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Surface oscillations of a Fermi liquid

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Surface oscillations of a normal Fermi liquid are analyzed using the Landau phenomenological theory. A scheme is formulated for describing surface effects using a kinetic equation with a self-consistent field; this scheme is based on successive separation of surface parts of various characteristics of a liquid. Boundary conditions on a free surface are derived. An expression is obtained for the surface tension. Oscillations of a Fermi liquid are studied under hydrodynamic and collisionless conditions and the distributions of the density and current in the presence of such oscillations are found.

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1. INTRODUCTION

Khodel' *et al.*[1,2] used the Fermi liquid approach[3] at absolute zero to develop a theory of surface oscillations of a Fermi liquid and they applied this theory to nuclear vibrational states. The theory describes successfully low collective nuclear states but considerable difficulties are encountered in the derivation of simple analytical expressions in the macroscopic limit of a large system such as liquid ^{3}He. It is important to stress that this approach does not require formulation of any boundary conditions on the free surface of a liquid. Fomin[4] formulated a scheme for describing surface phenomena in the collisionless case on the basis of a kinetic equation with a self-consistent field and certain boundary conditions on the free surface: the condition of specular reflection of quasiparticles and the ordinary hydrodynamic boundary condition with surface tension. Therefore, it would be interesting to investigate how these boundary conditions appear in a more general approach and also to describe surface phenomena not only in the collisionless case but also allowing for collisions.

Our aim is to develop a simply quasiclassical theory of surface oscillations of a normal Fermi liquid describing the dynamics of this liquid in the hydrodynamic and collisionless cases, and capable of dealing with surface phenomena in liquid ^{3}He. We shall use the Landau phenomenological theory of a Fermi liquid.[5] We shall formulate a description of surface phenomena using a kinetic equation with a self-consistent field. We shall derive boundary conditions on the free surface and these will be identical, apart from unimportant corrections, with the conditions used by Fomin.[4]

We shall obtain an expression for the surface tension which will be somewhat approximate: it will not include the quantum term arising because of the nonlocality of the relationship between the density of the system $\rho_0(\mathbf{r})$ and the self-consistent field $U_0(\mathbf{r})$ (Refs. 1 and 2). This occurs because of quasiclassical description given by a kinetic equation gives a poor description of the surface where $\rho_0(\mathbf{r})$ and $U_0(\mathbf{r})$ rapidly vary with the coordinates. However, the quasiclassical approach holds very well in the interior of a large system if $k \ll p_F$ (k is the oscillation wave vector and p_F is Fermi momentum) and it describes the dynamics of surface oscillations. Thus, all the inaccuracy of the quasiclassical approach reduces to the inaccuracy in the calculation of just one theoretical constant which is the surface tension.

We shall consider oscillations of a Fermi liquid in the hydrodynamic and collisionless case and find the distributions of the density and current in the presence of such oscillations. In the hydrodynamic case ($\omega\tau \ll 1$) the spectrum of surface waves is identical with the spectrum of capillary waves in viscous hydrodynamics, which is to be expected, and the motion of the liquid is described by laws of hydrodynamics. In the collisionless case ($\omega\tau \gg 1$), we shall show that there are two surface branches. The first low-lying branch is a continuation of a capillary hydrodynamic branch and is found to be simply damped. The second higher branch is a quantum analog of the Rayleigh surface waves in a solid[6] and it is considered in detail in Fomin's paper.[4] This second branch exists only if the constant in front of the first harmonic of the effective interaction is $F_1 > 6$, i.e., if transverse zero sound can travel in the system.

Application of the results obtained to liquid ^{3}He shows that both capillary and Rayleight oscillations are strongly damped in ^{3}He under collisionless conditions.

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The Rayleigh oscillations are damped because the velocity of transverse zero sound c_1 is close to V_F. This is due to the Landau damping mechanism, i.e., the decay of a collective mode into particle–hole pairs. A surface mode is a wave packet of plane waves whose spectral composition is found, for given values of ω and k, only from the requirement of disappearance of oscillations in the interior of the liquid. The components of a packet for which the ratio $s=\omega/qV_F$ (q is the wave vector of a plane wave in a packet) is less than unity experience the Landau damping. Therefore, the damping of the whole wave packet is governed by the relative weights of the components with $s<1$.

2. KINETIC EQUATION FOR A SYSTEM WITH A FREE SURFACE

In this section we shall derive the boundary conditions which should be satisfied by the distribution function on a free surface. We shall consider a Fermi sea bounded by the surface $x=0$ (the liquid is in the half-space $x>0$) and we shall describe this system using the Landau kinetic equation[5]:

$$\frac{\partial n}{\partial t}+\mathbf{V}\frac{\partial n}{\partial \mathbf{r}}-\frac{\partial n}{\partial \mathbf{p}}\frac{\partial U}{\partial \mathbf{r}}=I(n), \tag{1}$$

where $n(\mathbf{p}, \mathbf{r}, t)$ is the distribution function of quasiparticles in the sea, $I(n)$ is the collision integral,

$$\mathbf{V}(\mathbf{p},\mathbf{r},t)=\frac{\mathbf{p}}{m}+{}^1/_2\,\mathrm{Sp}_{\sigma'}\int \mathcal{F}(\mathbf{p},\mathbf{r};\mathbf{p}',\mathbf{r}')\frac{\partial n(\mathbf{p}',\mathbf{r}',t)}{\partial \mathbf{p}'}\,d\tau'\,d\mathbf{r}', \tag{2}$$

$$\frac{\partial U(\mathbf{p},\mathbf{r},t)}{\partial \mathbf{r}}={}^1/_2\,\mathrm{Sp}_{\sigma'}\int \mathcal{F}(\mathbf{p},\mathbf{r};\mathbf{p}',\mathbf{r}')\frac{\partial n(\mathbf{p}',\mathbf{r}',t)}{\partial \mathbf{r}'}\,d\tau'\,d\mathbf{r}'. \tag{3}$$

Here, $\mathcal{F}(\mathbf{p},\mathbf{r};\mathbf{p}',\mathbf{r}')$ is the effective interaction between the quasiparticles; $d\tau=2dp/(2\pi)^3$; the dependences on the spin variables (σ and σ') are not given explicitly.

The equilibrium solution of the kinetic equation is

$$n_0(\mathbf{p},\mathbf{r})=\left[\exp\left\{\frac{p^2/2m+U_0(\mathbf{p},\mathbf{r})-\mu}{T}\right\}+1\right]^{-1}, \tag{4}$$

where μ is the chemical potential and the self-consistent field $U_0(\mathbf{p},\mathbf{r})$ is defined in terms of $n_0(\mathbf{p},\mathbf{r})$ by Eq. (3). In view of the selected geometry, the equilibrium distribution function $n_0(\mathbf{p},\mathbf{r})$ and the self-consistent potential $U_0(\mathbf{p},\mathbf{r})$ vary rapidly with the coordinate near $x=0$, ranging from some finite value at $x>0$ to zero for $x<0$.

We shall consider small spin-symmetric deviations from equilibrium $n=n_0+\delta\tilde{n}$. Then, the linearized equation for $\delta\tilde{n}(\mathbf{p},\mathbf{r})$ is

$$\frac{\partial\delta\tilde{n}}{\partial t}+\mathbf{V}\frac{\partial\delta\tilde{n}}{\partial\mathbf{r}}+\frac{\partial n_0}{\partial\mathbf{r}}\frac{\partial\delta\tilde{U}}{\partial\mathbf{p}}-\frac{\partial U_0}{\partial\mathbf{r}}\frac{\partial\delta\tilde{n}}{\partial\mathbf{p}}-\frac{\partial n_0}{\partial\mathbf{p}}\frac{\partial\delta\tilde{U}}{\partial\mathbf{r}}=\hat{L}(\delta\tilde{n}), \tag{5}$$

where $\hat{L}(\delta\tilde{n})$ is the linearized collision integral

$$\delta\tilde{U}(\mathbf{p},\mathbf{r},t)=\int\mathcal{F}(\mathbf{p},\mathbf{r};\mathbf{p}',\mathbf{r}')\,\delta\tilde{n}(\mathbf{p}',\mathbf{r}',t)\,d\tau'\,d\mathbf{r}', \tag{6}$$

$$\mathbf{V}=\frac{\mathbf{p}}{m^*(\mathbf{r})}=\frac{\mathbf{p}}{m}+\int\mathcal{F}(\mathbf{p},\mathbf{r};\mathbf{p}',\mathbf{r}')\frac{\partial n_0(\mathbf{p}',\mathbf{r}')}{\partial\mathbf{p}'}\,d\tau'\,d\mathbf{r}', \tag{7}$$

$m^*(\mathbf{r})$ is the effective mass which generally has different values inside and outside the medium.

We shall point out that, in contrast to Eq. (2), V is defined in terms of the equilibrium distribution function. In Eqs. (6) and (7) the signs of summation with respect to the spin, $\mathrm{Tr}_{\sigma'}$, are omitted because here and later we shall consider only perturbations which are spin-symmetric.

We can easily demonstrate that Eq. (5) is translationally invariant, i.e., that a simple shift the whole system by the distance $\delta\mathbf{R}$,

$$\delta\tilde{n}=\frac{\partial n_0}{\partial\mathbf{r}}\delta\mathbf{R},$$

is a solution of this equation.

Equation (5) shows that a Fermi liquid with a free surface can be regarded as a system which is in some external confining field (equilibrium self-consistent field of the system) which is generally velocity-dependent. Then, the effective mass of a quasiparticle excitation is a function of the coordinates. However, in order to simplify the solution of the kinetic equation, we shall reduce the influence of this field to the boundary condition and use its characteristic coordinate dependence. This reduction is also of theoretical interest because it allows us to understand how surface tension appears in such a liquid.

We shall study free oscillations of our semi-infinite system

$$\delta\tilde{n}(\mathbf{p},\mathbf{r},t)=\delta\tilde{n}(\mathbf{p},x,\mathbf{k})\exp(-i\omega t+i\mathbf{k}\mathbf{r}_\perp), \tag{8}$$

where $\mathbf{k}=\{0,k_y,k_z\}$ and $\mathbf{r}_\perp=\{0,y,z\}$ are vectors lying in the plane of the surface. Substituting Eq. (8) into the linearized kinetic equation (5), we obtain

$$-i(\omega-\mathbf{k}\mathbf{V}_\perp)\delta\tilde{n}+V_x\frac{\partial\delta\tilde{n}}{\partial x}+\frac{dn_0}{d\varepsilon}\frac{\partial U_0}{\partial x}\frac{\partial\delta\tilde{U}}{\partial p_x}-\frac{\partial U_0}{\partial x}\frac{\partial\delta\tilde{n}}{\partial p_x}-\frac{dn_0}{d\varepsilon}\left(i\mathbf{k}\mathbf{V}_\perp+V_x\frac{\partial}{\partial x}\right)\delta\tilde{U}=\hat{L}(\delta\tilde{n}), \tag{9}$$

where

$$\delta\tilde{U}(\mathbf{p},x,\mathbf{k})=\int\mathcal{F}(\mathbf{p},x;\mathbf{p}',x';\mathbf{k})\,\delta\tilde{n}(\mathbf{p}',x',\mathbf{k})\,d\tau'\,dx', \tag{10}$$

i.e., it is defined in terms of the k component of the effective interaction

$$\mathcal{F}(\mathbf{p},x;\mathbf{p}',x';\mathbf{k})=\int\mathcal{F}(\mathbf{p},\mathbf{r};\mathbf{p}',\mathbf{r}')\exp\{-i\mathbf{k}(\mathbf{r}_\perp-\mathbf{r}_\perp')\}d(\mathbf{r}_\perp'-\mathbf{r}_\perp). \tag{11}$$

It should be noted that if the effective interaction is δ-like, $\mathcal{F}\sim\delta(\mathbf{r}-\mathbf{r}')$ then $\mathcal{F}(\mathbf{k})=\mathcal{F}(0)$, i.e., this interaction is independent of k and $\delta\tilde{n}(\mathbf{p},x,\mathbf{k})=(\partial n_0/\partial x)\delta X$ is a solution of Eq. (9) for any value of k when $\omega=0$. Such a solution describes static deformation of the liquid surface characterized by a wave vector k and a deformation amplitude δX ($\delta X=\mathrm{const}$). It follows that in this interaction the spectrum of surface waves is trivially degenerate: $\omega(\mathbf{k})=0$ and the surface tension of the system vanishes. For this reason we shall consider a weakly nonlocal interaction.

For simplicity, we shall assume that the effective interaction is Gaussian in respect of $(\mathbf{r}-\mathbf{r}')$, and we shall retain only the first two harmonics in respect of the angle between p and p′:

$$\mathcal{F}(\mathbf{p},\mathbf{r};\mathbf{p}',\mathbf{r}')=\left[\mathcal{F}_0(\rho)+\mathcal{F}_1(\rho)\frac{\mathbf{p}\mathbf{p}'}{p_F^2}\right][(2\pi)^{1/2}r_G]^{-3}\exp\left[-\frac{(\mathbf{r}-\mathbf{r}')^2}{2r_G^2}\right], \tag{12}$$

where p_F is the Fermi momentum; r_G is the parameter which governs the nonlocality of the effective interaction and is of the order of $1/p_F$, i.e., it is of the order of the average distance between the particles; $\mathcal{F}_0(\rho)$

and $\mathcal{F}_1(\rho)$ are the interaction constants for the zeroth and first harmonics, their dependences on the density of the system ρ implying that they are generally different for the medium and vacuum and at the surface of the system are in some way interpolated with respect to the density (see, for example, Refs. 2 and 3). The actual form of such interpolation is unimportant for our purpose. Using the expression for the interaction (12), we find that for $k \ll p_F$

$$\mathcal{F}(\mathbf{k}) \approx \mathcal{F}(\mathbf{k}=0)\,[1-{}^1/_2(kr_G)^2]. \tag{13}$$

It should be mentioned that the representation of $\mathcal{F}(\mathbf{k})$ in the form of an expansion in terms of a small parameter $(kr_G)^2$ is only a consequence of the smallness of the nonlocality parameter r_G of the effective interaction. The convenience of the Gaussian form of the interaction (12) is simply that the coefficients in front of 1 and in front of $(kr_G)^2$ [see Eq. (13)] are proportional to the same quantity $\mathcal{F}(\mathbf{k}=0)$, which simplifies subsequent operations. However, all these results can also be obtained for any other weakly nonlocal effective interaction. Since the correction due to the nonlocality is only small compared with unity in Eq. (13), we shall include this correction only in those cases when the main term of the expansion (13), which is $\mathcal{F}(\mathbf{k}=0)$ vanishes exactly.

It is clear from Eq. (9) that its solution is $\delta\tilde{n} \sim dn_0/d\varepsilon$ ($\varepsilon = p^2/2m + U_0$). We shall seek this solution in the form

$$\delta\tilde{n} = \delta n - \frac{\partial n_0}{\partial x}\delta X,$$

or, on the basis of the above discussion

$$\delta\tilde{n}(\mathbf{p}, x, \mathbf{k}) = \frac{dn_0}{d\varepsilon}\left[f(\mathbf{p}, x, \mathbf{k}) - \frac{\partial U_0}{\partial x}\delta X\right], \tag{14}$$

where δX ($\delta X = \text{const}$), is the amplitude of a surface oscillation,

$$\delta n = \frac{dn_0}{d\varepsilon} f, \qquad \frac{\partial n_0}{\partial x} = \frac{dn_0}{d\varepsilon}\frac{\partial U_0}{\partial x},$$

where the last expression follows from Eq. (4). Correct determination of the function f requires application of an additional condition which means that f does not include parts proportional to $\partial U_0/\partial x$. Therefore, we shall assume that

$$\int f \frac{\partial n_0}{\partial x}\, d\tau\, dx = 0. \tag{15}$$

It follows from Eq. (14) that a change in the self-consistent field also splits into two—bulk and surface—terms:

$$\delta U(\mathbf{p}, x, \mathbf{k}) = \delta U(\mathbf{p}, x, \mathbf{k}) - [1-{}^1/_2(kr_G)^2]\frac{\partial U_0}{\partial x}\delta X, \tag{16}$$

$$\delta U(\mathbf{p}, x, \mathbf{k}) = \int \mathcal{F}(\mathbf{k}=0)\frac{dn_0}{d\varepsilon} f\, d\tau'\, dx'$$

$$= \mathcal{F}_0\rho'(x, \mathbf{k}) + \frac{\mathcal{F}_1}{p_F^2}(\mathbf{pj}(x, \mathbf{k})); \tag{17}$$

the macroscopic characteristics of the system are introduced above and they are the bulk perturbation of the number of particles ρ' and the particle flux $\mathbf{j}$:

$$\rho'(x, \mathbf{k}) = \int \frac{dn_0}{d\varepsilon} f\, d\tau, \tag{18}$$

$$\mathbf{j}(x, \mathbf{k}) = \int \mathbf{p}\frac{dn_0}{d\varepsilon} f\, d\tau. \tag{19}$$

Equations (16) and (17) are derived using the self-consistency condition which follows from Eq. (3):

$$\frac{\partial U_0}{\partial x} = \int \mathcal{F}(\mathbf{k}=0)\frac{\partial n_0}{\partial x'}\, d\tau'\, dx', \tag{20}$$

and also ignoring the nonlocality of the effective interaction in the bulk term [see Eq. (17)], since allowance for this nonlocality in the bulk simply produces small corrections $\sim(kr_G)^2$.

Substituting Eqs. (14) and (16) in Eq. (9), we obtain

$$\left\{-i(\omega - \mathbf{kV}_\perp)f + V_x\frac{\partial f}{\partial x} - \left(i\mathbf{kV}_\perp + V_x\frac{\partial}{\partial x}\right)\delta U\right\}$$

$$+\left\{\frac{\partial U_0}{\partial x}\left[i\omega\delta X - \frac{\partial f}{\partial p_x} + \frac{\partial \delta U}{\partial p_x}\right]\right.$$

$$\left.+{}^1/_2(kr_G)^2\delta X\frac{\partial U_0}{\partial x}\frac{\partial}{\partial p_x}\frac{\partial U_0}{\partial x}\right.$$

$$\left.-{}^1/_2(kr_G)^2\delta X\left(i\mathbf{kV}_\perp + V_x\frac{\partial}{\partial x}\right)\frac{\partial U_0}{\partial x}\right\} = L(f), \tag{21}$$

where the collision integral $L(f)$ is defined in the natural manner:

$$\hat{L}\left(\frac{dn_0}{d\varepsilon} f\right) = \frac{dn_0}{d\varepsilon} L(f).$$

Equation (21) is derived using

$$\hat{L}\left(\frac{\partial n_0}{\partial x}\delta X\right) = \delta X\frac{\partial}{\partial x}\hat{L}(n_0) = 0.$$

We shall now utilize the fact that the function $\partial U_0/\partial x$ represents a sharp surface peak (at $x \sim 0$), whose width is of the order of the diffuseness of the edge of the system $\sim 1/p_F$, and vanishes inside and outside the system. Therefore, in the medium ($x > 0$) the term in the second set of braces in Eq. (21) disappears leaving

$$-i(\omega - \mathbf{kV}_\perp)f + V_x\frac{\partial f}{\partial x} - \left(i\mathbf{kV}_\perp + V_x\frac{\partial}{\partial x}\right)\delta U = L(f). \tag{22}$$

On the other hand, in the region of the surface diffuseness the term with $\partial U_0/\partial x$ predominates so that here Eq. (21) reduces to

$$\frac{\partial U_0}{\partial x}\left[i\omega\delta X - \frac{\partial f}{\partial p_x} + \frac{\partial \delta U}{\partial p_x}\right]$$

$$+{}^1/_2(kr_G)^2\delta X\frac{\partial U_0}{\partial x}\frac{\partial}{\partial p_x}\frac{\partial U_0}{\partial x}$$

$$-{}^1/_2(kr_G)^2\delta X\left(i\mathbf{kV}_\perp + V_x\frac{\partial}{\partial x}\right)\frac{\partial U_0}{\partial x} = 0. \tag{23}$$

We note immediately that $-i\omega\delta X = \mathcal{U}_0$ is the amplitude of the surface oscillation velocity. We shall assume that $x = 0$ in Eq. (23) and divide this equation by $(\partial U_0/\partial x)_{x=0}$, and then integrate with respect to dp_x between $-p_x$ and p_x. We must bear in mind that $U_0(\mathbf{p}, x)$ is an even function of p_x and all the terms in Eq. (23) are odd with respect to p_x, so that they vanish after such integration. We thus have

$$f(p_x) - f(-p_x) = -2p_x\mathcal{U}_0 + \delta U(p_x)$$

$$-\delta U(-p_x) + p_x(kr_G)^2\frac{\mathbf{kV}_\perp}{\omega}\mathcal{U}_0. \tag{24}$$

Substituting Eq. (17) in Eq. (24), we obtain

$$f(p_x) - f(-p_x) = 2p_x\left[-\mathcal{U}_0 + \frac{\mathcal{F}_1(x=0)}{p_F^2} j_x(x=0)\right]. \tag{25}$$

In Eq. (25) we have ignored the unimportant term $\sim(kr_G)^2$.

We now multiply Eq. (25) by $p_x dn_0/d\varepsilon$ and integrate:

$$2(2\pi)^{-3}\int_0^\infty dp_x \int d\mathbf{p}_\perp \dots .$$

A simple calculation yields the equation for $j_x(x=0)$:

$$j_x(x=0) = -m^*(0)\rho_0(0)\left[-\mathcal{U}_0 + \frac{\mathcal{F}_1(x=0)}{p_F^2} j_x(x=0)\right]; \qquad (26)$$

here,

$$\rho_0(x) = \int n_0(\mathbf{p}, x)\, d\tau$$

is the equilibrium density of the system,

$$m^*(x) = (1+\mathcal{F}_1 m^*\rho_0/p_F^2)\, m$$

is the effective mass.[1)] It follows from Eq. (26) that

$$j_x(x=0) = m\rho_0(x=0)\mathcal{U}_0. \qquad (27)$$

Finally, substituting $j_x(x=0)$ in Eq. (25), we obtain the first boundary condition of the $x=0$ surface:

$$f(p_x) - f(-p_x) = -2p_x \frac{\mathcal{U}_0}{1+{}^1/_3 F_1(x=0)}, \qquad (28)$$

where $F_1 = (3m^*\rho_0/p_F^2)\mathcal{F}_1$. This boundary condition has a very clear physical meaning: it is the condition for specular reflection of quasiparticles from the $x=0$ surface.

We shall now derive the second boundary condition. It is known[4] that the kinetic equation (1) yields the following equation for the conservation of the momentum flux:

$$\frac{\partial}{\partial t} j_i + \frac{\partial}{\partial r_k}\tilde{\Pi}_{ki} = 0, \qquad (29)$$

$$\frac{\partial}{\partial r_k}\tilde{\Pi}_{ki} = \int p_i \left(\frac{\partial n}{\partial r_k} V_k - \frac{\partial n}{\partial p_k}\frac{\partial U}{\partial r_k}\right) d\tau, \qquad (30)$$

where $\tilde{\Pi}_{ki}(\mathbf{r})$ is the momentum flux tensor.

We shall linearize the momentum flux tensor relative to $\delta\tilde{n}$:

$$\frac{\partial}{\partial r_k}\tilde{\Pi}_{ki} = \int p_i \left(\frac{\partial n_0}{\partial r_k}\frac{\partial \delta U}{\partial p_k} + \frac{\partial \delta\tilde{n}}{\partial r_k} V_k - \frac{\partial n_0}{\partial p_k}\frac{\partial \delta U}{\partial r_k} - \frac{\partial \delta\tilde{n}}{\partial p_k}\frac{\partial U_0}{\partial r_k}\right) d\tau. \qquad (31)$$

Next, separating the distribution function $\delta\tilde{n}$ into the bulk and surface term (14), we shall represent $\tilde{\Pi}_{ki}$ in the form of two terms: bulk and surface. We shall define the bulk momentum flux tensor by

$$\Pi_{ki} = \int p_i \left(\delta n V_k - \delta U \frac{\partial n_0}{\partial p_k}\right) d\tau. \qquad (32)$$

Using this definition, we can easily transform Eq. (29) to

$$-i\omega j_i + \frac{\partial}{\partial r_k}\Pi_{ki} - \delta_{i1}\left\{\int\left(\delta U\frac{\partial n_0}{\partial x} - \delta n\frac{\partial U_0}{\partial x}\right) d\tau + {}^1/_2(kr_G)^2 \int \frac{\partial n_0}{\partial x}\frac{\partial U_0}{\partial x} d\tau\delta X + {}^1/_2(kr_G)^2 \int p_x \frac{\partial}{\partial x}\left(\frac{\partial n_0}{\partial p_x}\frac{\partial U_0}{\partial x}\right) d\tau\delta X\right\} = 0. \qquad (33)$$

We shall consider Eq. (33) in the region of the transition (surface) layer $x \sim 0$. We can easily show that the boundary condition for specular reflection (28) makes the nondiagonal components of the bulk momentum flux tensor $\Pi_{yx} = \Pi_{zx} = 0$ vanish in the transition layer. Therefore,

$$\frac{\partial \Pi_{kx}}{\partial x_k} = \frac{\partial \Pi_{xx}}{\partial x}$$

at $x \sim 0$. We shall now find Π_{xx} in the direct vicinity of the surface on the side of the liquid, i.e., at $x=+0$. We shall do this by integrating Eq. (33) with respect to dx from $-\infty$ to $+0$ in such a way that the integration domain covers the whole of the transition layer. We then note that since $\partial n_0/\partial x$ and $\partial U_0/\partial x$ vanish outside the transition layer, integration in the expressions containing these quantities can be extended from $-\infty$ to $+\infty$. Using the expression $j_x = -i\omega m\rho_0\delta X$ in transition region [see Eq. (27)], and the orthogonality condition (15), we obtain the second boundary condition

$$\Pi_{xx}(x=+0) = -k^2\sigma\delta X + \omega^2 M_s \delta X, \qquad (34)$$

$$\sigma = -{}^1/_2 r_G^2 \int_{-\infty}^{\infty} \frac{\partial n_0}{\partial x}\frac{\partial U_0}{\partial x} d\tau dx, \qquad (35)$$

$$M_s = m\int_{-d}^{d} \rho_0(x)\, dx, \qquad (36)$$

where the integral in Eq. (36) is taken over the diffuse edge region $(-d, d)$, where $d \sim 1/p_F$.

In the coordinate-time representation the boundary condition (34) appears as follows:

$$\Pi_{xx}(x=+0) = \sigma\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\delta X(\mathbf{r}_\perp, t) - M_s\frac{\partial^2}{\partial t^2}\delta X(\mathbf{r}_\perp, t). \qquad (37)$$

Without the last term, Eq. (37) is the ordinary hydrodynamic boundary condition on the surface of a liquid, where σ is the surface tension. The last term in Eq. (37), which contains $\partial^2/\partial t^2$, reflects the fact that the surface layer is not only characterized by an elasticity σ but also by a certain mass (we can see from the definition that M_S is the mass per unit surface area). It follows that the surface mass increases on increase of diffuseness of the edge of the liquid. However, in all the cases considered above this inertial term is unimportant and, therefore, we shall ignore it.

The expression (35) obtained for the surface tension is somewhat inaccurate: it does not include the quantum term associated with the nonlocality of the relationship between the density $\rho_0(x)$ and the self-consistent field $U_0(x)$ (Refs. 1 and 2). This is a consequence of the fact that the quasiclassical expression for the equilibrium distribution function (4) provides a poor description of the surface of our system. The condition of validity of the quasiclassical approach $(\partial U/\partial x)/m^{1/2}U^{3/2} \ll 1$ is not obeyed in the surface region. In reality, this parameter is of the order of unity in the transition regions of nuclei and liquid ^{3}He. However, this is the only shortcoming of the quasiclassical approximation. On the other hand, Eq. (35) represents generalization of the classical term in the surface tension to the case $m^*/m \neq 1$ (Khodel' *et al.*[1,2] considered the case $m^* = m$).

3. SOLUTION OF THE KINETIC EQUATION

We shall now give the final formulation of the problem of finding free surface oscillations of the Fermi liquid. We shall consider the linearized kinetic equation

$$-i(\omega - \mathbf{kV}_\perp)f + V_x\frac{\partial f}{\partial x} - \left(i\mathbf{kV}_\perp + V_x\frac{d}{dx}\right)\left[\mathcal{F}_0\rho' + \frac{\mathcal{F}_1}{p_F^2}(\mathbf{pj})\right] = -\nu\left[f - \frac{1}{\langle n_0\rangle}\rho' - \frac{1}{\langle n_0 p^2\rangle}(\mathbf{pj})\right] \qquad (38)$$

in the half-space $x > 0$, where the perturbation of the density ρ' and the particle number flux $\mathbf{j}$ are given by Eqs. (18) and (19). The linearized collision integral in the τ approximation is selected in the same way as in the work of Khalatnikov and Abrikosov,[7] ν is the collision frequency

$$\langle n_0\rangle = \int \frac{dn_0}{d\varepsilon} d\tau = -\frac{m^* p_F}{\pi^2},$$

$$\langle n_0 p^2\rangle = {}^1/_3 \int \frac{dn_0}{d\varepsilon} p^2 d\tau = -\frac{m^* p_F}{\pi^2}\frac{p_F^2}{3}.$$

We shall seek the solution of Eq. (28) which decreases in the limit $x \to \infty$ and which satisfies the following boundary conditions at $x = 0$ [see Eqs. (25) and (34)]:

$$f(p_x) - f(-p_x) = 2p_x\left[-\mathcal{U}_0 + \frac{\mathcal{F}_1}{p_F^2} j_x(x=0)\right], \tag{39}$$

$$i\omega \Pi_{xx}(x=0) = k^2 \sigma \mathcal{U}_0. \tag{40}$$

We can see that this formulation of the problem is fully equivalent—with the exception of the collision integral on the right-hand side of the kinetic equation (38)—to the formulation adopted by Fomin.[4] For completeness, we shall briefly consider the type of solutions that are obtained in this formulation.

Equation (38) can be solved conveniently by a method suggested by Landau[8] in an investigation of the penetration of an electric field into a plasma. The kinetic equation (38) is reduced to a system of integral equations for the functions $\rho'(x, \mathbf{k})$ and $\mathbf{j}(x, \mathbf{k})$, defined in the half-space $x > 0$. Then, defining additionally the functions $\rho'(x, \mathbf{k})$ and $\mathbf{j}_\perp(x, \mathbf{k})$ as even and the function $j_x(x, \mathbf{k})$ as odd in the half-space $x < 0$, this system of equations is reduced to the form convenient for solution by the Fourier transformation method:

$$\rho'(\mathbf{q}) = \int_{-\infty}^{\infty} \rho'(x, \mathbf{k}) \exp(-iq_x x)\, dx, \tag{41}$$

and $\mathbf{j}(\mathbf{q})$ is defined similarly. Here, $\mathbf{q} = \{q_x, \mathbf{k}\}$, and $\rho'(x, \mathbf{k})$ and $\mathbf{j}(x, \mathbf{k})$ are understood to be the quantities extended to the half-space $x < 0$. A similar method for solving Eq. (38) in the collisionless case was used by Fomin.[4] Therefore, we shall give the final results without going into details of the solution method.

Solution of Eq. (38) subject to the condition of decrease in the limit $x \to \infty$ and the subject to the boundary condition of specular reflection (39) gives the following results:

$$\rho'(\mathbf{q}) = -i\frac{p_F^3}{\pi^2}\frac{\mathcal{U}_0}{V_F}\frac{1}{q\Delta_0(s)}\left\{2\left({}^1/_3 F_1 + i\frac{\nu}{\omega_1}\right) sw - (1+{}^1/_3 F_1) \times \frac{1}{s}\left[\left(1-\frac{q_x^2}{q^2}\right)(w+1) + \left(3\frac{q_x^2}{q^2}-1\right)s^2 w\right]\right\}, \tag{42}$$

$$\mathbf{j}(\mathbf{q}) = i\frac{p_F^3}{\pi^2} m^* \mathcal{U}_0 \left\{\frac{2}{\Delta_1(s)}(w - s^2 w + {}^1/_3)\frac{q_x}{q^4}[\mathbf{q}[\mathbf{k}\mathbf{e}_x]] + \frac{1}{\Delta_0(s)}\left[{}^2/_3\left(F_0 + i\frac{\nu}{\omega_1}\right)w + \left(1 - i\frac{\nu}{\omega_1}\right)\left\{\left(1-\frac{q_x^2}{q^2}\right)w + \left(3\frac{q_x^2}{q^2}-1\right)s^2 w\right\}\right]\frac{\mathbf{q}}{q^2}\right\}, \tag{43}$$

where $\omega_1 = \omega + iv$, $s = \omega_1/V_F q$, F_0 and F_1 are the dimensionless Fermi-liquid constants, $\mathbf{e}_x$ is a unit vector in the direction of the x axis,

$$w = {}^1/_2 s \ln\left((s+1)/(s-1)\right) - 1, \tag{44}$$

$$\Delta_0(s) = \left(1 - i\frac{\nu}{\omega_1}\right)(1 + {}^1/_3 F_1) - w\left[\left(F_0 + i\frac{\nu}{\omega_1}\right)(1 + {}^1/_3 F_1) + s^2\left(1 - i\frac{\nu}{\omega_1}\right)\left(F_1 + 3i\frac{\nu}{\omega_1}\right)\right], \tag{45}$$

$$\Delta_1(s) = {}^3/_2\left({}^1/_3 F_1 + i\frac{\nu}{\omega_1}\right)[(w+1) - s^2 w] - (1 - {}^1/_3 F_1). \tag{46}$$

Equations $\Delta_0(s) = 0$ and $\Delta_1(s) = 0$ define respectively the velocities ofpropagation of the longitudinal (c_0) and transverse (c_1) sound: $c_0 = s_0 V_F$, $c_1 = s_1 V_F$ (see Refs. 7 and 9). It is clear from Eqs. (42) and (43) that the propagation of a density perturbation and of the potential component of the current is governed by a pole of the longitudinal sound, whereas the propagation of the solenoidal part of the current is governed by a pole of the transverse current. The expression for the Fourier component of the even continuation to the half-space $x < 0$ of the momentum flux tensor $\Pi_{xx}(x, \mathbf{k})$ is of the form

$$\Pi_{xx}(\mathbf{q}) = \left(F_0 + i\frac{\nu}{\omega_1}\right) Z_0(\mathbf{q})\rho'(\mathbf{q}) + \left({}^1/_3 F_1 + i\frac{\nu}{\omega_1}\right)\left[Z_1(\mathbf{q})\frac{\mathbf{q}\mathbf{j}(\mathbf{q})}{q} + Z_2(\mathbf{q})\frac{[\mathbf{q}[\mathbf{k}\mathbf{e}_x]]\mathbf{j}(\mathbf{q})}{q^2 k^2}\right] + Z_3(\mathbf{q}); \tag{47}$$

$$Z_0(\mathbf{q}) = {}^1/_2 V_F p_F\left[\left(1 - \frac{q_x^2}{q^2}\right)(w+1) + \left(3\frac{q_x^2}{q^2} - 1\right)s^2 w\right],$$

$$Z_1(\mathbf{q}) = {}^1/_2 V_F s\left[\left(1 - \frac{q_x^2}{q^2}\right)w + \left(3\frac{q_x^2}{q^2} - 1\right)(s^2 w - {}^1/_3)\right],$$

$$Z_2(\mathbf{q}) = -V_F k^2 \frac{q_x}{q} s[w - s^2 w + {}^1/_3],$$

$$Z_3(\mathbf{q}) = 2i\mathcal{U}_0 \frac{p_F^3}{\pi^2}\frac{p_F}{q} s\left\{(s^2 w - {}^1/_3) + \frac{k^2}{q^2}[3w - 5(s^2 w - {}^1/_3)] + {}^1/_8 \frac{k^4}{q^4}\left[3\frac{1}{s^2}(w+1) - 30w + 35(s^2 w - {}^1/_3)\right]\right\}.$$

The spectrum of surface waves can be found by calculating the quantity

$$\Pi_{xx}(x=0, \mathbf{k}) = \int_{-\infty}^{\infty} \Pi_{xx}(\mathbf{q})\frac{dq_x}{2\pi} \tag{48}$$

and substituting it in the boundary condition (40). Omitting simple but time-consuming calculations, we shall give the final results for the two cases: hydrodynamic ($\omega/v \ll 1$) and collisionless ($\omega/v \gg 1$). The distribution of the current in the presence of oscillations will be described by means of two scalar potentials, $\varphi(\mathbf{r})$ and $\Phi(\mathbf{r})$, and these are used to describe the current and the curl of the current as follows:

$$\mathbf{j}(\mathbf{r}) = [\nabla[\mathbf{k}\mathbf{e}_x]]\Phi(\mathbf{r}) + \nabla\varphi(\mathbf{r}), \tag{49}$$

$$\operatorname{rot}\mathbf{j}(\mathbf{r}) = -[\mathbf{k}\mathbf{e}_x]\Delta\Phi(\mathbf{r}). \tag{50}$$

We can see that the potential $\varphi(\mathbf{r})$ describes the potential part of the current and $\Phi(\mathbf{r})$ describes the solenoidal part of the current.

1. In the hydrodynamic limit $\omega/v \ll 1$ we find, as expected, the hydrodynamic spectrum of capillary waves in a viscous liquid:

$$\left(2 - i\frac{\omega}{k^2\zeta}\right)^2 + \frac{\sigma}{m\rho_0\zeta^2 k} = 4\left(1 - i\frac{\omega}{k^2\zeta}\right)^{1/2}, \tag{51}$$

where $\zeta = \frac{1}{5}(1 + \frac{1}{3}F_1)V_F^2/v$ is the kinematic viscosity[7] and the branch of the square root on the right-hand side of Eq. (51) is found by postulating that its real part is

positive. The distributions of the density and current in these oscillations are

$$\rho'(\mathbf{r}) = -i\rho_0 \frac{\mathcal{U}_0}{c}\frac{\omega}{kc}\left(1+2i\frac{\zeta k^2}{\omega}\right)\exp(i\mathbf{k}\mathbf{r}_\perp - kx), \quad (52)$$

$$\varphi(\mathbf{r}) = -m\rho_0\mathcal{U}_0 k^{-1}\exp(i\mathbf{k}\mathbf{r}_\perp - kx), \quad (53)$$

$$\Phi(\mathbf{r}) = 2m\rho_0\mathcal{U}_0\frac{\zeta}{\omega}\left\{\exp\left[-\left(k^2 - i\frac{\omega}{\zeta}\right)^{1/2}x\right] - \exp(-kx)\right\}\exp(i\mathbf{k}\mathbf{r}_\perp), \quad (54)$$

where

$$c = [{}^1/_3(1+F_0)(1+{}^1/_3F_1)]^{1/2}V_F$$

is the velocity of hydrodynamic sound.[7]

In the long-wavelength limit $k^2 \ll \omega/\zeta$, the spectrum is

$$\omega^2 = \left[1 - 4i\zeta\left(\frac{m\rho_0}{\sigma}\right)^{1/2}k^{1/2}\right]\frac{\sigma k^3}{m\rho_0}.$$

The liquid then behaves as incompressible [an additional small factor ω/kc appears in $\rho'(\mathbf{r})$] and its motion is potential:

$$|\Phi(\mathbf{r})| \sim (k^2\zeta/\omega)|\varphi(\mathbf{r})| \ll |\varphi(\mathbf{r})|.$$

In the limit of short wavelengths $k^2 \gg \omega/\zeta$, the spectrum of capillary waves becomes damped, $\omega = -i\sigma k/2m\rho_0\zeta$, and the motion of the liquid becomes compressible and rotational.

2. In the collisionless limit $\omega/v \gg 1$ there are two branches of surface oscillations. The spectrum of the first low-lying branch is found from

$$\sigma k^2 = {}^1/_2\left[\left(\frac{3\pi}{8}\right)^2\frac{F_0}{1+F_0} - \frac{F_1}{1+F_1/3}\right]\frac{m^*\rho_0}{k}\omega^2 + {}^3/_4 im^*\rho_0 V_F\omega \quad (55)$$

and is purely damped. The distributions of the density and current in this case of perturbation of the Fermi liquid are

$$\rho'(\mathbf{r}) = {}^3/_2\frac{\rho_0}{1+F_0}\frac{\mathcal{U}_0}{V_F}(kx)K_1(kx)\exp(i\mathbf{k}\mathbf{r}_\perp), \quad (56)$$

$$\varphi(\mathbf{r}) = -m\rho_0\mathcal{U}_0 k^{-1}\exp(i\mathbf{k}\mathbf{r}_\perp - kx), \quad (57)$$

$$\Phi(\mathbf{r}) = im\rho_0\mathcal{U}_0\frac{(kx)}{k^2}\exp(i\mathbf{k}\mathbf{r}_\perp - kx), \quad (58)$$

where $K_1(z)$ is the first-order Macdonald function.

The second branch of surface oscillations corresponds to high-frequency waves of the same type as the Rayleigh waves in a solid.[6] These waves are similar to those studied by Fomin.[4]

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[1)]We can easily see that the expressions given here not only hold at $x = 0$ but for any value of x within the diffuse edge of the system. In particular, the expression for the effective mass shows how this mass changes from the value in the interior to the value in vacuum, where it is equal to the particle mass m.

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