space uniquely determines the sign of its curvature $\mathcal{K}$ and also restrictions on its size. Let R be the radius of curvature and D the diameter of the space, i.e., the distance between its most widely separated points. Then for $\mathcal{K}>0$ one finds $D>0.326R$, and for $\mathcal{K}<0$ apparently $D>1.128|R|$; for $\mathcal{K}=0$, the value of D is arbitrary. Further, Einstein's equations and astronomical data indicate that the modulus of the present-day radius of curvature of the Universe satisfies $|R|>0.5(c/H_0)\approx 0.9\times 10^{28}$ cm, where c is the velocity of light, and H_0 is the Hubble constant. Therefore, if observations show that the diameter of the Universe is $D_0 \ll 10^{28}$ cm, this will mean that as a whole our Universe is flat ($\mathcal{K}=0$). A model of a flat world is proposed which is closed in the form of a three-dimensional torus; all of its parameters (size, rate of expansion, mean matter density, etc.) are expressed in terms of atomic constants and a universal time. In this model, the present-day diameter of the Universe is $D_0=0.102(c/H_0)\approx 2\times 10^{27}$ cm, which does not contradict observational data.

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§1. INTRODUCTION

The aim of the present paper is to establish some connections between local and global properties of the Universe. The problem is analyzed on the basis of Einstein's general theory of relativity in the framework of locally isotropic and homogeneous cosmological models. It is well known that the exceptional isotropy of the cosmic microwave background enables one in conjunction with the "generalized Copernican principle" (i.e., a terrestrial observer is not distinguished) to introduce a universal time and three-dimensional space orthogonal to it, this space having at any time constant curvature in all directions.[1–3] We recall that in accordance with Schur's theorem local isotropy entails local homogeneity, namely, if at every point of a Riemannian manifold the curvature has the same value in all directions, then it also has a constant value as one moves from point to point.

In the construction of cosmological models based on three-dimensional spaces of constant curvature, physicists usually employ three degenerate types of space: Euclidean space E^3, Lobachevskiĭ space L^3, and the sphere S^3.[1–4] But the general classification of three-dimensional spaces of constant curvature gives 18 topologically different types of space with curvature $k=0$ and an infinite number of topological types with $k=-1$ and $k=+1$ (Ref. 5). All three-dimensional spaces with $k=+1$ are closed and orientable; among the flat three-dimensional spaces there are ten types which are closed (six are orientable[1)]) and eight types which are open (four of them orientable); the spaces with $k=-1$ contain an infinite number of closed types and an infinite number of orientable types.

The spaces of the types E^3, L^3, and S^3 are distinguished in this complete set by the fact that they are the only ones that are topologically simply connected. Therefore, it is only in them that each celestial object can be observed only in one direction at a particular stage of expansion of the world.[2)]

The multiply connected spaces of constant curvature can be obtained formally by specifying in E^3, L^3, and S^3 certain nonclosed manifolds (fundamental regions) whose boundaries are identified (or "glued") in accordance with definite laws (see Refs. 2 and 9; the precise formulations can be found in Appendix A). Then each light source can be joined to an observer by several geodesics

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rather than one, i.e., it will be observed simultaneously in several different directions.

The widespread neglect of the analysis of multiply connected spaces as cosmological models is purely traditional. The data of extragalactic astronomy are compatible with the hypothesis[9] that the same quasars, galaxies, and clusters of galaxies exist on the sky in the form of tens of images ("ghosts") which have not yet been identified because of purely observational difficulties. However, it is true that analysis of the observations show that the Universe is not small: the minimal identification parameter of the Universe (the distance to the nearest "ghost") satisfies $l_0 \geq 0.003\ (c/H_0)$, and the maximal identification parameter (the distance to the most distant "original") $L_0 \geq 0.1\ (c/H_0)$ (see Ref. 9). Here, c is the velocity of light, H_0 is the contemporary value of the expansion rate of the Universe (the Hubble constant), and $c/H_0 = 1.85 \times 10^{28}$ cm $\cdot$ $(H_0/50$ km/sec $\cdot$ Mpc$)^{-1}$ is the Hubble radius, which is approximately equal to the distance that light has traversed since the singularity to our epoch.

It must be emphasized that all "glued" spaces except one (the elliptical $El^3 \equiv S^3/\{\pm 1\}$) are globally nonisotropic, since the perpendiculars to the boundaries of the fundamental regions define distinguished directions. This does not contradict observations; for the homogeneity of space in conjunction with the isotropic (Friedmann!) nature of its expansion ensures both complete isotropy of the microwave background and statistical isotropy of the distribution of clusters of galaxies on the sky.[2,9]

However, global anisotropy of the Universe can influence the spectrum of primoridial density perturbations. In "glued" universe, the spectrum of large-scale, $x \sim l$, perturbations must be anisotropic and discrete, and in some directions modes with $x > l$ must be altogether absent. This interesting aspect of the problem was noted by Zel'dovich[10] and analyzed from different points of view by Sokolov and Starobinskiĭ.[11] Unfortunately, it has not yet been possible to improve the bounds on the parameters l and L as compared with Ref. 9.

The limits $l_0 \gtrsim 0.003c/H_0$ and $L_0 \gtrsim 0.1c/H_0$ were obtained in Ref. 9 on the basis of purely astronomical considerations, namely, an analysis of the distribution on the sky of "unique" clusters of galaxies. Below, one of these bounds is improved, and we show that if our Universe is not flat then $L_0 \gtrsim c/H_0$. This is due to two circumstances. On the one hand, mathematical methods have shown (Appendix A) that closed spaces of constant nonzero curvature have diameter D which is of the order of or greater than their radius of curvature $|R|$. (We recall that R is an imaginary number for $k=-1$; D is the distance between the most widely separated points of the space, $D \approx L$.) On the other hand, analysis of observational data shows (Appendix B) that the real Universe is characterized by $|R_0| > 0.5 \times c/H_0$. It follows that if $k \neq 0$, then $L_0 \approx D_0 \gtrsim c/H_0$.

Thus, only a flat closed space of nontrivial topological structure (for example, a three-dimensional torus) can have a diameter appreciably less than $c/H_0 \sim 10^{28}$ cm. In §§3 and 4, we present "aesthetic" arguments for our Universe's being constructed in this way. In §4, we then construct a model of the Universe that is closed in the form of a three-dimensional torus; the total number of nucleons in it is equal to the reciprocal of the square of the gravitational "fine structure constant." Its contemporary diameter is $D_0 = 0.102\ (c/H_0)$. This model does not contradict observations. It is interesting that it does not contain constants specific to cosmology; for all of them—the size of the Universe, the Hubble parameter, the mean matter density in the Universe, and so forth—can be expressed in terms of the universal time and atomic constants.

§2. BOUNDS ON THE VOLUME AND DIAMETER OF SPACES OF NONZERO CURVATURE

Let $\mathscr{P}^3$ and $\mathscr{N}^3$ be complete three-dimensional spaces of constant positive and negative curvature, R be their radius of curvature, and V and D their volume and diameter. The spaces $\mathscr{P}^3$ admit complete classification.[5] This makes it possible to obtain the inequalities (see Appendix A)

$$2\pi^2 R^3 \geqslant V(\mathscr{P}^3) > 0, \quad \pi R \geqslant D(\mathscr{P}^3)\ c_D{}^+ R, \tag{1}$$

where

$$c_D{}^+ = {}^1/_2 \arccos\left((1/\sqrt{3})\operatorname{ctg}\pi/5\right) \approx 0.326. \tag{1a}$$

It is important to emphasize that among the spaces of positive curvature there are topological types whose volume is arbitrarily small compared with R^3 whereas their diameter is always of order R.

The problem of classifying the spaces $\mathscr{N}^3$ has not yet been solved (see Refs. 5 and 12); it is not even known whether it has a solution. As is shown in Appendix A, the existence of bounds on the volume and diameter of $\mathscr{N}^3$ follows from Ref. 13, which relates to a quite different field of mathematics. Thus,

$$\begin{aligned} &\infty \geqslant V(\mathscr{N}^3) > c_V{}^- |R|^3, \quad c_V{}^- > 0; \\ &\infty \geqslant D(\mathscr{N}^3) > c_D{}^- |R|, \quad c_D{}^- > 0. \end{aligned} \tag{2}$$

By means of very crude arguments, one can find that certainly $c_V^- > 10^{-5}$ and $c_D^- > 1.3 \times 10^{-2}$. In reality, the hypothesis

$$c_V{}^-(\mathscr{N}^3) \approx 1, \quad c_D{}^-(\mathscr{N}^3) \approx 1, \tag{3}$$

for which arguments are given in the Appendix, is certainly true. The rigorous proof of the hypothesis (3) is a nontrivial problem. We note that in the two-dimensional case [see (A.2) and (A.6)]

$$c_V{}^-(\mathscr{N}^2) = 2\pi, \quad c_D{}^-(\mathscr{N}^2) = \operatorname{arcch} 4 \approx 2.1.$$

We have attempted to "guess" the actual form of the $\mathscr{N}^3$ space processing the smallest possible diameter [see Appendix (A.6)]. Apparently, it is a centrally symmetric, "almost regular" 24-hedron of negative curvature whose faces are identified in accordance with a definite law. According to numerical calculations, the diameter of this space is $D \approx 1.13\,|R|$ and the volume $V \approx 5\,|R|^3$. The required space can be selected by means of a computer from the class of "almost regular" polyhedrons capable of "paving" the Lobachevskiĭ plane L^3.

Thus, in all probability,

$$c_D^-(\mathcal{N}^3) \approx 1.13 \tag{3a}$$

in accordance with the hypothesis (3).

§3. TOPOLOGY "DICTATES" CURVATURE

As is shown in Appendix B, the Einstein equations and data of extragalactic astronomy lead to the following bound on the present-day radius of curvature of the Universe:

$$|R_0| > {}^1/_2(c/H_0), \tag{4a}$$

(H_0 is the Hubble constant), and probably

$$|R_0| > (c/H_0) \approx 1.8 \cdot 10^{28}\ \text{cm} \cdot (H_0/50\ \text{km/sec} \cdot \text{Mpc})^{-1}. \tag{4b}$$

On the other hand, it follows from Ref. 9 that the observational data do not rule out the possibility that the present-day diameter of the Universe is $D_0 \sim 0.1(c/H_0)$. An example of such a universe will be described below in §4.

Comparison of the expressions (4) with (1) and (3) makes the following clear. If observations of extragalactic sources show that the diameter of the Universe is significantly less than the Hubble radius $c/H_0 \sim 10^{28}$ cm, this will mean that our space is described on the average by a flat model. For it follows from (4) that the radius of curvature of the observable Universe is of the order or greater than the Hubble radius, and by virtue of (1) and (3) the diameter of a space of nonzero curvature is of the order of or greater than its radius of curvature. Therefore, the only way of obtaining a small universe is to "glue it together" out of a flat Euclidean space.

It is here appropriate to emphasize that no refinements of the current observational values of the expansion rate H_0, the critical parameter Ω_0, the deceleration parameter q_0, etc., can establish that the mean curvature of the Universe is strictly zero; they can only improve the inequality $|R_0| > 0.5(c/H_0)$. The basic possibility of establishing the fact $R = \infty$ is due to the circumstance that among closed spaces only the flat space does not enforce a connection between the dimensions and the curvature.

In Appendix A, we prove a more general assertion: The topological structure of any closed three-dimensional manifold uniquely determines the sign of its curvature. In other words, topology is capable of "dictating" zero (and nonzero) mean curvature of the physical space.

Speaking figuratively, one cannot "sprinkle" into a closed flat world a little matter so as to make its curvature "slightly positive"; for this would lead to an abrupt change in the topological properties of the Universe (in this hypothetical experiment, we are considering "sprinkling" matter into unit volume of a space expanding with given velocity). In contrast, adding a small amount of matter to a space of positive or negative curvature would not lead to "catastrophic consequences"; the radius of curvature would change smoothly and the topological properties of the space would remain as before.

Thus, from the topological point of view the closed flat model is distinguished.

§4. ON THE AESTHETIC PREFERENCE FOR THE MODEL OF A FLAT CLOSED UNIVERSE

Models of a flat Universe are distinguished not only from the point of view of topology but also aesthetically. The point is that contemporary observational bounds on the density parameter Ω, which determines the curvature of the Universe, are not impressive: $0.02 < \Omega_0 < 5$. However, they lead to the conclusion that at the epoch of element synthesis, when the temperature of the Universe was 10^{10} °K,

$$|\Omega^* - 1| = \left|\frac{\Omega_0 - 1}{1 + \Omega_0 z^*}\right| < 2 \cdot 10^{-8}.$$

Here, $z^2 = 3 \times 10^9$ is the red shift corresponding to $T^* = 10^{10}$ °K. Would it not be more natural to have the condition $\Omega^* \equiv 1$, i.e., a strictly flat Universe, rather than a condition of the type $\Omega^* = 1 - 10^{-8}$? It is interesting to note that for $|\Omega^* - 1| \geq 10^{-6}$ stars could not form in the Universe and, therefore, life would not have arisen (see Refs. 14 and 15) . . . .

Another aesthetic consideration is the circumstance that it is only in a flat Universe with zero cosmological constant Λ that the expansion rate $H(t)$ of space is uniquely determined by the mean matter density $\rho(t)$ and is not a free parameter. It is therefore only in the flat model of the Universe that Einstein's original idea of a *unique* connection between the properties of space and matter is realized. "From the standpoint of epistomology it is more satisfying to have the mechanical properties of space *completely* determined by matter," wrote Einstein and continued " . . . this is the case only in a space-bounded Universe."[16]

Although both of Einstein's theses now appear very debatable, we wish to draw attention to a model in which they are satisfied.

We are attracted to a model of the Universe consisting of a flat space and closed in the form of a regular three-torus. (We recall that such a torus is obtained from the Euclidean space E^3 by specifying as fundamental region a cube in which the opposite faces are identified. The corresponding manifold is flat, closed, orientable, and globally homogeneous.) Its diameter is

$$D = (\sqrt{3}/2)\, V^{1/3},$$

where V is the volume of space. Denoting by N_0 the total number of elementary particles (nucleons) in the Universe, by m_p the mass of one nucleon, and by $\rho(t)$ the mean matter density at the time t, we obtain for the diameter of the Universe

$$D(t) = (\sqrt{3}/2)\,[N_0 m_p/\rho(t)]^{1/3}. \tag{5}$$

The number N_0 can be regarded as a new universal constant. However, it is more attractive to regard N_0, not as an independent quantity, but as a function of the other universal constants. For example, a sensible estimate for the dimensions of the Universe can be obtained by equating N_0 to the reciprocal of the square of the "gravitational fine structure constant":

$$N_0=(\alpha_{gr})^{-2}\equiv(Gm_p^2/\hbar c)^{-2}\approx 2.87\cdot 10^{76}. \quad (6)$$

Substituting (6) in (5) and remembering that in a flat world the density varies in accordance with the law[3] (Refs. 1 and 2)

$$\rho(t)=1/6\pi G t^2,$$

we find

$$D(t)=3^{5/6}\left(\frac{\pi}{4}\right)^{1/3}\left(\frac{\hbar^2c^2}{G}\right)^{1/3}\frac{t^{2/3}}{m_p} \quad (7)$$

or, in the Planck system of units $\hbar=c=G=1$,

$$[D(t)]\approx 2.305[m_p]^{-1}[t]^{2/3}. \quad (7a)$$

At the contemporary epoch, we find from (7)

$$D\left(t_0=\frac{2}{3}\frac{1}{H_0}\approx 13\cdot 10^9\text{ years}\right)\approx 0.102\left(\frac{c}{H_0}\right)\left(\frac{H_0}{50\text{ km/sec}\cdot\text{Mpc}}\right)^{1/3}\approx 2\cdot 10^{27}\text{ cm}. \quad (8)$$

This value for the diameter of the Universe does not contradict the data of extragalactic astronomy and is in principle observationally verifiable (see Ref. 9). We emphasize that in accordance with (8) all celestial sources having red shift $z>0.102$ are "ghosts" of nearer objects.

The proposed model of the Universe does not contain constants specific to cosmology; all of them—the diameter of the Universe, the Hubble constant, the mean matter density, etc.,—can be expressed in terms of the universal time and the atomic constants. Using (5) and (6), we readily find from the relation $D(t_c)=ct_c$ one further interesting parameter, namely, the time of complete causal connection of the Universe:

$$[t_c]=(3^{5/2}\pi/4)N_0^{3/4},\quad t_c\approx 1.5\cdot 10^{15}\text{ sec}. \quad (9)$$

The estimate (9) corresponds to a red shift $z_c\approx 45$, i.e., causal connection was established long before the epoch of galaxy separation, which was at $z_s\sim 5-10$.

Finally, to avoid misunderstandings, we emphasize the following. The above model of the Universe is undoubtedly a varient of the Large Number Hypothesis, but it also differs radically from Dirac's hypothesis.[18,19] Dirac's theory, which postulates a variable gravitational constant G, is an alternative to Einstein's general theory of relativity. The model proposed above is entirely constructed on the basis of the Friedmann solution of the Einstein equations.

APPENDIX A

Diameter and volume of three-dimensional spaces of constant curvature

1. *Introductory remarks.* As usual, we shall define a closed space as a compact Riemannian manifold without edge; the diameter of such a space is defined as the maximal distance between its points. Below, we shall consider complete three-dimensional spaces X of constant curvature k. We take $k=1$, 0, -1 (i.e., the radius of curvature R is 1 or ∞).

It is well known[5] that among all the spaces X the simply connected manifolds U are the sphere S^3 (for $k=1$), the Euclidean space E^3 (for $k=0$), and the Lobachevskiĭ space L^3 (for $k=-1$). Any space X can be obtained as the factor space U/Γ, where Γ is a discrete group of motions of the space U that does not possess fixed points (see Ref. 5, §2.4).

A connected subset Y of the manifold U containing all points of U that cannot be carried into one another by the action of Γ is called a fundamental region of the space X. Every point x on the manifold U can be associated with the "canonical" fundamental region $Y_0(x)$, which consists of points y such that $\rho(x,y)\leqslant\rho(x,\gamma y)$ for all γ in Γ.

2. *Topology "dictates" curvature.* In this section, we shall show that the topology of a closed space X uniquely determines the sign of its curvature k.

If $k=1$, then Γ is a finite group, since the space S^3 is closed. If $k=0$, then Γ contains a commutative subgroup with three independent elements, since it follows from Bieberbach's theorem (see Refs. 5, §3.2) that Γ contains shifts in three independent directions. If $k=-1$, the group Γ is infinite, but any commutative subgroup in Γ has not more than two independent elements. It follows from this that the six-dimensional group $G(L^3)$ of all orientable motions of the Lobachevskiĭ space L^3 is isomorphic to the Lorentz group, i.e., to the group $SL(2,C)/\{\pm 1\}$. In this group, for any element γ not equal to the identity all the elements which commute with it form a subgroup of dimension 2, i.e., any commutative discrete subgroup in $G(L^3)$ has not more than two independent elements.

Thus, the structure of the group Γ is different for the cases $k=1$, 0, -1. Since the group Γ is determined by the topology of the space X (Γ is simply the fundamental group of X), the topological structure of X is also different for $k=1,0,-1$. In other words, the topology of the closed space X uniquely determines the sign of its curvature.

3. *The size of two-dimensional spaces.* Below, for methodological purposes, we choose the two-dimensional case. Let X^2 be a two-dimensional closed Riemannian space of constant curvature k. If X^2 is orientable, then by the Gauss-Bonnet theorem[20]

$$\int_{X^2} k\,dV=4\pi(1-g),$$

where V is the area of the manifold, and g is the genus of the manifold, i.e., the number of handles. Hence, for $k\neq 0$ we find

$$V(X^2)=|4\pi(1-g)|\geqslant V(S^2)=4\pi. \quad (A.1)$$

For a nonorientable manifold X^2 there always exists a two-sheeted orientable covering $\tilde{X}^2$ (so that to every point in X^2 there correspond two points in $\tilde{X}^2$). Therefore, in this case

$$V(X^2)=V(\tilde{X}^2)/2\geqslant V(S^2)/2=2\pi. \quad (A.2)$$

Now let D be the diameter of the manifold X^2. Then a disk B_D of radius D contains a fundamental region for X^2, so that

$$V(B_D)\geqslant V(X^2)\geqslant 2\pi.$$

For $k=1$, we have $V(B_D)=2\pi(1-\cos D)$, whence

$$D \geqslant \pi/2 \approx 1{,}57. \quad (A.3)$$

For $k=-1$, we have $V(B_D)=2\pi(\cosh D-1)$, whence

$$D > \operatorname{arcch} 2 \approx 1.32. \quad (A.4)$$

Note that the bound (A.3) is precise ($D=\pi/2$ for the projective plane), while (A.4) is certainly an underestimation. The point is that the fundamental regions for X^2 when $k=-1$ are not disks but $2p$-gons with $p \geqslant 3$ and sum of the internal angles equal to 2π.[21] These figures on the Lobachevskiĭ plane L^2 recall stars, and have long "rays." The diameter of such $2p$-gons is minimal in the cases when they are regular. It is readily seen that in such a case the diameter is

$$D(p) = \operatorname{arcch}(\operatorname{ctg} \pi/2p)^2, \quad (A.5)$$

whence

$$D(X^2) \geqslant D(p=3) = \operatorname{arcch} 4 \approx 2.06. \quad (A.6)$$

We emphasize that the difference between the estimates (A.4) and (A.6) is due solely to the "rays."

4. *Diameter of three-dimensional spaces of positive curvature.* The Gauss-Bonnet theorem has a generalization only for spaces of even dimension.[22] Therefore, the estimate of the diameter of three-dimensional manifolds is not trivial. For $k=+1$, the situation is facilitated by the circumstance that in this case the spaces X have a complete classification (see Ref. 5, §7.5).

We realize X as S^3/Γ, where Γ is a finite group of motions of S^3. If x and y are points of S^3, the X distance between them is $\rho_X(x,y)=\min \rho_3(x,\gamma y)$, where the minimum is taken over all elements γ of the group Γ, and ρ_3 denotes the distance on the sphere S^3.

We shall regard a point x of the sphere S^3 as a pair of complex numbers (z,w), where $|z|^2+|w|^2=1$. We associate it with a point of the three-dimensional space $\varkappa(x)=(u,v,t)$, where $u=2\,\mathrm{Re}(z,\bar{w})$, $v=2\,\mathrm{Im}(z,\bar{w})$, $t=2|z|^2-1$. Since $u^2+v^2+t^2=1$, the point $\varkappa(x)$ lies on the two-dimensional sphere S^2. It is easy to show that $\rho_2(\varkappa(x),\varkappa(y)) \leqslant 2\rho_3(x,y)$, where ρ_2 is the distance on the sphere S^2.

To each unitary 2×2 matrix γ there corresponds a rotation of the sphere S^3. It is readily verified that it corresponds to a rotation $\tilde{\gamma}$ of the sphere S^2 for which $\tilde{\gamma}(\varkappa(x))=\varkappa(\gamma(x))$ for all points x. It follows from the classification of Wolf[5] that one can realize the elements of the group Γ by unitary matrices and, hence, transfer their action to the sphere S^2. Thus,

$$\rho_X(x,y) = \min \rho_3(x,\gamma y) \geqslant {}^1/_2 \min \rho_2(\varkappa(x), \tilde{\gamma}(\varkappa(y))). \quad (A.7)$$

As is shown in §2.6 of Ref. 5, the group Γ of rotations of S^2 conserves either a regular n-gon (inscribed in the equatorial circumference of S^2) or a regular polyhedron: tetrahedron, cube, or dodecahedron (inscribed in S^2). We consider a point a on S^2 projected to the center of a face of the polyhedron and a point b coinciding with one of the vertices of this face; then we choose points x and y on S^3 such that $\varkappa(x)=a$ and $\varkappa(y)=b$ Then for the diameter D we obtain the estimate

$$D(X) \geqslant \rho_X(x,y) \geqslant {}^1/_2 \min \rho_2(a, \tilde{\gamma} b) = {}^1/_2\rho_2(a,b). \quad (A.8)$$

It is easy to show that

$$\rho_2(a,b) = \arccos((1/\sqrt{3})\operatorname{ctg}\pi/m), \quad (A.9)$$

and for the n-gon, tetrahedron, cube, and dodecahedron $m=2,3,4,5$ [respectively, $\frac{1}{2}\rho_2(a,b) \approx 0.785, 0.615, 0.478, 0.326$]. One can show that for each of these cases the diameter D is always greater than $\frac{1}{2}\rho_2(a,b)$ but may be arbitrarily close to this value. Thus, finally

$$D(X) > {}^1/_2 \arccos((1/\sqrt{3})\operatorname{ctg}\pi/5) \approx 0.326. \quad (A.10)$$

In contrast to the diameter $D(X)$, the volume $V(X)$ is not bounded below. This follows from the classification of the spaces X, in accordance with which the group Γ can have an arbitrarily large number of elements.

5. *Diameter of three-dimensional spaces of negative curvature.* The problem of classifying the spaces X for $k=-1$ has not yet been solved (see Refs. 5 and 12). Therefore, to estimate their volume and diameter we shall have recourse to indirect methods.

Let X be orientable, so that Γ is contained in the six-dimensional group $G=G(L^3)$ of orientable motions of the Lobachevskiĭ space L^3. We shall represent L^3 as the factor space of the group G with the respect of the subgroup C consisting of rotations of L^3 about a point. Then $V(X)=V(G/\Gamma)/V(C)$, where $V(X)$ and $V(C)$, are the three-volumes, and $V(G/\Gamma)$ is the six-volume. As is shown in Ref. 13, one can choose an open set G_0 in the group G such that the sets $G_0\gamma$ for different elements γ of Γ do not intersect. From this there follows the inequality $V(G/\Gamma) > V(G_0)$, i.e., $V(X) > V(G_0)/V(C)$. In Ref. 13, an estimate for $V(G_0)$ was not sought. If we make arguments in accordance with the scheme of Ref. 13 but with numerical constants, we obtain the estimate $V(X) > 2\times 10^{-5}$. For a nonorientable manifold X, arguing as in (A.3), we have $V(X) > 10^{-5}$.

The fact that the volume $V(X)$ is bounded below for $k=-1$ is of fundamental importance, but it is very difficult to obtain a bound by the presented method. In the two-dimensional case, such a method would lead to the inequality $V(X^2) > 2\times 10^{-3}$ instead of the $V(X^2) \geqslant 2\pi$ obtained in (A.3). It is natural to assume that an estimate of the type

$$V(X) \geqslant V(S^3)/2 = \pi^2, \quad (A.11)$$

which is analogous to (A.2) in the two-dimensional case, holds.

Now let D be the diameter of the space X. A sphere B_D of radius D in the Lobachevskiĭ space L^3 contains a fundamental region for X, so that

$$V(X) < V(B_D) = \pi(\operatorname{sh} 2D - 2D). \quad (A.12)$$

Therefore, it follows from the estimate $V(X) > 10^{-5}$ that $D > 1.3\times 10^{-2}$. But if we take the estimate $V(X) \gtrsim \pi^2$, we obtain $D \gtrsim 1.2$.

Note that the estimate of $D(X)$ obtained by circumscribing a sphere is crude, since the fundamental region for X when $k=-1$ recalls, as in the two-dimensional case, a starlike polyhedron rather than a sphere [cf (A.4) and (A.6)]. Therefore, even if $V(X)$ for a number of spaces is several times smaller than π^2, the inequality $D(X) > 1$

for $k=-1$ is probably always satisfied.

6. *Example of a small three-dimensional space of negative curvature*. In this section, we describe an example of a space that apparently has the smallest of the diameters possible when $k=-1$. In constructing this example, we use the circumstance that the canonical fundamental region Y_0 for a space of the smallest diameter can be chosen in the form of a fairly regular polyhedron. At the same time, the complete space L^3 must be composed of polyhedrons congruent to Y_0. These circumstances impose fairly stringent restrictions of the form of Y_0.

In the example that we construct, Y_0 is a centrally symmetric 23-hedron with 26 vertices. All the dihedral angles of Y_0 are equal to $2\pi/3$, and all the vertices of Y_0 are at approximately the same distance from the center (between 1.015 and 1.128) and the lengths of all edges are not greater than this distance. Thus, the polyhedron Y_0 is fairly close to a sphere of radius $|R|=1$. The space X corresponding to Y_0 has the following parameters [cf. (A.11) and (A.12)]:

$$2\mathscr{L}_0>D(X)>\mathscr{L}_0=1.1284\ldots\ (\text{in fact } D(X)\approx\mathscr{L}_0),\qquad V(X)=V(Y_0)\approx V(B_{D=1})=\pi(\operatorname{sh}2-2)\approx 5. \tag{A.13}$$

We now describe the example. Regular polyhedrons are intimately related to groups generated by reflections. Therefore, we shall construct the space X as follows. We consider a discrete group $\tilde{\Gamma}$ of motions of the Lobachevskiĭ space L^3 generated by reflections. In it, we choose a subgroup Γ that acts on L^3 without fixed points, and set $X=L^3/\Gamma$.

It is known[23] that there exist nine compact hyperbolic types of group $\tilde{\Gamma}$ for which the fundamental regions are simplexes (tetrahedrons) T, the diameter of the space $L^3/\tilde{\Gamma}$ being equal to the length $\mathscr{L}$ of the maximal edge of the simplex T. Therefore, $D(X)=D(L^3/\Gamma)\geq D(L^3/\tilde{\Gamma})=\mathscr{L}$. Using the Bourbaki classification,[23] we chose from the nine types of group $\tilde{\Gamma}$ the one for which $\mathscr{L}$ is minimal (the lengths of all the edges of the simplexes T for each of the nine cases were calculated on a computer).

We denote by x_1, x_2, x_3, x_4 the vertices of the simplex T and by P_1, P_2, P_3, P_4 the faces opposite them. The simplex T_0 with minimal $\mathscr{L}=\mathscr{L}_0$ is characterized by the circumstance that the dihedral angles between P_i and P_j have the form π/m_{ij}, where $m_{24}=m_{13}=2$, $m_{12}=m_{14}=m_{34}=3$, $m_{23}=4$. At the same time, $\rho(x_1,x_3)=\rho(x_2,x_4)\approx\rho(x_1,x_4)\approx 1.128$, $\rho(x_1,x_2)=\rho(x_3,x_4)\approx 1.015$, $\rho(x_2,x_3)\approx 0.769$. Thus, $\mathscr{L}_0\approx 1.128$.

Thus, the complete space L^3 is divided up into simplexes congruent to T_0. We denote by Y_0 the polyhedron composed of all simplexes T_0 containing the point x_1 (there are altogether 48 of them). One can show that there exists a subgroup Γ_0 in Γ that acts without fixed points and for which Y_0 is a fundamental region. On the other hand, any subgroup Γ in $\tilde{\Gamma}$ that acts without fixed points has fundamental region Y containing Y_0. Thus, Y_0 cannot be decreased.

The required polyhedron Y_0 has 24 tetragonal faces, each of which consists of two faces of the simplexes T_0; all angles between the faces of Y_0 are equal to $2\pi/3$. Note that it is impossible to make all the dihedral angles of the fundamental region greater than $2\pi/3$, since not less than three dihedral angles must converge on every edge when L^3 is filled with polyhedrons. This is a further argument for our examples, describing the smallest of the possible spaces of negative curvature.

APPENDIX B

Observational bounds on the radius of curvature of the universe

In the isotropic and homogeneous case, i.e., in a space of constant curvature, Einstein's equations reduce to the form

$$\mathscr{K}=\frac{8\pi G\rho}{3c^2}-\frac{1}{c^2}\left(\frac{\dot R}{R}\right)^2+\frac{\Lambda}{3}, \tag{B.1}$$

$$\frac{\ddot R}{R}=-\frac{4\pi G}{3c^2}(\rho c^2+3p)+c^2\frac{\Lambda}{3}. \tag{B.2}$$

Here, $\mathscr{K}$ is the Gaussian curvature of space, R is the radius of curvature for $\mathscr{K}\neq 0$ ($R^{-1}\equiv\mathscr{K}^{1/2}$) and an arbitrary scale factor for $\mathscr{K}=0$, G is the gravitational constant, Λ is the cosmological constant, ρ is the mean density of all forms of matter in the Universe, and p is the averaged pressure of the matter. Introducing the parameters that are observed in astronomy,

$$H=\dot R/R,\quad q=-R\ddot R/\dot R^2,\quad \Omega=\rho/\rho_{cr}=8\pi G\rho/3H^2, \tag{B.3}$$

we transform Eq. (B.2) to the form

$$q=\frac{\Omega}{2\gamma}-\frac{\Lambda}{3}\left(\frac{c}{H}\right)^2,\quad \gamma=\left(1+\frac{3p}{\rho c^2}\right)^{-1}. \tag{B.4}$$

Since the equation of state of the matter is contained in the range $0<p<\rho c^2/3$, we have $1>\gamma>\frac{1}{2}$. From (B.1) and (B.4), we obtain for the radius of curvature

$$R=\frac{c/H}{[\Omega(1+1/2\gamma)-q-1]^{1/2}}=\frac{c/H}{[2q\gamma-1+{}^1/_3\Lambda(c/H)^2(2\gamma+1)]^{1/2}}. \tag{B.5}$$

The data of extragalactic astronomy yield the bounds (Refs. 2, 8, and 24–26)

$$35<H_0<120\ \text{km/sec}\cdot\text{Mpc},\quad \Omega_0>0.02,\quad |q_0|<2.5; \tag{B.6}$$

$$-3(H_0/c)^2q_0<\Lambda<6(H_0/c)^2. \tag{B.7}$$

The left-hand side of the inequality (B.7) follows from (B.4). The upper bound on the right-hand side of (B.7) is obtained in Ref. 8 for the case of the Lemaître model in the acceleration stage ($q_0<-0.5$) after the "plateau" stage. Bearing this in mind, we can obtain from (B.5)–(B.7) the following bound on the modulus of the radius of curvature of the Universe:

$$|R_0|>{}^1/_2(c/H_0). \tag{B.8}$$

It is probable that the cosmological term vanishes, $\Lambda=0$, and the density parameter satisfies $\Omega_0<2$. In accordance with (B.5) and (B.6), we then have

$$|R_0(\Lambda=0)|=\left|\frac{c/H_0}{(\Omega_0-1)^{1/2}}\right|>c/H_0\approx 1.85\cdot 10^{28}\left(\frac{H_0}{50\,\text{km/sec}\cdot\text{Mpc}}\right)^{-1}\ \text{cm}. \tag{B.9}$$

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1) There are arguments supporting the view that the real physical space msut be orientable.[6,7]

2) In the Lemaître model with cosmological constant $\Lambda > 0$ and $\mathscr{K} > 0$, the observation of the same object in two (opposite)directions is possible in the late stages of expansion; see, for example, Refs. 2 and 8.

3) Below, it is assumed that the cosmological constant vanishes, $\Lambda = 0$, and, in addition, the equation of state of the matter is $p = 0$, i.e., we ignore radiation pressure. We recall that in the standard model of a hot Universe the total density of the background radiation at the contemporary epoch is $\rho_{m=0} < 2\times10^{-33}$ g/cm^3,[17] i.e., it is much smaller than the critical density $\rho_{cr} \approx 2\times10^{-29}$ g/cm^3. Here, $\rho_{m=0}$ includes the density of photons, gravitons, all types of neutrinos, and also all as yet unkown particles with zero rest mass that have survived from the superdense phase. If, however, $m_{0\nu} \approx 30$ eV, then today $p_\nu \ll \varepsilon_\nu/3$, i.e., we still have $p_\Sigma \approx 0$.

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Hyperfine structure of the energy levels of μ-mesic molecules of the hydrogen isotopes

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In the first order of perturbation theory in α^2, a calculation accurate to $\sim 10^{-3}$ eV is made of the hyperfine structure of the energy levels of all stationary states with quantum numbers $J \leq 1$ and $v \leq 1$ of the total orbital angular momentum and of the vibrational motion, respectively, for mesic molecules of the hydrogen isotopes. The solutions to the nonrelativistic problem of the bound states of a system of three particles with Coulomb interaction found in the adiabatic representation are chosen as the zeroth approximation. Expressions are given for the probability amplitudes of the different values of the total spin of the nuclei and the total spin of the μ-mesic molecules in the stationary states of the hyperfine structure. Calculations are made of the populations of the stationary states of the hyperfine structure of the μ-mesic molecules formed in collisions of the mesic atoms $p\mu$, $d\mu$, and $t\mu$ in the parastate or orthostate with the nuclei p, d, and t.

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1. INTRODUCTION

Recent experiments on the resonance formation of $dd\mu$ and $dt\mu$ mesic molecules[1] have confirmed the theoretical predictions in Ref. 2 that these mesic molecules should have excited weakly bound states with quantum number $J = 1$ for the total orbital angular momentum and quantum number $v = 1$ for the vibrational motion and

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