Motion of a linear vortex singularity

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A formula is obtained for the velocity of a certain element of a linear vortex in a Bose gas and in helium II in terms of the contribution made to the wave function by the remaining elements of the same wave vortex and of other perturbations of the ground state of the medium. It is shown that, besides the Magnus force, the vortex is acted upon by a force proportional to the condensate-density gradient and directed opposite to the gradient. A generalization is possible to linear vortex singularities of relativistic fields that do not describe any condensed medium at all.

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1. In the hydrodynamics of a classical ideal incompressible liquid, the velocity of a free (unpinned) vortex is equal to the velocity of the liquid at the vortex point less its own (unpinned) contribution—the vortex is dragged by the stream. It is natural to assume that in more complicated cases, for example quantum liquids, it is also possible to establish a general law for the vortex velocity in terms of the characteristic states of the liquid, minus the contribution of the vortex itself (more accurately of that vortex element whose velocity is being determined). We confirm this assumption in Sec. 2, using as the example a degenerate weakly interacting Fermi gas. In Secs. 3 and 4 we consider vortex motion in helium II, in Sec. 5 we present some concrete examples, and in Sec. 6 we describe vortex singularities of complex scalar fields that do not necessarily describe the density and velocity distribution in some condensed medium.

The determination of the motion of the vortex in quantized liquids is considered in the present article as a particular case of a more general problem of defining the motion of the singularity of a certain field by the equation of this field. It is appropriate to mention in this connection that in investigations devoted to the derivation of the law of particle motion from the equations of a gravitational field, Einstein referred to Helmholtz's theory of vortex motion as an example of a field specifying the motion of its own singularity.

2. A vortex in a degenerate weakly interacting Bose gas was first considered by Pitaevskiĭ,[1] who used the Gross-Pitaevskiĭ equation[2),1] for the condensate wave function $a_0 \equiv \psi e^{-t/2}$. We write this equation in the dimensionless form[1)]

$$2i\,\partial\psi/\partial t+\Delta\psi+\psi-|\psi|^2\psi=0. \tag{1}$$

The function ψ is complex, $\psi \equiv f\exp(i\varphi)$, and its modulus describes the density of the condensate ($f^2=n/n_0$), and the phase its flow velocity ($\mathbf{v}=\nabla\varphi$).

Equation (1) has a solution that describes an isolated straight vortex line in an infinite medium[1]:

$$\psi_L=f_L\exp(i\varphi_L),$$

where $\varphi_L=n\alpha$, and $f_L(r)$ is the solution of the equation

$$\frac{d^2f_L}{dr^2}+\frac{1}{r}\frac{df_L}{dr}+f_L-f_L^3=\frac{n^2}{r^2}f_L \tag{2}$$

subject to the boundary condition $f_L(\infty)=1$. We have used here the cylindrical coordinates r, α, and z, with the z axis aligned with the vortex; n is the number of circulation quanta. As $r\to 0$ we have

$$f_L\to\text{const}\, r^{|n|}. \tag{3}$$

A natural generalization of the function ψ_L is the wave function ψ_v that describes an isolated straight vortex moving uniformly (perpendicularly to itself) in an infinite medium that is immobile at infinity:

$$\begin{gathered}\psi_v=f_v\exp(i\varphi_v),\\ \varphi_v=n\alpha_v,\quad \alpha_v=\operatorname{arc\,tg}\frac{y-v_yt}{x-v_xt},\\ f_v=f_L(r_v),\quad \mathbf{r}_v=\mathbf{r}-\mathbf{v}t,\end{gathered} \tag{4}$$

$\mathbf{v}\equiv(v_x, v_y)$ is the vortex velocity. Using the definition (4) and Eq. (2), we easily verify that the function ψ_v satisfies the following equation[2)]:

$$2i\frac{\partial\psi_v}{\partial t}+\Delta\psi_v+\psi_v-|\psi_v|^2\psi_v=-2i\mathbf{v}\nabla\psi_v. \tag{5}$$

We consider now a state of the medium such that it contains a vortex filament (a linear vortex singularity[3)]), but the filament is not generally speaking a straight line and (or) there are other factors that cause a difference between ψ and ψ_L. Examples are other singularities that are spatially separated from the considered vortex, boundary surfaces, flow at infinity, and others. We chose a certain vortex-filament element (sometimes called for brevity a vortex point or simply a vortex), and choose a reference frame in which the z axis is directed along this element and the origin coincides with it. Without loss of generality we can express ψ in the form

$$\psi=\psi_v\tilde\psi=\exp(\ln f_v+\ln\tilde f+in\,\alpha_v+i\tilde\varphi), \tag{6}$$

where $\tilde\psi$ has by definition no singularity in the considered vortex point and is a solution of Eq. (1) from which the contribution of this point is separated (multiplicatively for ψ and additively from the complex phase $\ln f+i\varphi$; we note that $\tilde\psi$ is not a solution in the absence of the vortex).

Substituting expression (6) in Eq. (1) and taking (5) into account, we obtain

$$2(\nabla\tilde\psi-i\mathbf{v}\tilde\psi)\cdot\nabla\psi_v+\psi_v\left[2i\frac{\partial\tilde\psi}{\partial t}+\Delta\tilde\psi+|\psi_v|^2\tilde\psi(1-|\tilde\psi|^2)\right]=0. \tag{7}$$

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We investigate this equation in the vicinity of the vortex point. Using the property (3) of the function f_L and the absence of singularities of the function $\tilde{\psi}$ at $r=r_v$, we can easily verify that when the left-hand side of (7) is expanded in powers of r_v the principal terms stem from the first member of the series. The real part of the expansion is

$$\tilde{f}_0 r_v^{|n|-2}\,\mathbf{r}_v\{|n|\nabla\ln\tilde{f}+n[\mathbf{k}\times,\nabla\tilde{\varphi}-\mathbf{v}]\}_0+\ldots=0, \qquad (7a)$$

and the imaginary part

$$\tilde{f}_0 r_v^{|n|-2}\,\mathbf{r}_v\{-n[\mathbf{k}\times,\nabla\ln\tilde{f}]+|n|(\nabla\tilde{\varphi}-\mathbf{v})\}_0+\ldots=0, \qquad (7b)$$

where $\mathbf{k}$ is the unit vector of the z axis, and the dots stand for terms proportional to r_v raised to a power not lower than $|n|$, while the subscript 0 means that the expression in the curly brackets is calculated at the point $r_v=0$.

We stipulate that expression (6) be a regular solution of Eq. (7) in the vicinity of the vortex point, i.e., that Eqs. (7a) and (7b) be satisfied regardless of the direction of $\mathbf{r}_v$. This calls for the vanishing of the expressions in the curly brackets, which is equivalent to an equation that determines the vortex velocity

$$\mathbf{v}=\mathbf{\Phi}-\frac{n}{|n|}[\mathbf{k}\times\mathbf{G}], \qquad (8)$$

$$\mathbf{\Phi}=(\partial\tilde{\varphi}/\partial x,\ \partial\tilde{\varphi}/\partial y)_0,\qquad \mathbf{G}=(\partial\ln\tilde{f}/\partial x,\ \partial\ln\tilde{f}/\partial y)_0.$$

If $\Delta\tilde{f}=0$ at the vortex point, then $\mathbf{G}=0$ and formula (8) goes over into the expression $\mathbf{v}=\mathbf{\Phi}$ known from classical hydrodynamics of an ideal incompressible liquid. At $[\mathbf{k}\times\nabla\tilde{\varphi}]_0=0$, in contrast to the case of an incompressible liquid, $\mathbf{v}\neq 0$ and the vortex velocity is $v=-(n/|n|)\mathbf{k}\times\mathbf{G}$ perpendicular to the condensate-density gradient.

The physical meaning of formula (8) is clearer if it is expressed in the form

$$n[\mathbf{k}\times\mathbf{v}-\mathbf{\Phi}]-|n|\mathbf{G}=0, \qquad (8a)$$

which can be intepreted as the vanishing of the sum of the forces acting on the vortex point, namely the Magnus force[4] and the force proportional to $|n|\mathbf{G}$. The existence of the last force, directed opposite to the condensate-density gradient (from which the contribution of the considered vortex element is excluded), is physically due to the fact that the vortex whose presence generates f produces a small perturbation of the state of the medium in the region of small $\tilde{f}$.

3. We consider now the motion of vortices in superfluid helium II. We use for this purpose the Ginzburg-Pitaevskii equation[3]

$$(\nabla-i\mathbf{v}_n)^2\psi+\psi-|\psi|^2\psi=0. \qquad (9)$$

The wave function ψ describes here the density of the superfluid component and its velocity:

$$\psi=fe^{i\varphi},\quad f^2=\rho_s/\rho_{sb},\quad \nabla\varphi=\mathbf{v}_s,$$

ρ_{sb} is the equilibrium value of ρ_s. Equation (9) is written in the following units: $(\rho_{sb}/m)^{1/2}$ for the wave function (m is the helium-atom mass), the coherence length ξ for the distances, and $\hbar/m\xi$ for the velocity.

In contrast to the time-dependent equation (1), expression (9) is a balance equation and is valid only in the absence of dissipative processes. The vortex velocity should therefore coincide with the velocity of the normal component at the vortex point: $\mathbf{v}=\mathbf{v}_{n0}$. At the same time, Eq. (9) can be subjected to the procedure employed in Sec. 2 to investigate Eq. (1) in the vicinity of a vortex point. It is easy to verify that the result of the requirement that Eq. (9) be regular in the vicinity of such a point is an equation of the type (8) for $\mathbf{v}_n$. Thus, equilibrium (nondissipative) motion of the vortex in helium II takes place at a velocity

$$\mathbf{v}=(v_{nx},v_{ny})_0=\mathbf{\Phi}-\frac{n}{|n|}[\mathbf{k}\times\mathbf{G}]. \qquad (10)$$

Actually, however, the velocity of the normal component is set by the boundary conditions and generally speaking does not coincide with the right-hand side of (10). In those cases when this equation does not hold, the balance equation (9) is not satisfied and the vortex motion is not in equilibrium. It is in equilibrium only for a restricted number of boundary conditions, and furthermore if the vortices are located at points where the normal flow velocity coincides with the equilibrium vortex velocity determined by Eq. (10).

4. The nonequilibrium motion of the vortex can be determined by the time-dependent equation for the ψ function.[4] The requirement that this equation be regular in the vicinity of the vortex point leads to the formula

$$\mathbf{v}=\mathbf{\Phi}-\frac{n}{|n|}[\mathbf{k}\times\mathbf{G}]-\Lambda\left(\mathbf{G}+\frac{n}{|n|}[\mathbf{k}\times\mathbf{\Phi}-\mathbf{v}_{n0}]\right), \qquad (11)$$

where Λ is a coefficient connected in a known manner with the relaxation time of the order parameter ψ. From the point of view from the requirement that the sum of the forces acting on the vortex be zero, Eq. (11) can be supplemented, in comparison with formula (8a), by dissipative forces and can be rewritten in the form

$$n[\mathbf{k}\times\mathbf{v}-\mathbf{\Phi}]-|n|\mathbf{G}+\Lambda(n[\mathbf{k}\times\mathbf{G}]-|n|[\mathbf{k}\times[\mathbf{k}\times\mathbf{v}_{n0}-\mathbf{\Phi}]])=0. \qquad (11a)$$

Let us compare it with an equation having an identical meaning, derived for vortex dynamics in helium II by Hall and Vinen (see, e.g., Refs. 5–7). The forces proportional to the gradient of $\tilde{f}$ (i.e., to the gradient of ρ_s with the contribution of the vortex itself subtracted) do not appear in the formulas of Hall and Vinen. This is natural, since they assume the vortex to be placed in a liquid of constant density (in many cases, if the vortex is not close to a solid surface or to another vortex, this assumption is fully justified). The last term of (11a) corresponds to the a mutual-friction force component (with coefficient B) directed along $(\mathbf{v}_n-\mathbf{v}_s)_\perp$, where the subscript $\perp$ denotes the projection on the (x,y) plane (perpendicular to the vortex):

$$-\Lambda|n|[\mathbf{k}\times[\mathbf{k}\times\mathbf{v}_{n0}-\mathbf{\Phi}]]=\Lambda|n|(\mathbf{v}_n-\mathbf{v}_s)_{0\perp}.$$

5. We consider now several concrete examples. In view of the nonlinearity of Eqs. (1) and (9) it is quite difficult to obtain for them complete solutions corresponding to the complicated boundary conditions, and we confine ourselves to examples in which the form of the function $\tilde{\psi}$ is approximately determined in the vicinity of the considered vortex. In all cases the vortex in question is assumed straight and having a positive

circulation: $n > 0$.

Example 1. Near the vortex there is a similar one with negative circulation $n_1 = -n$. Assume that at the considered instant of the vortex is at the origin and the antivortex at the point $y = -b$. If $b < 1$ (i.e., the distance between the vortices is less than the coherence length), then

$$\tilde{\psi} \approx f_L(r_v) \exp(-in\alpha_v),$$
$$r_v^2 = (x - vt)^2 + (y + b)^2, \quad \alpha_v = \mathrm{arc\,tg}\,\frac{y+b}{x-vt},$$

and the function $f_L(r_v)$ at $x \approx vt$ and $y \approx 0$ is determined by formula (3). The vortex velocity calculated from formulas (8) or (19) is directed along the x axis and is equal to

$$v \approx n/b + n/b = 2n/b \quad (b \ll 1).$$

In dimensional notation, this quantity equals $\Gamma/\pi b$ ($\Gamma = 2\pi n\hbar/m$ is the circulation), or double the velocity of a vortex pair in a classical ideal incompressible liquid. If $b \gg 1$, then $G \ll \Phi$ and $v \approx n/b$, just as in an incompressible liquid.

Example 2. Halfway between two vortices arranged as in the preceding example, is placed a solid boundary—the plane $y = -b/2$ (the antivortex is then a reflection of the vortex). Then

$$\tilde{\psi} \approx \mathrm{th}\,\frac{y + b/2}{2^{1/2}} f_L(r_v) \exp(-in\alpha_v).$$

If the vortex is close to the surface, then the first factor is approximately equal to the argument of the hyperbolic tangent, and the remaining factors are those determined in the preceding example. The vortex velocity is again directed along x and is equal to

$$v \approx 2/b + n/b + n/b = 2(n+1)/b = (n+1)/a \quad (a \ll 1),$$

where $a = b/2$ is the distance from the vortex to the wall. The vortex can be at equilibrium if: 1) $v_n = (n+1)/a$ at a distance a from the wall, or 2) a countercurrent of the superfluid component flows at a velocity $v_s(\infty) = -(n+1)/a$, while the normal component is immobile. It is clear that even at an arbitrarily large normal-component flow one can find a sufficiently small a such that $v_n < (n+1)/a$ and no equilibrium flow of the vortex along the solid surface is possible.

Example 3. The vortices are arranged in the same manner as in example 1, but their circulations are equal, $n_1 = n$. The vortex velocity calculated by formulas (8) or (10) is then

$$v \approx -n/b + n/b = 0,$$

i.e., as $b \to 0$ the vortex velocity decreases to zero. In a classical ideal incompressible liquid, closely located identical vortices rotate about a common center with velocity n/b (if $b \gg 1$ then we have also in our case $v \approx -n/b$, since $G \ll \Phi$).

6. If we change to a reference frame in which the vortex is immobile, then formula (8) can be rewritten in the form of the equalities

$$|n|\left(\frac{\partial \ln \tilde{f}}{\partial x}\right)_0 = n\left(\frac{\partial \tilde{\varphi}}{\partial y}\right)_0, \quad |n|\left(\frac{\partial \ln \tilde{f}}{\partial y}\right)_0 = -n\left(\frac{\partial \tilde{\varphi}}{\partial x}\right)_0. \qquad (12)$$

The condition for the regularity of the solution $\psi = \psi_L \tilde{\psi}$ of Eq. (1) in the vicinity of the vortex point consists therefore of conditions of the Cauchy-Riemann type, which must be satisfied at the vortex point itself: the complex phase of the nonsingular part of the solution $\tilde{\psi}$ must have at this point the analytic properties of the complex phase of the function $r^{|n|}e^{in\alpha}$, to which ψ_L tends as $r \to 0$. According to (3) we have at $n > 0$

$$\ln f_L + i\varphi_L \to n \ln(x + iy) + \mathrm{const},$$

and at $n < 0$ we get

$$\varphi_L + i \ln f_L \to -in \ln(x + iy) + \mathrm{const}.$$

The conditions (12) are correspondingly the analyticity conditions at the vortex point, of the function $\ln\tilde{f} + i\tilde{\varphi}$ at $n > 0$ and of the function $\tilde{\varphi} + i \ln \tilde{f}$ at $n < 0$.

Obviously, an investigation of the vicinity of the vortex similar to that carried out in Sec. 2, is possible also for the equations for physically different fields, including those for which neither the field functions, nor their gradients, nor the expressions associated with them have the meaning of the velocity of a certain medium. It is easy to verify that for a field of any physical nature it is necessary here to satisfy Eqs. (12) (which are valid in the reference frame in which the vortex is immobile), if the following conditions are satisfied:

a) the field is described by a scalar wave function $\psi = f \exp(i\varphi)$;

b) the equation has a solution ψ_L that describes an immobile straight vortex and has the property (3);

c) the field equation contains the term $\Delta\psi$ and contains no other terms of the same or lower order in r than the principal terms separated from $\Delta\psi$.

Consider a vortex in a certain field described by a Lorentz-invariant equation possessing the properties a), b) and c). In a certain reference frame, arbitrarily called "immobile," the conditions (12), generally speaking, are not satisfied. We obtain a ("moving") reference frame in which they are satisfied:

$$|n| G_x' = n\Phi_y', \quad |n| G_y' = -n\Phi_x'.$$

The velocity of this ("primed") reference frame is in fact the vortex velocity. It can therefore be determined from the system of equations:

$$|n|(G_x + vG_t) = n(1 - v^2)^{1/2}\Phi_y,$$
$$(1 - v^2)^{1/2}|n| G_y = -n(\Phi_x + v\Phi_t). \qquad (13)$$

Here

$$G_t \equiv (\partial \ln \tilde{f}/\partial t)_0, \quad \Phi_t \equiv (\partial \tilde{\varphi}/\partial t)_0,$$

v is the vortex velocity. It is directed along the x axis, but the direction of this axis relative to the (specified) gradients $\Delta\tilde{\varphi}$ and $\Delta\tilde{f}$ is unknown. The conditions (13) are the system of equations for the determination of the value of v and of the direction of the x axis (i.e., of the direction of $\mathbf{v}$).

We plan to deal with vortex motion in relativistic fields elsewhere. We note only that, as can be shown with the aid of (13), the relative velocity of two identical straight vortices tends to zero as they approach each other, and the velocity of a vortex-antivortex pair tends to that of light.

[1)] The following units are used: $\hbar/(2mgn_0)^{1/2}\equiv\xi$ for distances, $\hbar/2gn_0$ for time, and $n_0^{1/2}$ for the wave function, where m is the mass of the atom and n_0 is the equilibrium density of the condensate. The constant g characterizes the interaction between the particles and is assumed to be small enough to make the coherence length ξ larger than the interatomic distances.

[2)] The function ψ_v is not a solution of Eq. (1), except at $v=0$, inasmuch as in a medium that is immobile at infinity a straight vortex, being the only perturbation of its ground state, is immobile (see below).

[3)] It is defined as the line the circuit around which (around each of its elements) changes the phase by $2\pi n (n=\pm1, \pm2, \ldots)$. In the immediate vicinity of such a line we have $\psi\propto\psi_v$.

[4)] It is possible to separate in the gradient of the phase Φ the various contributions (of the flux, of another vortex, of a reflected vortex, of other elements of the considered vortex, etc.) and resolve the Magnus force into several forces (interaction with external flux, interaction with another vortex, interaction with the wall-image force, interaction with other elements of the same vortex-rectifying force acting on a bent vortex).

[1] L. P. Pitaevskiĭ, Zh. Eksp. Teor. Fiz. **40**, 646 (1961) [Sov. Phys. JETP **13**, 451 (1961)].

[2] E. P. Gross, Phys. Rev. **106**, 161 (1957).

[3] V. L. Ginzburg and L. P. Pitaevskiĭ, Zh. Eksp. Teor. Fiz. **34**, 1240 (1958) [Sov. Phys. JETP **7**, 858 (1958)].

[4] L. P. Pitaevskii, *ibid.* **35**, 408 (1958) [8, 282 (1959)].

[5] W. F. Vinen, Progr. in Low Temp. Phys., North-Holland, V. 3, Chap. 1, 1961.

[6] I. M. Khalatnikov, Teoriya svertekuchesti (Theory of Superfluidity), Nauka, 1971.

[7] E. L. Andronikashvili, Yu. G. Mamaladze, and Dzh. S. Tsakadze, Fizika sverkhtekucheskogo geliya (Physics of Superfluid Helium), book II, Metsinereba, Tbilisi, 1978.

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Role of spatial dispersion in light absorption by excitons

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A phenomenological analysis is presented of resonant absorption of light by excitons with spatial dispersion taken into account. It is shown that the form of the absorption line contour and its characteristics change drastically when the damping constant goes through a certain critical value. Analytic formulas are obtained for the absorption coefficient, for the integral absorption, for the maximum value of the absorption coefficient, and for the equivalent line width at damping-constant values larger and smaller than critical.

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1. INTRODUCTION

Light absorption by excitons has been the subject of a considerable number of experimental[1-5] (see also the references in these papers) and theoretical[6-10] studies. It has been established that a change in the sample temperature is accompanied by a change in the absorption-line shape and the line parameters such as the half-width, maximum of the line, and integral absorption. One of the distinguishing features of the observed spectra is the decrease of the integral absorption with temperature. There is still no complete theoretical explanation of all the features of the absorption spectra, although it is clear that the experimentally observed changes are connected with the dependence of the phenomenological damping constant ν on the temperature and on the frequency.[1] The role played by spatial dispersion in the decrease of the integral absorption with decreasing temperature was recently considered in Refs. 9 and 10.

We report here a detailed quantitative investigation of the dependence of the absorption line shape and of the line parameters on ν in a crystal model in which spatial dispersion is taken into account. We confine ourselves here to a macroscopic solution of the problem and assume ν to be an independent variable; this assumption is justified because ν has a monotonic dependence on T in a wide temperature interval. In addition, to reveal the role played in absorption by spatial dispersion "in pure form," we assume ν to be constant over the entire range of frequencies to interest to us. Effects connected with the frequency dependence of ν are discussed briefly at the end of the article.

2. ABSORPTION COEFFICIENT

We consider an isotropic nongyrotropic crystal whose dielectric constant in the vicinity of an isolated exciton absorption line can be represented in the form

$$\varepsilon(\omega, \mathbf{K})=\varepsilon_0+\frac{p\omega_0^2}{\omega_0^2-\omega^2-i\omega\nu+\beta c^2\mathbf{K}^2}, \quad (1)$$

where ε_0 is the contribution made to the dielectric constant by other resonances, p is the oscillator strength, ω_0 is the natural frequency of the mechanical exciton at $\mathbf{K}=0$, $\beta=\hbar\omega_0/m_e^*c^2$, and m_e^* is the effective mass of the exciton. Let the crystal be a plane-parallel plate of thickness d. At normal incidence of the light on the face of the plate there will propagate inside the plate, at a given frequency ω, two waves with refractive indices $\tilde{n}_1=n_1+i\varkappa_1$ and $\tilde{n}_2=n_2+i\varkappa_2$. The complete formula for the amplitude transmission coefficient of the plate, with allowance for the multiple re-