Production of phonons in an isotropic universe

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A quantum theory of irrotational perturbations of an ideal fluid in a spatially flat isotropic homogeneous cosmological model is constructed. Two gauge-invariant canonically conjugate scalars are obtained that describe the evolution of the two physical degrees of freedom of irrotational perturbations of the matter in an expanding universe; they are analogs of the velocity potential and the density perturbation in a stationary nongravitating medium [E.M. Lifshitz and L. P. Pitaevskiĭ, Statistical Physics, Part 2, 1978, §24]. A Lagrangian and an equation of motion of second order are obtained. Canonical quantization is performed, and the concept of phonons (sound quanta in a nonstationary universe) is introduced. Conformal noninvariance of sound waves in an isotropic universe is proved. It is shown that the mechanism of spontaneous production of phonons in the process of the cosmological expansion is the cause of the formation of the initial spectrum of adiabatic perturbations of the matter density in an isotropic Friedmann universe. In the subsequent evolution in the stage after hydrogen recombination in an expanding universe the long-wavelength density fluctuations may lead to the formation of galaxies and clusters of galaxies. Estimates are made of the spectrum of the primordial density fluctuations. It it shown that they are in agreement with the requirements imposed on the initial spectrum by the adiabatic theory of the origin of galaxies.

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In this paper, irrotational perturbations of an ideal fluid in a Friedmann cosmology will be canonically quantized. The main reason why this problem has not hitherto been solved is the difficulty of constructing two canonically conjugate scalar operators describing invariant (independent of the coordinate system) perturbations of the matter in a nonstationary universe. These scalars are the analogs of the operators of the velocity potential and the density perturbation of matter, on which the algebra of quantum perturbations of an ideal stationary nongravitating medium is based.[1]

Adiabatic perturbations of the matter density in homogeneous isotropic cosmological models were investigated for the first time by Lifshitz.[2] Using the Fourier method, Lifshitz reduced the problem to two second-order equations describing the behavior of two "fictitious" modes (which can be eliminated by transition to a different coordinate system) and two physical (invariant) modes of irrotational perturbations. Field and Shepley,[3] using Lifshitz's method, obtained an equation of second order for the evolution of the Fourier components of the physical perturbations of the matter density. In the present paper, we construct a Hamiltonian formalism of irrotational perturbations of an ideal fluid in the spatially flat Friedmann cosmological model and in this framework carry out a canonical quantization of these perturbations.[4] The sections of the paper are as follows:

§1. Irrotational motions of an ideal fluid in the general theory of relativity. Lagrangian of the linearized equations.

§2. Irrotational perturbations of the matter in a Friedmann cosmology: Hamiltonian formalism.

§3. Sound quanta (phonons) and their conformal noninvariance in an isotropic universe.

§4. Primordial spectrum of adiabatic perturbations of the matter density.

§1. IRROTATIONAL MOTIONS OF AN IDEAL FLUID IN THE GENERAL THEORY OF RELATIVITY. LAGRANGIAN OF THE LINEARIZED EQUATIONS

Irrotational motions of a perfect fluid in the general theory of relativity can be described by a single scalar function—the potential $\varphi = \varphi(x)$, whose gradient is proportional to the four-momentum of a particle of the matter[1]:

$$p_i = w u_i = \varphi_{,i},\tag{1.1}$$

where u^i is the four-velocity of the matter $(u_i u^i = 1)$,

$$w = (\varepsilon + p)/n = (\varphi_{,i}\varphi_{,k}g^{ik})^{1/2}\tag{1.2}$$

is the specific enthalpy, p is the pressure, ε is the matter density, and

$$n = \frac{dp}{dw} = \exp \int \frac{d\varepsilon}{\varepsilon + p}\tag{1.3}$$

is the particle number density. Among the four quantities w, p, ε, and n, only one is independent and they are related to each other through the equation of state of the matter:

$$p = p(w).\tag{1.4}$$

In what follows, we shall not require notation for the temperature and specific entropy.

Equation (1.1) is an exact integral of the Euler equations

$$u^k(p_{i,k} - p_{k,i}) = 0,\tag{1.5}$$

which are the projection of the hydrodynamic equations onto the directions orthogonal to the velocity vector[2]:

$$(\delta_i^k - u_i u^k)T^l_{k;l} = 0,$$

where $T_i^k = (\varepsilon + p)u_i u^k - p\delta_i^k$ is the energy–momentum tensor of the matter. The continuity equation ($u_i T^k_{i;k} = 0$) expresses the law of conservation of the particle number in a volume that is comoving with the matter:

$$(nu^i)_{;i} = 0.\tag{1.6}$$

The semicolon denotes the convariant derivative in the metric g_{ik}.

Despite the apparent simplicity of the definition (1.1), irrotational motions present a complicated hydrodynamic picture with the occurrence of shock waves,[3] hydrodynamic discontinuities, etc. The self-gravitation of the matter flows influences the geometry of space, and the motions (1.1) generate gravitational waves of a special structure that do not lead to the occurrence of a solenoidal component of the momentum p^i (Thomson's theorem). The Einstein equations $G_i^k = T_i^k$, which describe these processes, are obtained by equating to zero the first variation of the action ($c = \hbar = 8\pi G = 1$)

$$W = W[\varphi, g^{ik}] = \int (p - R/2)(-g)^{1/2} d^4x \qquad (1.7)$$

with respect to the metric g^{ik} for fixed function φ. [The continuity Eq. (1.6) follows from a $\delta W = 0$ variation with respect to φ for constant g^{ik}.] In (1.7), $g = \det\{g_{ik}\}$, and $R = R_i^i = -G_i^i$ is the scalar curvature. Equations (1.2) and (1.4) determine the pressure p as a function of the potential φ and the metric g^{ik}.

One of the main problems that arises in this direction is the investigation of small perturbations of certain exact solutions of the type (1.1). In this case, the function φ is the sum of the known function $\varphi^{(0)}$, which determines the background solution, and a small function $\delta\varphi = \Phi$, which is the subject of the analysis:

$$\varphi = \varphi^{(0)} + \Phi, \quad g^{ik} = g^{ik(0)} - h^{ik}. \qquad (1.8)$$

The small tensor h^{ik} is defined in the background space $g_{ik}^{(0)}$ and is a linear function of the scalar Φ. (In the first order in Φ, gravitational waves are not generated.) This problem is topical for the following two reasons:

a) the most important solutions possessing a definite symmetry are special cases of (1.1) (for example, the simplest anisotropic cosmologies, the Friedmann models, spherically symmetric collapse, and so forth);

b) the problem of the evolution and the initial spectrum of adiabatic perturbations of the matter density in a homogeneous isotropic universe is the key to the explanation of the origin of structure in the Universe and the expected small anisotropy of the microwave background associated with it.

The Lagrangian of the linearized Einstein equations, which connect the perturbations of the metric h_i^k to the potential Φ, is obtained by expanding the integrand of (1.7) up to second order in Φ and h_i^k (see Appendix I):

$$W^{(2)} = W^{(2)}[\Phi, \psi_i^k] = \int L(-g^{(0)})^{1/2} d^4x,$$
$$L = \frac{\varepsilon+p}{2}(v_i v^i - 2v_i \psi_k^i u^k - \varkappa^2(1-\beta^{-2})) + \frac{\varepsilon-p}{8}\left(\psi_{ik}\psi^{ik} - \frac{\psi^2}{2}\right)$$
$$+ \frac{1}{8}(\psi_{ik;l}\psi^{ik;l} - 2\psi_{ik;l}\psi^{il;k} - \frac{1}{2}\psi_{,l}\psi^{,l}), \qquad (1.9)$$

where

$$\psi_i^k = h_i^k - \delta_i^k h/2, \quad \psi = \psi_i^i = -h, \quad v_i = \Phi_{,i}/w,$$
$$\varkappa = v_i u^i - h_i^k u^i u_k/2, \quad \beta = (dp/d\varepsilon)^{1/2} = (d\ln w/d\ln n)^{1/2},$$

and β is the speed of sound. [All the operations are performed in the background metric g_{ik}, and here and in what follows the superscript (0) is omitted.] The

perturbed Einstein equations [and the continuity Eq. (1.6), which is a first integral of them], are obtained from a $\delta W^{(2)} = 0$ variation with respect to ψ^{ik} (with respect to Φ) for fixed $\Phi(\psi_i^k)$ and background metric:

$$\psi_{i;l}^{k;l} - \psi_{i;l}^{l;k} - \psi^{kl}_{;il} - \frac{1}{2}\delta_i^k \psi_{;l}^{;l} = (\varepsilon - p)\left(\psi_i^k - \frac{1}{2}\delta_i^k \psi\right)$$
$$+ 2(\varepsilon+p)\left[-v_i u^k - v^k u_i + \varkappa(1-\beta^{-2})\left(u_i u^k - \frac{1}{2}\delta_i^k\right)\right], \qquad (1.10)$$

where

$$[n(v^i - \psi_k^i u^k - \varkappa(1-\beta^{-2})u^i)]_{,i} = 0.$$

In what follows, we shall be interested in the Cauchy problem for Eqs. (1.10).

§2. IRROTATIONAL PERTURBATIONS OF THE MATTER IN A FRIEDMANN COSMOLOGY: HAMILTONIAN FORMALISM

To quantize the irrotational perturbations (sound waves) in the Friedmann cosmology, it is necessary to separate the physical degrees of freedom, of which there are obviously two: For the second-order hyperbolic Eqs. (1.10) it is necessary to specify on the initial Cauchy hypersurface Σ two functions, for example, the velocity potential Φ and the density perturbation $\delta\varepsilon$ (the derivative of Φ along the normal to Σ).

We introduce a synchronous coordinate system in which the scalar $\varphi^{(0)}$ depends on the universal time t:

$$ds^2 = dt^2 - a^2(\delta_{\alpha\beta} + h_{\alpha\beta})dx^\alpha dx^\beta. \qquad (2.1)$$

Here, $\delta_{\alpha\beta}$ is the unit tensor, and the function $a = a(t)$ determines the expansion rate of the background space:

$$3\left(\frac{\dot{a}}{a}\right)^2 = \varepsilon, \quad -2\left(\frac{\dot{a}}{a}\right)^{\cdot} = \varepsilon + p, \qquad (2.2)$$

where the dot denotes differentiation with respect to t. For the unperturbed four-velocity and particle number density we obtain from (2.1) and (2.2)

$$u^i = \delta_0^i, \quad na^3 = \text{const}. \qquad (2.3)$$

For the perturbed hydrodynamic quantities in the first order in Φ we have

$$\frac{\delta p}{\varepsilon+p} = \frac{\delta w}{w} = \frac{\dot{\Phi}}{w} = \dot{v} - 3\beta^2 \frac{\dot{a}}{a} v, \quad \delta u^0 = 0, \quad u_\alpha = v_{,\alpha}, \qquad (2.4)$$

where $v = \Phi/w$. The small tensor $h_{\alpha\beta}$ in the Euclidean three-space $\mathbf{x} = \{x^\alpha\} \equiv (x, y, z)$ is determined by Eq. (2.1) up to a term of the form

$$f(t)\delta_{\alpha\beta}, \qquad (2.5)$$

which vanishes after redefinition of the constants of integration of Eqs. (2.2) for the function $a(t)$. The general form of $h_{\alpha\beta}$ (A and B are scalars and linear in Φ) is

$$h_{\alpha\beta} = A\delta_{\alpha\beta} + B_{,\alpha\beta}. \qquad (2.6)$$

(All operations with Greek indices are performed by means of $\delta_{\alpha\beta}$. For example,

$$u^\alpha = -a^{-2}v^{,\alpha}, \quad h_\alpha^\beta = A\delta_\alpha^\beta + B_{,\alpha}^{,\beta}$$

etc.)

Thus, we have expressed all the perturbed quantities in terms of the three scalars v, A, and B. The gauge freedom in the choice of these scalars is due to the ar-

bitrariness in the construction of the synchronous coordinate system (2.1) (Ref. 5):

$$\tilde{t} = t - \frac{1}{2}F, \quad \tilde{\mathbf{x}} = \mathbf{x} - \frac{1}{2}\left(\nabla F \int \frac{dt}{a^2} + \nabla H\right), \quad \nabla \equiv \left\{\frac{\partial}{\partial x^\alpha}\right\}, \quad (2.7)$$

where $F = F(\mathbf{x})$ and $H = H(\mathbf{x})$ are arbitrary functions of the spatial coordinates (F and ∇H are small quantities), and it has the form ($u_i dx^i = \text{inv}, g_{ik} dx^i dx^k = \text{inv}$)

$$\tilde{v} = v + \frac{1}{2}F, \quad \tilde{A} = A + \frac{\dot{a}}{a}F, \quad \tilde{B} = B + F \int \frac{dt}{a^2} + H. \quad (2.8)$$

It follows from the transformations (2.7) that the scalar $q = q(t, \mathbf{x})$,

$$q = 3\left(\frac{\dot{a}}{a}v - \frac{1}{2}A\right), \quad (2.9)$$

is gauge invariant (this means that q is a four-scalar; v, A, and B are three-scalars).

We now proceed as follows: Using some of the Eqs. (1.10), we express the original scalars v, A, and B in terms of the invariant scalar q, after which, substituting in (1.9), we find the Lagrangian $\tilde{L} = \tilde{L}(q)$ and an equation of motion of second order for the function q, which describes the evolution of the two physical degrees of freedom of the irrotational perturbations of the ideal fluid in the flat Friedmann model.

We rewrite Eqs. (1.10) in the coordinate system (2.1):

$$\delta G_0^0 = \delta \varepsilon = \frac{\dot{a}}{a}h - \frac{1}{a^2}\Delta A, \quad 2\delta R_0^0 = (1 + 3\beta^2)\delta\varepsilon = -\left(\dot{h} + 2\frac{\dot{a}}{a}h\right), \quad (2.10)$$

$$G_{0\alpha} = (\varepsilon + p)v_{,\alpha} = -A_{,\alpha}, \quad (2.11)$$

$$\delta\left(G_\alpha^\beta - \frac{1}{3}G_\gamma^\gamma\delta_\alpha^\beta\right) = 0 = -\frac{1}{2a^2}\left(C_{,\alpha\beta} - \frac{1}{3}\delta_{\alpha\beta}\Delta C\right) \quad (2.12)$$

where

$$h = 3A + \Delta B, \quad C = A - \frac{(Ba^3)^{\cdot}}{a}, \quad \Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

It follows from Eq. (2.12) that ΔC is an arbitrary function of the time, which, by virtue of (2.5), can be set equal to zero. Further, since the scalar B determines the perturbations of the metric only through the second derivatives $B_{,\alpha\beta}$, it is defined only up to an additive function linear in x. It follows from this that Eq. (2.12) has only the trivial solution

$$C = 0. \quad (2.13)$$

Equations (2.11) and (2.13) enable us to express the scalars v, A, and B directly in terms of the function q:

$$v = \frac{a}{3\dot{a}}q - \frac{1}{2}Q, \quad A = -\frac{\dot{a}}{a}Q, \quad B = \frac{1}{a^3}P - \frac{1}{a^2}Q, \quad (2.14)$$

where

$$Q = Q(t, \mathbf{x}) = \int \gamma q \, dt, \quad P = P(t, \mathbf{x}) = \int a\gamma q \, dt, \quad \gamma = \gamma(t) = 1 + p/\varepsilon.$$

The functions Q and P, which are integrals with respect to the universal time t, are defined up to arbitrary additive functions of the spatial coordinates. This arbitrariness in the definition of the function Q is an expression of the gauge freedom (2.8) and can be eliminated by going over to a suitable frame of reference. We shall consider more carefully the indeterminacy of the integral P.

Substituting Eqs. (2.14) in Eqs. (2.10), we obtain an expression for the perturbation of the matter density,

$$\frac{\delta\varepsilon}{\varepsilon} = \frac{\Delta P}{3\dot{a}a^2} - 3\frac{\dot{a}}{a}\gamma v, \quad (2.15)$$

and an equation that relates $\dot{q}$ to the Laplacian of the function P,

$$\gamma a^3 \dot{q} = \beta^2 \Delta P. \quad (2.16)$$

It follows from Eq. (2.16) that for $\beta \neq 0$ the integral $P(t, \mathbf{x})$ is determined from the known function $q(t, \mathbf{x})$ up to an additive harmonic function $P_0(\mathbf{x})$ of the spatial coordinates:

$$\Delta P_0 = 0. \quad (2.17)$$

Thus, Eqs. (2.14) enable us to find the functions v, A, and B from the given scalar $q(t, \mathbf{x})$ up to a particular solution that does not vanish under gauge transformations:

$$v = A = 0, \quad B = P_0 \int a^{-3} dt. \quad (2.18)$$

A solution of Eq. (2.17) satisfying the condition of uniform boundedness in the $\mathbf{x}$ space for quantities of the type $P_{0,\alpha\beta}P_0^{,\alpha\beta}$ [if this condition is not satisfied for the solution of the Cauchy problem, Eqs. (2.10)–(2.12) of linear perturbation theory are invalid] is the bilinear form

$$P_0 = a_{\alpha\beta}x^\alpha x^\beta, \quad a_{\alpha\beta} = \text{const} \quad (2.19)$$

with vanishing trace $a_\alpha^\alpha = 0$. The solutions (2.19) describe irrotational perturbations of infinite scale in the flat Friedmann model.[6] They are homogeneous and of the first type in the Bianchi classification; the only observational manifestation of them is a quadrupole anisotropy of the microwave background (see Appendix II). It follows from Eqs. (2.2) and (2.5) that the function A is defined up to a constant term. Hence, we obtain the general form of irrotational perturbations with infinite scale of variation in terms of the invariant scalar q:

$$q = \text{const.} \quad (2.20)$$

For $\beta = 0$, the function $P_0(\mathbf{x})$ is arbitrary, and the solution (2.18) determines a damped mode for dust. The general solution with $\beta = 0$ in a synchronous comoving frame ($v = 0$) has the form

$$q = q(\mathbf{x}), \quad Q = 2aq/3\dot{a}, \quad (2.21)$$

$$P = q \int a\gamma \, dt + P_0(\mathbf{x}), \quad \delta\varepsilon/\varepsilon = \Delta P/3\dot{a}a^2.$$

The growing mode for dust is completely specified by the scalar q, which is an arbitrary function of the $\mathbf{x}$ coordinates.

Thus, with allowance for what we have said above we may conclude that for $\beta \neq 0$ the scalar q describes the evolution of the two physical degrees of freedom of irrotational perturbations of the matter in the spatially flat Friedmann cosmological model. Differentiating (2.16), we obtain a second-order equation for the scalar field $q(t, \mathbf{x})$:

$$q'' + 2\frac{\alpha'}{\alpha}q' - \beta^2\Delta q = 0, \quad (2.22)$$

where the prime denotes the operation $a\partial/\partial t$ and $\alpha = (a/\beta)(\gamma/3)^{1/2}$. To solve this equation on the initial Cauchy hypersurface $t = \text{const}$, we must specify two arbitrary small functions q and $q' = \Delta P/3\alpha^2$ of the spatial coordinates [see (2.16), (2.14), and (2.15)].

Equation (2.22) for the scalar field q can be obtained by equating to zero the variation with respect to q of the action

$$W^{(2)}[q] = \int \tilde{L}a^3\, dt\, d^3x, \quad \tilde{L} = \frac{\alpha^2}{2a^4}(q'^2 - \beta^2 q_{,\alpha}q^{,\alpha}). \tag{2.23}$$

After the substitution of (2.14) in (1.9), fairly lengthy calculations lead to the conclusion that the Lagrangians L and $\tilde{L}$ differ only by a divergence term.

We denote by σ the gauge-invariant scalar canonically conjugate to q:

$$\sigma = \sigma(t,\mathbf{x}) = \partial a^3 \tilde{L}/\partial \dot{q} = \alpha^2 q' = \Delta P/3. \tag{2.24}$$

We can use σ to express the density perturbations [see (2.15)]:

$$\delta\varepsilon/\varepsilon = \gamma\delta n/n = \sigma/\dot{a}a^2 - 3\dot{a}\gamma v/a. \tag{2.25}$$

The constructed scalars q and σ describe the invariant variations of the velocity potential and the perturbations of the matter density in an expanding universe. Note that the invariant perturbations of dust $(\beta = 0)$ can be expressed in terms of q and σ. From Eqs. (2.21) and (2.25) we obtain a damped mode, $q = 0$, $\sigma = \sigma(\mathbf{x})$, and a growing mode, $q = q(\mathbf{x})$, $\sigma = \Delta q \int a\gamma dt/3$.

§3. SOUND QUANTA (PHONONS) AND THEIR CONFORMAL NONINVARIANCE IN AN ISOTROPIC UNIVERSE

The canonical quantization of the irrotational perturbations (sound waves) in an expanding universe is based on the Lagrangian L and uses the equal-time commutation relation for the canonically conjugate operators q and σ:

$$[q(t,\mathbf{x})\sigma(t,\mathbf{y})] = q\sigma - \sigma q = i\delta(\mathbf{x}-\mathbf{y}).$$

This relation is analogous to the commutation rule between the operators of the velocity potential and the density perturbation in a stationary medium,[1] the operators q (2.14) and σ (2.25) playing the corresponding parts in the nonstationary universe.

The concept of phonons (quasiparticles, sound quanta) arises when we go over to the Fock representation of the algebra of the q field [see Eqs. (2.22) and (3.1)[4]]. We introduce the space of the state vectors of the q field with definite number of phonons having three-momentum k. In this representation, the operator of the q field has the form $(k = |\mathbf{k}|)$

$$q = \frac{1}{\alpha(2\pi)^{3/2}} \int_{-\infty}^{\infty} d^3k\,(a_{\mathbf{k}}e^{i\mathbf{k}\mathbf{x}}v_k(t) + a_{\mathbf{k}}^+ e^{-i\mathbf{k}\mathbf{x}}v_k^*(t)), \tag{3.2}$$

where $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^+$ are the operators of annihilation and creation of phonons with momentum k, and they satisfy the Bose commutation relations [see (3.1)]:

$$[a_{\mathbf{k}_1}a_{\mathbf{k}_2}] = 0, [a_{\mathbf{k}_1} a_{\mathbf{k}_2}^+] = \delta(\mathbf{k}_1 - \mathbf{k}_2). \tag{3.3}$$

The basis of the Fock space consists of eigenvectors of the phonon number operator $N = a_{\mathbf{k}}^+ a_{\mathbf{k}}$. The functions $v_k = v_k(t)$ satisfy the equation

$$v_k'' + (\omega^2 - U)v_k = 0, \tag{3.4}$$

where $\omega = \beta k$ and $U = \alpha''/\alpha$ are functions of the time and are normalized by the condition

$$v_k v_k''^* - v_k^* v_k' = i. \tag{3.5}$$

The gauge invariance of the scalars q and σ makes it possible to rewrite Eqs. (2.22)–(3.5) in generally covariant form (see Appendix III).

An important property of Eq. (2.22), which describes the evolution of the sound waves in an expanding universe, is its conformal noninvariance with respect to the equation for propagation of sound in flat space-time. We introduce the new function $\bar{q} = \alpha q$. Then the equation for the gauge-invariant scalar $\bar{q}$ in the conformally Euclidean coordinates $(\eta, \mathbf{x})$, where $\eta = \int dt/\alpha$, has the form

$$\bar{q}'' - \beta^2 \Delta\bar{q} = \frac{\alpha''}{\alpha}\bar{q}. \tag{3.6}$$

It follows from this that sound waves in a nonstationary isotropic universe are conformally invariant only when $\alpha'' = 0$. A universe filled with ultrarelativistic particles with equation of state $p = \varepsilon/3$ has this property. In such a universe, the processes of nonadiabatic amplification of sound waves and the production of phonons are impossible. In all cases when $\alpha'' \neq 0$, spontaneous production of acoustic oscillations accompanies the process of cosmological expansion. This mechanism forms the initial spectrum of perturbations of the matter density in an isotropic universe. The long-wavelength fluctuations of the density may be the cause of the formation of galaxies and clusters of galaxies in the framework of adiabatic theory.[10]

The finding of the correlation functions of the density fluctuations, the metric, and other quantities requires the construction of the quantum propagator (Green's function) of the field q, which we define as follows:

$$K_0(x_1, x_2) = K_0(x_2, x_1) = \langle 0|T\bar{q}(x_1)\bar{q}(x_2)|0\rangle, \tag{3.7}$$

where T is the operator of time ordering:

$$Tq(x_1)q(x_2) = \begin{cases} q(x_1)q(x_2) & \text{for } t_1 > t_2, \\ q(x_2)q(x_1) & \text{for } t_1 < t_2. \end{cases} \tag{3.8}$$

This operation is generally covariant, since the commutator

$$iG(x_1, x_2) = [q(x_1)q(x_2)] \tag{3.9}$$

vanishes if the points x_1 and x_2 are separated by a spacelike interval. The averaging in Eq. (3.8) is performed with respect to the quantum ground state $|0\rangle$ of the field q, in which the quasiparticles of (3.2) are absent: $a_{\mathbf{k}}|0\rangle = 0$. Substituting (3.2) in (3.7), we obtain

$$K_0(x_1, x_2) = -\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} d^3k\, e^{i\mathbf{k}\Delta\mathbf{x}}\mu_k = \frac{1}{2\pi^2}\int_0^{\infty} k\, dk\, \mu_k \frac{\sin kr}{r}, \tag{3.10}$$

$$t = (t_1 + t_2)/2, \quad \tau = |t_1 - t_2|/2, \quad r = |\mathbf{x}_1 - \mathbf{x}_2|, \quad \mu_k = v_k(t+\tau)v_k^*(t-\tau).$$

We must here say something about the energy–momentum tensor of the acoustic oscillations of the matter density. In the preceding sections we constructed the Lagrangians L and $\tilde{L}$, by means of which we can obtain the field Eqs. (1.10) and (2.22). In the derivation of these Lagrangians, the background metric was assumed to be known: In L (1.9), we used the equations $G_{ik}^{(0)} = T_{ik}^{(0)}$ for the background metric, and in $\tilde{L}$ (2.23) the explicit form of the metric (2.1). For this reason, the

Lagrangians L and $\tilde{L}$ cannot in general be used to obtain the energy–momentum tensor $T_{ik}^{(2)}$ of the phonons, which describes the gravitational influence of the acoustic oscillations of the matter density on the background metric.[5] This tensor can be calculated by means of the Lagrangian $L^{(2)}$ obtained in Appendix I.

To calculate the effect of the back reaction of the produced phonons on the rate of the cosmological expansion, the tensor $T_{ik}^{(2)}$, averaged with respect to the Heisenberg state vector of the field q, must be substituted in the right-hand side of the Einstein equations describing the evolution of the background metric [see Eq. (2.2)].[6] In what follows, we shall not need the expression for $T_{ik}^{(2)}$, since to estimate the primordial spectrum of adiabatic perturbations it is sufficient to have the operators of the perturbations of the metric $h_{\alpha\beta}$ and the matter density $\delta\varepsilon/\varepsilon$ [see (2.6), (2.14), and (2.25)].

To conclude this section, we draw attention to the scale invariance of the constructed quantities. The background metric (2.1) is invariant under a change in scale of the spatial coordinates x and corresponding multiplication of the function $a(t)$ by a constant, under which the vector ax remains unchanged. Under such a transformation, the following quantities are invariant: $p, \varepsilon, v, A, Ba^2, h, q, \sigma a^{-3}, K_0 a^{-2}, ka^{-1}, \omega a^{-1}, \nu_k a^{1/2}, \hat{a}_k a^{-3/2}$. We shall use this later to choose a convenient normalization for the wave vector k.

§4. PRIMORDIAL SPECTRUM OF ADIABATIC PERTURBATIONS OF THE MATTER DENSITY

The conformal noninvariance of acoustic waves in an isotropic universe casts a new light on one of the basic problems concerning the origin of structure in the Universe (galaxies, groups and clusters of galaxies), namely, the problem of the initial conditions. Particularly attractive from this point of view is the idea of obtaining the initial spectrum of density fluctuations, which lead to the formation of galaxies in the later evolution, from an initially unperturbed, maximally symmetric state of the matter in an isotropic homogeneous universe. In this approach, finite fluctuations of the geometrical and hydrodynamic quantities arise in the process of the cosmological expansion as a result of nonadiabatic amplification of the zero-point vibrations of the phonon vacuum. The actual form of the resulting spectra must be calculated in each case in accordance with the particular model of the singularity that is used, for example, a model with a bounce, a model in which the equation of state of the matter changes during the expansion (as a result of the production of short-lived supermassive particles, phase transitions, etc.), or a model of an explosion from an initially stationary state of the universe, etc.

Below, we shall obtain the correlation functions of the main physical variables as a function of a single parameter β_k, which is the coefficient of nonadiabatic amplification calculated in the classical problem of scattering of a $\bar{q}$ wave [see (3.6)] by the effective potential $U = \alpha''/\alpha$.

Suppose that after a certain time $t_0 > 1$ [in the usual units, $t_0 > t_{Pl} = (8\pi G\hbar/c^5)^{1/2}$] the equation of state of the matter has the form $p = \varepsilon/3$. In this stage of the cosmological expansion $\alpha'' = 0$, and we can define the energy ground state $|\text{out}\rangle$ of the field q for $t > t_0$ as the state in which there are no phonons with energy $\omega > 0$ [see (3.2)]:

$$a_k|\text{out}\rangle = 0, \quad v_k = (2\omega)^{-1/2} e^{-i\omega\eta}, \tag{4.1}$$

where $\omega = k/\sqrt{3}$. We denote by $|\text{in}\rangle$ the Heisenberg state vector in which the field q is. In the limit $t \to -\infty$, this state $|\text{in}\rangle$ is the ground state in accordance with the cosmological hypothesis of a maximally symmetric, isotropic, and homogeneous initial state of the Universe. Mathematically, this can be expressed in the form $\bar{a}_k|\text{in}\rangle = 0$, where the operators $\bar{a}_k$ and $\bar{a}_k^+$ are operators of creation and annihilation of phonons only as $t \to -\infty$. The normal modes $\bar{v}_k$ corresponding to the $\bar{a}_k$ quasiparticles [see (III.3)–(III.5)] for $t > t_0$ have the form

$$\bar{v}_k = \alpha_k v_k + \beta_k v_k^*, \quad |\alpha_k|^2 - |\beta_k|^2 = 1, \tag{4.2}$$

where α_k and β_k are constants determined in the solution of Eq. (3.6) with the effective potential $U(t)$ corresponding to the employed model of the cosmological singularity. The operators a_k and $\bar{a}_k$ are related by the Bogolyubov transformation $a_k = \alpha_k \bar{a}_k + \beta_k^* \bar{a}_{-k}^+$.

The correlation functions quadratic in the field operator $q(x)$ in which we are interested, taken at different points x_1 and x_2 and averaged with respect to the field state $|\text{in}\rangle$, can be calculated by means of the function $K_{\text{in}}(x_1, x_2)$ [see (3.7)]. They diverge in the limit $x_1 \to x_2$ of coincident points because of the unphysical contribution of the zero-point vibrations of the "vacuum" (i.e., the state $|\text{out}\rangle$ in which phonons are absent) in the stage $t > t_0$. This divergence is eliminated by subtracting from the function $K_{\text{in}}(x_1, x_2)$ the "vacuum" Green's function in this stage

$$K_{\text{out}}(x_1, x_2) = -\frac{\sqrt{3}}{4\pi^2}(s^{-2} + i\pi\delta(s^2)),$$

the conformal interval $s^2 = (\eta_1 - \eta_2)_3^2 - r^2$ vanishing on the sound cone. The real symmetric function

$$T(x_1, x_2) = T(x_2, x_1) = K_{\text{in}}(x_1, x_2) - K_{\text{out}}(x_1, x_2)$$

is finite for $x_1 \to x_2$. Substituting here Eqs. (4.1) and (4.2), we obtain [see (3.10)]

$$T(x_1, x_2) = \frac{\sqrt{3}}{2\pi^2} \int_0^\infty dk \frac{\sin kr}{r} \{|\beta_k|^2 \cos(\omega\Delta\eta) + \text{Re}(\alpha_k \beta_k^* e^{-2i\omega\eta})\}. \tag{4.3}$$

The first term in the integral depends on the time distance $\Delta\eta = \eta_1 - \eta_2$ between the points x_1 and x_2 and determines the contribution to the function $T(x_1, x_2)$ of real phonons. The number density of the phonons in the interval of momenta d^3k is proportional to $(2\pi a)^{-3} \times |\beta_k|^2 d^3k$. The second term determines the nonlocal part of the vacuum polarization $[\eta = (\eta_1 + \eta_2)/2]$, and it is linear in β_k. For perturbations that are below the sound horizon $(\omega\eta \gg 1)$, the contribution of the second term to $T(x_1, x_2)$ is exponentially small.

To obtain the correlation functions of the fluctuations of the density, metric, and other quantities, it is necessary to carry out the corresponding operations of dif-

ferentiation and integration of the function $T(x_1, x_2)$ at the points x_1 and x_2 [see (3.7)]. If in the final result we go to the limit $\Delta\eta = 0$ or $\Delta\eta = \gamma = 0$, we obtain, respectively, the correlation functions on the section $t = \text{const}$ and the mean squares of the fluctuating physical quantities at the point x.

We rewrite Eqs. (2.14) and (2.25) in the stage $t > t_0$ $(a = t^{1/2} = \eta/2)$ in a more convenient form, eliminating the gauge degrees of freedom and the infinite mode (2.18), which are not quantized:

$$\alpha = -\frac{2a}{\sqrt{3}}, \quad \bar{q} = aq, \quad Q = -\frac{2}{\sqrt{3}}\int \bar{q}\,d\eta, \quad Pa^{-1} = Q - \frac{1}{\eta}\int Q\,d\eta,$$

$$v = \frac{2}{3}tq - \frac{1}{2}Q, \quad A = -\frac{1}{2t}Q, \quad B = -2\int\frac{d\eta}{\eta^2}\left(\int Q\,d\eta\right), \quad (4.4)$$

$$\sigma = \alpha\left(\bar{q}' - \frac{1}{\eta}\bar{q}\right), \quad \Delta P = 3\sigma, \quad \frac{\delta\varepsilon}{\varepsilon} = \frac{4}{\sqrt{3}}\bar{q}' - \frac{8}{3}q + \frac{1}{t}Q.$$

We then obtain for the mean squares of the perturbations at the point x [we retain below only the first term in (4.3), since it makes the main contribution to $T(x_1, x_2)$ for $\eta \gg k^{-1}$]

$$\langle\bar{q}^2\rangle = \frac{\sqrt{3}}{2\pi^2}\int_0^\infty k\,dk|\beta_k|^2, \quad \langle Q^2\rangle = \frac{2\sqrt{3}}{\pi^2}\int_0^\infty\frac{dk}{k}|\beta_k|^2,$$

$$\langle\bar{q}'q\rangle = \langle Qq\rangle = 0, \quad \langle\bar{q}'^2\rangle = \frac{1}{2\sqrt{3}\pi^2}\int_0^\infty k^3\,dk|\beta_k|^2, \quad (4.5)$$

$$\langle v^2\rangle = \frac{1}{3}t\langle\bar{q}^2\rangle + \frac{1}{4}\langle Q^2\rangle, \quad \left\langle\left(\frac{\delta\varepsilon}{\varepsilon}\right)^2\right\rangle = \frac{16}{3}\langle\bar{q}'^2\rangle + \frac{1}{t^2}\langle Q^2\rangle,$$

$$\langle A^2\rangle = \frac{1}{4t^2}\langle Q^2\rangle, \quad \langle\bar{q}'Q\rangle = -\frac{2}{\sqrt{3}}\langle\bar{q}^2\rangle, \quad \langle(\Delta B)^2\rangle = \frac{3}{t^3}\langle\bar{q}^2\rangle,$$

etc. The correlation functions on the section $t = \text{const}$ are obtained from Eqs. (4.5) by introducing the factor $\sin(kr)/kr$ in the spectral integrals. The long-wavelength density fluctuations due to the contribution of the produced phonons and which subsequently lead to the formation of galaxies are formed during the hydrodynamic time $\eta \sim \omega^{-1}$, during which the sound traverses a distance equal to the diameter of the perturbation [it is only for $\omega\eta \gg 1$ that the nonlocal contribution of the vacuum polarization can be ignored in $T(x_1, x_2)$ (4.3)]. From (4.5), we obtain for such perturbations

$$\left\langle\left(\frac{\delta\varepsilon}{\varepsilon}\right)^2\right\rangle = \pi^{-2}\left(\frac{4}{3}\right)^{1/2}\int_{k_D}^\infty k^3\,dk|\beta_k|^2. \quad (4.6)$$

The wave vector k is normalized as follows: at the time t for perturbations that have the Jeans scale $(\omega_D\eta = 1)$, $k_D = (3/4t)^{1/2}$.

The coefficient β_k depends on the evolution of the cosmological model in the stage $t < t_0$ and will be given for different models in a following paper.[7] The most optimistic estimate is obtained in an initially stationary cosmological model. In this model, the Friedmann universe $(a = t^{1/2})$ arises as a result of an explosion from an initially stationary state with constant space-time curvature $H \lesssim 1$ $(a \sim e^{Ht}, t < t_0 = 1/2H)$.[13] Approximation of this model (taking into account the contribution of conformal fields in the single-loop approximation) in the language of the Einstein equations gives a rough estimate for the mean square fluctuations of the density in the

long-wavelength region (the state $|\text{in}\rangle$ is selected by the condition that in the stage $\beta^2 = -1$ there should be no exponentially increasing solutions as $t \to -\infty$):

$$\left\langle\left(\frac{\delta\varepsilon}{\varepsilon}\right)^2\right\rangle \sim \frac{H^2}{6\pi^2}\int\frac{dk}{k}.$$

Thus, when the radiation-dominated stage of the expansion ends the density perturbations on the scale $k_D \ll k \ll \sqrt{H}$ have a flat spectrum.[8] If the fundamental dimensionless constant H is not too small, these perturbations are responsible for the formation of galaxies.[10,14] Note that our estimate must be made more precise, since the field equations near the singularity differ from the Einstein equations. However, the fundamental cause of the initial perturbations of the matter density— the conformal noninvariance of irrotational perturbations in an isotropic universe—does not depend on the modification of the equations at Planck curvatures.

APPENDIX I

Assuming that Eqs. (1.8) are exact, we expand the following quantities up to second order in Φ and h^{ik}:[9]

$$\frac{\delta w}{w^{(0)}} = \frac{1}{w^{(0)}}\delta(\varphi_{,i}\varphi_{,k}g^{ik})^{1/2} = \varkappa - \frac{1}{2}\varkappa^2 - v_i h^{ik}u_k + \frac{1}{2}v_i v^i, \quad (I.1)$$

$$\frac{\delta p}{\varepsilon^{(0)} + p^{(0)}} = \frac{\delta w}{w^{(0)}} + \frac{1}{2\beta^2}\left(\frac{\delta w}{w^{(0)}}\right)^2, \quad (I.2)$$

$$\delta g_{ik} = h_{ik} + h_i^l h_{lk}, \quad \delta\sqrt{-g} = \sqrt{-g^{(0)}}\left(\frac{1}{2}h + \frac{1}{4}h_{ik}h^{ik} + \frac{1}{8}h^2\right), \quad (I.3)$$

$$2\delta\Gamma_{kl}^i = h_{k;l}^i + h_{l;k}^i - h_{kl}^{;i} + h_k^m h_{m;l}^i + h_l^m h_{m;k}^i + h_m^i h_{kl}^{;m} - (h_{km}h_l^m)^{;i}, \quad (I.4)$$

$$\delta R_{ik} = (\delta\Gamma_{ik}^l)_{;l} - (\delta\Gamma_{il}^l)_{;k} + (\delta\Gamma_{ik}^l)(\delta\Gamma_{lm}^m) - (\delta\Gamma_{il}^m)(\delta\Gamma_{km}^l), \quad (I.5)$$

where

$$\delta f = f - f^{(0)}, \quad v_i = \Phi_{,i}/w^{(0)}, \quad \varkappa = v^i u_i^{(0)} - u_i^{(0)}u_k^{(0)}h^{ik}/2.$$

All the operations are performed in the background metric $g_{ik}^{(0)}$. [In deriving (I.3), we used the exact equation $d\ln(-g) = g^{ik}dg_{ik}$, from which it follows that $\ln(g/g^{(0)}) = h + h_k^i h_i^k/2$.]

Substituting (I.1)–(I.5) in Eq. (1.7), we obtain

$$\left(p - \frac{1}{2}R\right)\left(\frac{g}{g^{(0)}}\right)^{1/2}$$
$$= p^{(0)} - \frac{1}{2}R^{(0)} + \frac{1}{2}h^{ik}(G_{ik}^{(0)} - T_{ik}^{(0)}) + (-g^{(0)})^{-1/2}S_{,i}^i + L^{(2)},$$

$$S^i = \frac{1}{2}\sqrt{-g}(g^{il}\delta\Gamma_{lk}^k - g^{ik}\delta\Gamma_{lk}^l + 2\Phi n^{(0)}u^{i(0)}); \quad (I.6)$$

$$L^{(2)} = \frac{\varepsilon^{(0)} + p^{(0)}}{2}(v_i v^i - 2v_i\psi_k^i u^k - \varkappa^2(1-\beta^{-2}))$$
$$+ \frac{1}{4}\left(p^{(0)} - \frac{1}{2}R^{(0)}\right)\left(\psi_{ik}\psi^{ik} - \frac{1}{2}\psi^2\right)$$
$$+ \frac{1}{8}\left(\psi_{ik;l}\psi^{ik;l} - 2\psi_{ik;l}\psi^{il;k} - \frac{1}{2}\psi_{,l}\psi^{,l}\right) + \frac{1}{4}hh^{ik}(G_{ik}^{(0)} - T_{ik}^{(0)}). \quad (I.7)$$

In obtaining the field Eqs. (1.10) from $L^{(2)}$, we do not vary the background metric. Assuming that it is known and substituting $G_{ik}^{(0)} = T_{ik}^{(0)}$ in $L^{(2)}$, we obtain the Lagrangian L (1.9).

APPENDIX II

In this appendix, we consider irrotational perturbations of infinitely large scale in the spatially flat Friedmann cosmological model. It follows from (2.14)–(2.19) that there always exists a coordinate system in which the solution (2.18) has the form

$$v=0, \qquad h_{\alpha\beta}=2a a_{\alpha\beta}\int a^{-3}\,dt, \qquad \text{(II.1)}$$

where $a_{\alpha\beta}=\text{const}, a_{\alpha}^{\alpha}=0 \;(h=0)$. We shall say that this coordinate system is homogeneous. It follows from (II.1) that the invariant perturbations of the hydrodynamic variables in the homogeneous system are zero, $\delta\varepsilon=u_{\alpha}=0$. Obviously, a linear superposition of perturbations of infinite scale is also a solution of (II.1).

The expressions in (II.1) are an asymptotic expansion (for large t) linear in $a_{\alpha\beta}$ of the exact solution

$$ds^2=dt^2-g_{\alpha\beta}\,dx^{\alpha}\,dx^{\beta}, \quad \{g_{\alpha\beta}\}=a^2\exp\left(2\bar{A}\int\frac{dt}{a^3}\right), \qquad \text{(II.2)}$$

which is completely defined by the constant 3×3 matrix

$$\bar{A}=\{a_{\alpha\beta}\}, \qquad \text{Sp}(\bar{A})=0. \qquad \text{(II.3)}$$

The function $a=a(t)$ is found from the equations

$$3\left(\frac{\dot{a}}{a}\right)^2=\varepsilon+\frac{1}{2a^6}\text{Sp}(\bar{A})^2, \quad -2\left(\frac{\dot{a}}{a}\right)^{\cdot}=\varepsilon+p+\frac{1}{a^6}\text{Sp}(\bar{A})^2, \quad \text{(II.4)}$$

which are identical to Eqs. (2.2) in the approximation linear in $\bar{A}$. As $t\to0$, we have

$$a^3=\Lambda t, \qquad \Lambda=\left(\frac{3}{2}\text{Sp}(\bar{A})^2\right)^{1/2}=\text{const.} \qquad \text{(II.5)}$$

The asymptotic behavior (II.5) describes the Kasner singularity, and the principal values of the metric tensor $g_{\alpha\beta}$ are proportional to t^{2p_a}, where p_a are the Kasner indices $(p_1+p_2+p_3=p_1^2+p_2^2+p_3^2=1)$. Obviously, the eigenvalues λ_a of the matrix $\bar{A}$, which can be found from the equation

$$\det\{a_{\alpha\beta}-\lambda\delta_{\alpha\beta}\}=0,$$

are related to the p_a by the simple equation

$$\lambda_a=\Lambda(p_a-1/3). \qquad \text{(II.6)}$$

The solution (II.2)–(II.4) describes a cosmological model of the first Bianchi type with comoving Euclidean three-space. Note that irrotational perturbations of infinite scale in the spatially flat Friedmann model, which give rise to anisotropy in the deformation of the expansion, do not perturb the spatial curvature. Perturbations of vortex and gravitational-wave type of infinite scale[6] also lead to the solution (II.2)–(II.4).

APPENDIX III

From the quantities that characterize the flat Friedmann model we construct the symmetric tensor

$$D_{ik}=\gamma(\beta^{-2}u_iu_k+P_{ik})/3, \qquad \text{(III.1)}$$

where $P_{ik}=g_{ik}-u_iu_k$ is a projection tensor. Obviously, the background scalars $(\beta,\gamma,u^i{}_{;i}, \text{etc.})$ depend as functions of the coordinates x^i on the universal time $t=\int u_i\,dx^i=\int d\varphi/w(\varphi)$; all the other background quantities can be expressed in terms of these scalars, the metric g_{ik}, and the vector u^i (for example, $u_{i;k}=u^l{}_{;l}P_{ik}/3, P^l_{i;l}=-u_iu^l_{;l}$ etc.). Using D_{ik}, we can rewrite

Eqs. (2.22) and (2.23) in the generally covariant form

$$\hat{L}(q)=D^{ik}q_{,i}q_{,k}/2, \qquad (D^{ik}q_{,i})_{;k}=0. \qquad \text{(III.2)}$$

The relations (3.1)–(3.5) are generalized for arbitrary Cauchy hypersurface by means of a bilinear form composed of any two (classical) solutions q_1 and q_2 of Eq. (III.2):

$$J_{12}{}^i=J_{21}{}^{i*}=iD^{ik}(q_1^* q_{2,k}-q^*_{1,k}q_2). \qquad \text{(III.3)}$$

By virtue of the conservation law $J^i_{12;i}=0$, the integral

$$(q_1q_2)=\int_{\Sigma} J_{12}{}^i\,d\Sigma_i, \qquad \text{(III.4)}$$

where $d\Sigma^i$ is an invariant measure on Σ, does not depend on the choice of the Cauchy hypersurface Σ. For $q_k=\nu_k(t)e^{i\mathbf{k}\mathbf{x}}/\alpha(2\pi)^{3/2}$ [see (3.2)] we have

$$(q_kq_{k'})=\delta(\mathbf{k}-\mathbf{k}'), \qquad (q_k^*q_{k'}^*)=-\delta(\mathbf{k}-\mathbf{k}'), \qquad \text{(III.5)}$$

The functions q_k and q_k^* determine a basis of the Hilbert space of all classical solutions of Eq. (III.2) and divide this space into two subspaces: the positive-frequency (q_k) and negative-frequency (q_k^*).

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[1] The worldline of a particle (the element) of the matter is a matter tube whose walls are frozen into the matter. The indeterminacy of the amount of matter that forms the element is expressed mathematically in the fact that the potential φ and, therefore, the functions w and n^{-1} are defined up to multiplication by a constant; p, ε, β, and the other hydrodynamic variables do not have this ambiguity.

[2] These directions are tangent to the surfaces = const. In the regions of space-time where $w\neq0$, one can always introduce a comoving coordinate system with synchronous time φ:

$$ds^2=d\varphi^2/w^2-\gamma_{\alpha\beta}dx^{\alpha}dx^{\beta}, \quad u^i=w\delta_0^i.$$

[3] Note that the condition (1.1) may be violated on a shock front. We do not consider such processes here.

[4] In this case, the quantization procedure is analogous to quantization of a real scalar field in an isotropic universe.[7,8,9] Note that in the ideal-fluid approximation dust $(\beta=0)$ is always classical: Constancy of the pressure (and the temperature) when the matter density changes means that thermal excitations (phonons) are absent.

[5] An error at this place in the paper of Ford and Parker[11] in their derivation of the energy–momentum tensor of gravitational waves was pointed out by A. D. Popova.

[6] The energy–momentum tensor is regularized by the standard methods developed for free quantum fields in curved space-time.

[7] Note that because Eq. (2.22) has the same form as the corresponding equation for the amplitude of gravitational waves in an isotropic universe it is possible to extend to the case of phonons the main conclusions for the long-wavelength part of the spectrum from Refs. 11 and 12 (for $\beta=\text{const}, U=\alpha''/\alpha =a''/a$).

[8] This spectrum explains naturally the homogeneity of the Universe on large scales, since the amplitude of perturbations for $k<k_D$ decreases when their scale increases (Refs. 2, 5, 10, and 14) and these perturbations do not succeed in growing in the $p=0$ stage. Note that in the conversion to perturbations of the metric $(t\sim t_0)$ the gravitational waves also have a flat spectrum.[15]

[9] All the notation for the metric quantities, including the Ricci tensor R_{ik} and the Einstein tensor $G_{ik}=R_{ik}-Rg_{ik}/2$, corresponds to the notation adopted in Ref. 5.

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On the connection between the size of the Universe and its curvature

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Closed three-dimensional Riemannian spaces with curvature that is constant in all directions are considered. It is shown that the topological structure of any such space uniquely determines the sign of its curvature $\mathscr{K}$ and also restrictions on its size. Let R be the radius of curvature and D the diameter of the space, i.e., the distance between its most widely separated points. Then for $\mathscr{K} > 0$ one finds $D > 0.326R$, and for $\mathscr{K} < 0$ apparently $D > 1.128|R|$; for $\mathscr{K} = 0$, the value of D is arbitrary. Further, Einstein's equations and astronomical data indicate that the modulus of the present-day radius of curvature of the Universe satisfies $|R| > 0.5(c/H_0) \approx 0.9 \times 10^{28}$ cm, where c is the velocity of light, and H_0 is the Hubble constant. Therefore, if observations show that the diameter of the Universe is $D_0 < 10^{28}$ cm, this will mean that as a whole our Universe is flat ($\mathscr{K} = 0$). A model of a flat world is proposed which is closed in the form of a three-dimensional torus; all of its parameters (size, rate of expansion, mean matter density, etc.) are expressed in terms of atomic constants and a universal time. In this model, the present-day diameter of the Universe is $D_0 = 0.102(c/H_0) \approx 2 \times 10^{27}$ cm, which does not contradict observational data.

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§1. INTRODUCTION

The aim of the present paper is to establish some connections between local and global properties of the Universe. The problem is analyzed on the basis of Einstein's general theory of relativity in the framework of locally isotropic and homogeneous cosmological models. It is well known that the exceptional isotropy of the cosmic microwave background enables one in conjunction with the "generalized Copernican principle" (i.e., a terrestrial observer is not distinguished) to introduce a universal time and three-dimensional space orthogonal to it, this space having at any time constant curvature in all directions.[1-3] We recall that in accordance with Schur's theorem local isotropy entails local homogeneity, namely, if at every point of a Riemannian manifold the curvature has the same value in all directions, then it also has a constant value as one moves from point to point.

In the construction of cosmological models based on three-dimensional spaces of constant curvature, physicists usually employ three degenerate types of space: Euclidean space E^3, Lobachevskiĭ space L^3, and the sphere S^3.[1-4] But the general classification of three-dimensional spaces of constant curvature gives 18 topologically different types of space with curvature $k = 0$ and an infinite number of topological types with $k = -1$ and $k = +1$ (Ref. 5). All three-dimensional spaces with $k = +1$ are closed and orientable; among the flat three-dimensional spaces there are ten types, which are closed (six are orientable[1)]) and eight types which are open (four of them orientable); the spaces with $k = -1$ contain an infinite number of closed types and an infinite number of orientable types.

The spaces of the types E^3, L^3, and S^3 are distinguished in this complete set by the fact that they are the only ones that are topologically simply connected. Therefore, it is only in them that each celestial object can be observed only in one direction at a particular stage of expansion of the world.[2)]

The multiply connected spaces of constant curvature can be obtained formally by specifying in E^3, L^3, and S^3 certain nonclosed manifolds (fundamental regions) whose boundaries are identified (or "glued") in accordance with definite laws (see Refs. 2 and 9; the precise formulations can be found in Appendix A). Then each light source can be joined to an observer by several geodesics

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