tion allows us to assume that both phenomena are realized in one spatial region and are due to a single cause, the two-plasma parametric instability that sets in when the threshold flux is exceeded.[9] Accordingly, the electron Langmuir waves of frequency $\omega_0/2$ generated in the plasma, first, produce the $(\frac{3}{2})\omega_0$ harmonic by combining with the laser radiation and, second, lead to generation of the hot electrons when absorbed as a result of the Cerenkov interaction with the electrons. All this allows us to state that under the conditions of experiment with the "Mishen'-1" facility the hot electrons are generated in a region with densities close to one-quarter the critical density, not as a result of the superheating of the plasma but because of the Cerenkov acceleration of the electrons by the electron Langmuir waves, which are the products of two-plasmon parametric instability. It seems to us that the approach described in the present communication is quite general and is of particular interest under conditions of interaction between the plasma corona and the CO_2-laser radiation, where the problem of anomalous heat transport is still quite vital because of the low densities in the absorption band.

[1] K. A. Brueckner, Nucl. Fusion 17, 1257 (1977). S. J. Gitomer and D. B. Henderson, Phys. Fluids 22, 364 (1979).

[2] N. E. Andreev, V. P. Silin, G. L. Stenchikov, Pis'ma Zh. Eksp. Teor. Fiz. 28, 533 (1978) [JETP Lett. 28, 494 (1978)]. N. E. Andreev, V. P. Silin, and G. L. Stenchikov, Nonlinear Interaction of Radiation with Lasma under Conditions of Deformation of the Density by a Ponderomotive Force, FIAN Preprint No. 113, 1979.

[3] R. Benattar, C. Popovics, R. Sigel, and J. Virmont, Phys. Rev. Lett. 42, 766 (1979). A. Raven and O. Willi, *ibid.* 43, 278 (1979).

[4] V. V. Aleksandrov, E. P. Velikhov, A. G. Galygin, N. G. Kiselev, N. G. Koval'skiĭ, V. V. Korobkin, P. P. Pashinin, M. I. Pergament, A. M. Prokhorov, and A. I. Yaroslavskiĭ, Proc. 4th European Conf. on Plasma Physics, Vol. 2, JINR, 1974, p. 407.

[5] V. V. Aleksandrov, V. D. Vikharev, V. V. Gavrilov, A. V. Senik, and A. I. Yaroslvaskiĭ, Abstracts, 2nd All-Union Conf. on High-Temp. Plasma Diagonstics, Khar'kov, 1977, Atomizdat, p. 19.

[6] V. V. Aleksandrov, M. V. Brenner, V. D. Vikharev, V. P. Zotov, N. G. Koval'skiĭ, M. I. Pergament, and V. N. Yufa, Features of Plasma Emission at Frequencies $2\omega_0$ and $(3/2)\omega_0$ in Interaction of Short Laser Pulses with Solid Targets, Preprint IAE-2852, M., 1977.

[7] V. V. Aleksandrov, V. D. Vokharev, V. V. Gavrilov, Yu. S. Petrykin, A. V. Senik, and A. I. Yaroslavskiĭ, Investigation of X Rays from a Laser Plasma in the "Mishen'-1" Facility, Preprint IAE-3258, M., 1979.

[8] V. Yu. Bychenkov, V. P. Silin, and V. T. Tikhonchuk, Fizika plazmy 3, 1314 (1977) [Sov. J. Plasma Phys. 3, 730 (1977)].

[9] V. P. Silin, Parametricheskoe vozdeistvie islucheniya bol'shoĭ moshchnosti na plazmu (Parametric action of High-Power Radiation on a Plasma), Nauka, 1973.

Translated by J. G. Adashko

Dynamics of the modulational instability of a broad spectrum of Langmuir waves

B. N. Breĭzman and V. M. Malkin

Institute of Nuclear Physics, Siberian Division, Academy of Sciences of the USSR

(Submitted 19 January 1980)
Zh. Eksp. Teor. Fiz. 79, 857–869 (September 1980)

We consider the modulational instability of a Langmuir turbulence spectrum in which the group velocities of the waves are large compared to the ion sound speed. We obtain a dispersion relation which enables us to determine the threshold and growth rate of the instability for an arbitrary ratio of the spatial scale of the modulation to the characteristic wavelength of the Langmuir oscillations. We derive and solve by the inverse scattering method an equation which describes the nonlinear stage of the instability of one-dimensional long-wavelength perturbations at small excess above threshold. We establish that the transition of the instability into the nonlinear regime is qualitatively similar to the hard excitation of turbulence in hydrodynamics.

PACS numbers: 52.35.Py, 52.35.Ra, 52.35.Mw

1. INTRODUCTION

The present paper is devoted to the non-linear stage of the one-dimensional modulational instability of a spectrum of Langmuir waves with random phases.[1,2] The special feature of the one-dimensional problem is that here the possibility of Langmuir collapse is excluded, i.e., non-linear effects necessarily lead to a stabilization of the growth of the modulational perturbations (see Ref. 3). Depending on how the amplitude of the non-linear oscillations behaves in the transition through the instability threshold, one can distinguish between two stabilization regimes: soft and hard. In the first case the amplitude just above threshold turns out to be small, and in the second case it reaches a finite magnitude at arbitrarily small excess above threshold. The problem of which of these two regimes is realized in the case of the modulational instability was not clear until recently. The present paper contains an answer to that question: if the instability threshold corresponds to long-wavelength perturbations, the regime is hard. This conclusion is based upon results given in section 3, where we obtain and solve a non-linear equation which describes the evolu-

0038-5646/80/090435-07$02.40

tion of long-wavelength small-amplitude modulational perturbations [see Eq. (19)]. From the solutions constructed here it follows, in particular, that the instability develops until an appreciable part of the Langmuir waves turns out to be captured in the potential wells formed by the plasma density inhomogeneities.

The statement of the non-linear problem which interests us requires, as shall become clear from what follows, a preliminary improvement of the linear theory of the modulational instability. We give this in section 2. The principal element here will be a generalization of the results obtained earlier in the long-wavelength limit by Bedenov and Rudakov[1] to the case of modulational perturbations of arbitrary wavelengths.

2. LINEAR APPROXIMATION

1. Derivation of the dispersion relation

Vedenov and Rudakov,[1] who started the study of the modulational instability of Langmuir oscillations on the basis of the kinetic equation for plasmons

$$\frac{\partial N(\mathbf{k},\mathbf{r},t)}{\partial t}+3\omega_p r_D^2\mathbf{k}\frac{\partial N(\mathbf{k},\mathbf{r},t)}{\partial \mathbf{r}}-\frac{1}{2}\frac{\omega_p}{n_0}\frac{\partial n}{\partial \mathbf{r}}\frac{\partial N(\mathbf{k},\mathbf{r},t)}{\partial \mathbf{k}}=0 \quad (1)$$

and of the equation for sound waves, taking the high-frequency force into account,[4]

$$\frac{\partial^2 n}{\partial t^2}-c_s^2\nabla^2 n=\frac{\omega_p}{2M}\nabla^2\int N(\mathbf{k},\mathbf{r},t)\,d\mathbf{k}, \quad (2)$$

showed that for a sufficiently high level of Langmuir turbulence in the plasma density perturbations start to grow spontaneously. It follows from their results[1] that the growth rate of this instability increases with decreasing wavelength of the perturbations. At the same time, Eq. (1), in the derivation of which the geometric optics approximation for the plasmons is used, does not enable us to analyze perturbations with small spatial scales. For the evaluation of the maximum growth rate (just as for the determination of the threshold of the instability) it is thus necessary to change to a more exact description of the Langmuir waves.

It is natural here to use the equation averaged over the "fast" time, for the amplitude of the high-frequency electrical field **E**. and the equation for the perturbations of the plasma density n (see Refs. 5, 6):

$$i\frac{\partial \mathbf{E}}{\partial t}+\frac{3}{2}\omega_p r_D^2\nabla(\nabla\mathbf{E})-\frac{1}{2}\frac{c^2}{\omega_p}[\nabla\times[\nabla\times\mathbf{E}]]=\frac{\omega_p}{2n_0}n\mathbf{E}, \quad (3)$$

$$\frac{\partial^2 n}{\partial t^2}-c_s^2\nabla^2 n=\frac{1}{16\pi M}\nabla^2|\mathbf{E}|^2. \quad (4)$$

The set (3), (4) and sets similar to it have been used before[5-7] in studies of the instability of monochromatic waves. Each such wave is a stationary solution with $n=0$, but a superposition of waves does not have this property, as the high-frequency force contains interference terms. As the initial state in the form of a set of waves is non-stationary, the statement of the problem itself of its stability must be made more precise. If, however, the spectrum is sufficiently broad, its rearrangment due to taken the interference contribution to the high-frequency force into account takes place relatively slowly. The corresponding characteristic time is equal to the time τ of the decay process in which high-frequency waves and ion sound take part. When studying faster instabilities (and it is just in these that we shall be interested) we can neglect the non-stationarity of the initial spectrum.

The lower limit for the instability growth rate γ, above which our results will be valid, is thus given by the following inequality:

$$\gamma\tau\gg 1. \quad (5)$$

We note that for the turbulence spectra discussed below

$$\tau=\left[\omega_p\frac{W}{n_0 T}\frac{m}{M}\frac{1}{k_0^2 r_D^2}\right]^{-1},$$

where W is the energy density and k_0 a characteristic wavenumber value for the Langmuir oscillations (see, e.g., Ref. 8, p. 104).

The limitation from above of the growth rate γ is connected with the requirement of randomness of phases of the high-frequency waves. We shall assume that the initial phases are random and, moreover, that the time of phase mixing is of the order or magnitude of $(\delta\omega)^{-1}$, where $\delta\omega$ is the width of the spectrum, which is small compared to the time for the evolution of the modulational instability, i.e.,

$$\gamma\ll\delta\omega. \quad (6)$$

Taking into account what we have said, we can replace the high-frequency force in Eq. (14) by its value averaged over the phases of the waves. The set (3), (4) then has stationary solutions in which $n=0$, while the electrical field is an arbitrary set of Langmuir and electromagnetic waves. We now consider a perturbation of the ion density

$$n\propto\cos(\mathbf{qr}-\Omega t)$$

and find from Eq. (3) the correction to the electrical field connected with this perturbation. Substituting the correction into the linearized right-hand side of Eq. (4) and averaging over the phases of the unperturbed field we get the following dispersion relation:

$$\Omega^2=q^2c_s^2+\frac{q^2\omega_p^2}{4Mn_0}\sum_{\lambda,\lambda'}\int d\mathbf{k}\left|\mathbf{S}_\lambda\left(\mathbf{k}+\frac{\mathbf{q}}{2}\right)\mathbf{S}_{\lambda'}^*\left(\mathbf{k}-\frac{\mathbf{q}}{2}\right)\right|^2 \times\frac{N_\lambda(\mathbf{k}+\mathbf{q}/2)-N_{\lambda'}(\mathbf{k}-\mathbf{q}/2)}{\omega_\lambda(\mathbf{k}+\mathbf{q}/2)-\omega_{\lambda'}(\mathbf{k}-\mathbf{q}/2)-\Omega-i0}. \quad (7)$$

Here $\omega_\lambda(\mathbf{k})$ is the difference between the frequency of the wave and the plasma frequency ω_p; $N_\lambda(\mathbf{k})$ is the spectral density of the waves of the branch λ, normalized by the condition

$$\int N_\lambda(\mathbf{k})\,d\mathbf{k}=W_\lambda/\omega_p,$$

where W_λ is the energy density of the waves; we denote by $\mathbf{S}_\lambda(\mathbf{k})$ the polarization vector of the wave. The index λ in Eq. (7) takes on three values, where[1)]

$$\omega_1={}^3/_2\omega_p k^2 r_D^2, \quad \mathbf{S}_1=\mathbf{k}/k;$$
$$\omega_{2,3}=k^2c^2/2\omega_p, \quad \mathbf{S}_{2,3}\perp\mathbf{k}.$$

We note that Eq. (7) allows a simple generalization to the case of a plasma in a weak external magnetic field ($\omega_{He}\ll\omega_p$). When there is a field, it is only necessary to correct appropriately the dispersion laws ω_λ and the polarization vectors $\mathbf{S}_\lambda$ and also to take into account the effect of the field on the ion motion, for which one must multiply the left-hand side of (7) by the follow-fraction (see Ref. 9).

$$\frac{\Omega^2-\omega_{Hi}^2}{\Omega^2-(\mathbf{qH}/qH)^2\,\omega_{Hi}^2}.$$

The equation obtained generalizes Vedenov and Rudakov's dispersion relation[1,2] to the case of short-wavelength perturbations and arbitrary dispersion relations for high-frequency waves. At the same time it describes correctly the modulational instability of a monochromatic wave, as one can easily verify by putting $N_\lambda(\mathbf{k})\propto\delta(\mathbf{k}-\mathbf{k}_0)$ and comparing the result with the dispersion relations in the papers by Zakharov[5] and Kuznetsov.[6]

2. Instability of the Langmuir turbulence spectrum

A detailed study of Eq. (7) will be given separately. Here we restrict ourselves to only those conclusions which refer to Langmuir turbulence in a plasma without a magnetic field. We analyze the stability of a Langmuir-wave spectrum with a width Δk of the same order of magnitude as the characteristic wavenumber k_0. We shall assume for the sake of simplicity that the spectral function $N(\mathbf{k})$ is even, and about k_0 we shall assume that

$$k_0 r_D \gg (m/M)^{1/4} \tag{8}$$

(this inequality means that the plasmon group velocity is much larger than the sound speed).

We turn first of all to long-wavelength perturbations with $q\ll k_0$. The instability threshold for them lies, according to Refs. 1,2 at $W/n_0T\sim k_0^2r_D^2$. If we are not too far above threshold so that

$$W/n_0T \ll Mk_0^4r_D^4/m, \tag{9}$$

we can show that the instability develops adiabatically slowly: in a time equal to its inverse growth rate, plasmons moving with the group velocity traverse a distance appreciably longer than the wavelength of the perturbation. This enables us to neglect the quantity Ω on the right-hand side of Eq. (7). We use also the fact that the ratio q/k_0 is small and we expand the right-hand side of (7) in a series up to terms of order $(q/k_0)^4$. As a result we get

$$\Omega^2=q^2c_s^2\left(1+\frac{WI_1}{12n_0Tk_0^2r_D^2}+\frac{q^2}{k_0^2}\frac{WI_2}{12n_0Tk_0^2r_D^2}\right), \tag{10}$$

where

$$I_1=\frac{k_0\,\omega_p}{W}\int d\mathbf{k}\frac{\mathbf{n}}{(\mathbf{kn})}\frac{\partial N}{\partial\mathbf{k}},$$

$$I_2=\frac{k_0^4\omega_p}{W}\int\frac{d\mathbf{k}}{(\mathbf{kn})}\left\{\frac{n_\alpha n_\beta n_\gamma}{24}\frac{\partial^3N}{\partial k_\alpha\,\partial k_\beta\,\partial k_\gamma}+\frac{1}{k^2}\left[\left(\frac{\mathbf{kn}}{k}\right)^2-1\right]\mathbf{n}\frac{\partial N}{\partial\mathbf{k}}\right\},\ \mathbf{n}=\frac{\mathbf{q}}{q}.$$

The dimensionless functions I_1 and I_2 in Eq. (10) are introduced such that their characteristic values would have a modulus of order of magnitude unity. Depending on the shape of the spectrum of the Langmuir oscillations and the direction of the vector $\mathbf{q}$, these functions can take on both positive and negative values.

We consider in more detail the threshold regime of the instability, with the implication that the lowest threshold corresponds to perturbations with $q\to 0$ (just this situation will be discussed in section 3 when we solve the non-linear problem). In that case the value of I_1 which is minimal with respect to $\mathbf{n}$ is negative and the value of I_2 for the corresponding $\mathbf{n}$ is positive. Satisfying the conditions $I_1<0, I_2>0$ does, generally speaking, not guarantee that it is precisely the long-wavelength perturbations that grow at the threshold (these are only necessary conditions). However, one can easily construct many examples of spectra for which also sufficient conditions are satisfied.

Denoting by ε the relative excess of the energy of the Langmuir oscillations over their threshold value we can rewrite Eq. (10) as follows:

$$\Omega^2=q^2c_s^2\left[-\varepsilon+\frac{I_1(\mathbf{n})-I_1(\mathbf{n}_0)}{I_1(\mathbf{n}_0)}+\frac{q^2}{k_0^2}\frac{I_2(\mathbf{n}_0)}{|I_1(\mathbf{n}_0)|}\right], \tag{11}$$

where

$$I_1(\mathbf{n}_0)=\min_{\mathbf{n}} I_1(\mathbf{n}).$$

From this it is clear that the maximum growth rate is reached when

$$q=k_0|\varepsilon I_1(\mathbf{n}_0)/2I_2(\mathbf{n}_0)|^{1/2}\ll k_0$$

and is equal to

$$\tfrac{1}{2}k_0c_s\varepsilon|I_1(\mathbf{n}_0)/I_2(\mathbf{n}_0)|^{1/2},$$

while the instability region in q stretches from $q=0$ to

$$q=k_0|\varepsilon I_1(\mathbf{n}_0)/I_2(\mathbf{n}_0)|^{1/2}$$

(see Fig. 1).

For a number of turbulence spectra the threshold value of q turns out to be different from zero and to be of the order of magnitude of the quantity k_0 [this occurs, in particular, for those spectra for which $I_1(\mathbf{n}_0)>0$ or $I_2(\mathbf{n}_0)<0$]. In that case it is convenient to expand the right-hand side of Eq. (7) in the vicinity of the largest "unstable" value of $\mathbf{q}$ which we denote by $\mathbf{q}_0$. As a result we get

$$\Omega^2=q_0^2c_s^2[-\varepsilon+k_0^{-2}A_{\alpha\beta}(q_\alpha-q_{0\alpha})(q_\beta-q_{0\beta})], \tag{12}$$

where the $A_{\alpha\beta}$ are coefficients of order unity which depend on the shape of the turbulence spectrum. From the meaning of the expansion (12) it follows that the matrix $A_{\alpha\beta}$ is positive definite. The width of the instability zone in $\mathbf{q}-\mathbf{q}_0$ in the present case is estimated to remain the same as for the long-wavelength instability but the maximum growth rate is now proportional to ε rather than to $\varepsilon^{1/2}$.

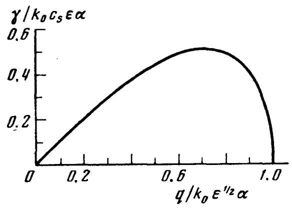

FIG. 1. Growth rate of the long-wavelength modulational instability γ averaged over directions near threshold ($\varepsilon\ll 1$) as function of the wavenumber q of the perturbation. The quantity $|I_1/I_2|^{1/2}$ is indicated by α; k_0 is a characteristic value of the wavevector of the Langmuir oscillations.

Equations (11) and (12), which correspond to a small excess above threshold, are suitable up to $\varepsilon \sim 1$. At the limit of their applicability ($W/n_0T \sim k_0^2 r_D^2$) the maximum growth rate is reached at $q \approx k_0$ and is estimated at

$$\gamma_{max} \sim \omega_p \left(\frac{W}{n_0 T}\frac{m}{M}\right)^{1/2}. \tag{13}$$

As W increases the instability region encompasses ever larger values of q. For example, when $W/n_0T \gg k_0^2 r_D^2$ perturbations with $q \gg k_0$ become unstable. For those perturbations the dispersion relation (7) can be considerably simplified:

$$(\Omega^2 - q^2 c_s^2)[\Omega^2 - \omega_i^2(q)] = q^2 c_s^2 \omega_i^2(q) \frac{\omega_p}{\omega_i(q)} \frac{\mu W}{2n_0T},$$
$$\mu = \frac{\omega_p}{W}\int d\mathbf{k} \left(\frac{\mathbf{kq}}{kq}\right)^2 N(\mathbf{k}). \tag{14}$$

We note that this formula, but with a different value of μ ($\mu = 1$), is true also in the case $q \ll k_0$, $|\Omega|/q \gg \omega_p k_0 r_D^2$. In this case it describes the instability of a cold plasmon gas, observed in Ref. 1.

The dispersion relation (14) enables us to determine the upper boundary of the instability region q_*:

$$q_*^2 = \frac{1}{3} r_D^{-2} \max_{\mathbf{n}} \frac{\omega_p}{n_0 T}\int d\mathbf{k}\left(\frac{\mathbf{kn}}{k}\right)^2 N(\mathbf{k}).$$

Moreover, it follows from it that up to $W/n_0T \sim Mk_0^4 r_D^4/m$ the maximum value of the growth rate is reached at $q \sim k_0$ and, as before, is given by the estimate (13).

In the region $W/n_0T \gtrsim Mk_0^4 r_D^4/m$ the condition for phase mixing (6) is violated and the formulation of the problem of the instability of the spectrum needs to be made more precise. The discussion of that limiting case goes beyond the framework of the present paper.

3. NON-LINEAR STAGE OF THE INSTABILITY

1. Basic equation

In this section we turn to the problem of the possibility of suppression of the modulational instability by a small non-linearity. As we have already noted we are interested in such a situation at a small excess above threshold ($\varepsilon \ll 1$) and the threshold corresponds to perturbations with $q \to 0$ (see Fig. 1). We restrict ourselves here to a discussion only of one-dimensional perturbations. However, the non-linear equation for the ion density which is then obtained [see Eq. (19)] can easily be generalized also to the three-dimensional case. To derive the equation in which we are interested we turn to the dispersion relation (11). Multiplying both sides by $n_{q\Omega}$ and performing the inverse Fourier transformation we get

$$\frac{\partial^2 n}{\partial t^2} + \varepsilon c_s^2 \frac{\partial^2 n}{\partial x^2} + \frac{I_2}{|I_1|}\frac{c_s^2}{k_0^2}\frac{\partial^4 n}{\partial x^4} = 0, \tag{15}$$

where the x-coordinate is reckoned in the direction corresponding to the maximum growth rate.

A comparison of Eq. (15) with the initial equation for n [see (4)] shows that in the linear problem the role of the high-frequency force is reduced to a renormalization of the sound velocity and an additional dispersion. Dispersion in the present case is caused by small corrections of order $(q/k_0)^2$. Therefore, when evaluating the non-linear corrections to the high-frequency force we can neglect it. This enables us to determine the required correction by means of Eq. (1). Since the instability develops adiabatically slowly, the plasmon distribution manages to adjust itself to the plasma density perturbation. The correction linear in n to the spectral function $N(\mathbf{k})$ of the plasmons then adds to the high-frequency force a contribution that leads to Eq. (15) with zero dispersion. However, the quadratic correction to $N(\mathbf{k}, \mathbf{r}, t)$ has the form

$$N^{(2)}(\mathbf{k}, x, t) = \frac{n^2 - \langle n^2\rangle}{72 n_0^2 r_D^4}\frac{1}{k_x}\frac{\partial}{\partial k_x}\frac{1}{k_x}\frac{\partial}{\partial k_x}N(\mathbf{k}), \tag{16}$$

where $N(\mathbf{k})$ is the unperturbed spectral function while the angle brackets indicate spatial averaging. The expansion here is in the parameter $n/n_0 k_0^2 r_D^2$, whose smallness means that the number of "trapped" plasmons is small compared to the number of untrapped ones. In order that the presence of trapped plasmons not hinder the use of perturbation theory, the initial spectrum $N(\mathbf{k})$ must be sufficiently smooth in the small wavenumber region.

Allowance for the corrections quadratic in n to the spectral function of the plasmons leads to replacement of the right-hand side of Eq. (15) by the quantity

$$\frac{\partial^2}{\partial x^2}\frac{\omega_p}{2M}\int N^{(2)}(\mathbf{k}, x, t)\, d\mathbf{k}.$$

As a result we have

$$\frac{\partial^2 n}{\partial t^2} + \varepsilon c_s^2 \frac{\partial^2 n}{\partial x^2} + \frac{I_2}{|I_1|}\frac{c_s^2}{k_0^2}\frac{\partial^4 n}{\partial x^4} = \frac{c_s^2 I_3}{n_0 k_0^2 r_D^2}\frac{\partial^2 n^2}{\partial x^2}, \tag{17}$$

where

$$I_3 = \frac{\omega_p k_0^2}{144 n_0 T r_D^2}\int \frac{d\mathbf{k}}{k_x}\frac{\partial}{\partial k_x}\frac{1}{k_x}\frac{\partial}{\partial k_x}N(\mathbf{k}). \tag{18}$$

The integral I_3 is estimated at $W/nTk_0^2 r_D^2$; for the threshold value of W it is a number of order unity. Depending on the shape of the spectrum, this number can be either positive or negative, but in the one-dimensional problem $I_3 > 0$, as in that case

$$I_3 = \frac{W}{12 n_0 T k_0^2 r_D^2} I_2.$$

It is convenient to change in Eq. (17) to dimensionless variables through the following substitutions:

$$x \to \frac{2}{k_0}\left|\frac{I_2}{I_1 \varepsilon}\right|^{1/2} x, \quad t \to \frac{2}{k_0 c_s \varepsilon}\left|\frac{I_2}{I_1}\right|^{1/2} t,$$
$$n \to \frac{3}{4}\frac{n_0 k_0^2 r_D^2}{I_3}\varepsilon u.$$

In the new variables this equation takes the following form:

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} + \frac{1}{4}\frac{\partial^4 u}{\partial x^4} = \frac{3}{4}\frac{\partial^2 u^2}{\partial x^2}. \tag{19}$$

For reference we give here the three-dimensional analogs of Eq. (19). If the initial spectrum of the Langmuir waves is isotropic, the second derivatives with respect to x must be replaced by the Laplacian operator:

$$\partial^2 u/\partial t^2 + \Delta u + {}^1/_4 \Delta\Delta u = {}^3/_4 \Delta u^2.$$

However, in the case of an anisotropic spectrum with an anisotropy of order unity we have

$$\frac{\partial^2 u}{\partial t^2}+\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}-\frac{\partial^2 u}{\partial z^2}+\frac{1}{4}\frac{\partial^4 u}{\partial x^4}=\frac{3}{4}\frac{\partial^2 u^2}{\partial x^2},$$

where the x-coordinate [as in Eq. (19)] corresponds to the unstable direction. In the last equation the scale lengths in the x-, y-, and z-directions are different.

We emphasize that Eq. (19) does not contain the small parameter ε. Therefore the boundedness in time of all its solutions, corresponding to small initial perturbations, would mean that a soft regime of instability saturation occurs. On the contrary, when infinitely growing solutions are present, a hard regime must exist.

2. Evolution of unstable perturbations

Equation (19) pertains to a number of non-linear equations which are integrable by the inverse-scattering method. Its integrability was established by Zakharov and Shabat,[10] but the solutions themselves were not obtained in that case. We construct them using Shabat's scheme[10,11] and introduce the auxiliary integral equation

$$K(x,y,t)=F(x,y,t)+\int_x^\infty K(x,s,t)F(s,y,t)\,ds, \tag{20}$$

which in shortened form has the form

$$K=F+K*F,$$

and consider a pair of differential operators:

$$D_1=\frac{\partial^3}{\partial x^3}+\frac{\partial^3}{\partial y^3}+\frac{\partial}{\partial x}+\frac{\partial}{\partial y},\quad D_2=\frac{3^{1/2}}{2}\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)+\frac{\partial}{\partial t}.$$

Each of the operators D_i $(i=1,2)$ corresponds to a "dressed" operator $\tilde{D}_i$ defined such that

$$\tilde{D}_iK=D_iF+\tilde{D}_iK*F+K*D_iF. \tag{21}$$

The operators D are given by the following formulae:[10]

$$\tilde{D}_1=D_1-\frac{3}{4}\left(u\frac{\partial}{\partial x}+\frac{\partial}{\partial x}u\right)+\frac{3}{2}w,\quad \tilde{D}_2=D_2-\frac{3^{1/2}}{2}u,$$

where

$$u=-2\frac{d}{dx}K(x,x,t),$$
$$w=\frac{d}{dx}\left\{K^2(x,x,t)+\left[\frac{\partial}{\partial x}K(x,y,t)-\frac{\partial}{\partial y}K(x,y,t)\right]_{y=x}\right\}. \tag{22}$$

Equation (21) shows that with appropriate limitations on the functions K and F the equations $D_iF=0$ and $\tilde{D}_iK=0$ are equivalent. Hence, if F satisfies the two equations

$$D_1F=0,\quad D_2F=0, \tag{23}$$

which are necessarily compatible, K must be a solution of the set

$$\tilde{D}_1K=0,\quad \tilde{D}_2K=0. \tag{24}$$

The compatibility condition of this set reduces to Eq. (19). Thus, each solution of Eq. (20) with kernel F that satisfies conditions (23) generates a solution of Eq. (19).

In the problem considered by us, of most interest physically are those solutions of Eq. (19) which at the initial instant are a set of unstable sinusoidal small-amplitude waves. An obstruction to the construction of such solutions is given both by the usual difficulties of studying a problem with initial conditions and by the fact that Shabat's scheme in its standard form enables one only to look for solutions which decrease as $x\to+\infty$ (or as $x\to-\infty$). In the present case, however, it is possible to circumvent these difficulties: we shall show that the required solutions are obtained by going to the limit of solutions corresponding to a degenerate kernel F that decreases as $x\to+\infty$.

We consider the degenerate kernel

$$F(x,y,t)=\sum_n f_n(x,t)\varphi_n(y,t). \tag{25}$$

Substituting it into Eq. (23) we find the functions f_n and φ_n:

$$f_n=a_n\exp\left\{ik_{nx}x+\frac{3^{1/2}}{2}k_{nx}^2t+i\theta_n\right\},\quad \varphi_n=\exp\left\{ik_{ny}y-\frac{3^{1/2}}{2}k_{ny}^2t\right\}. \tag{26}$$

Here a_n and θ_n are arbitrary real constants, and k_{nx} and k_{ny} are complex numbers connected by the relation

$$k_{nx}^2-k_{nx}k_{ny}+k_{ny}^2=1. \tag{27}$$

We assume additionally for k_{nx} and k_{ny} that

$$\operatorname{Im}k_{nx}>0,\quad \operatorname{Im}k_{ny}>0. \tag{28}$$

In what follows we assume that the kernel F consists of pairs of complex conjugate terms $f_n\varphi_n+f_n^*\varphi_n^*$, so that it is automatically real.

We write the solution of the integral equation (20) for the kernel (25) in the following form:

$$K(x,y,t)=\sum_n \psi_n(x,t)\varphi_n(y,t),$$

where the functions ψ_n are determined from the set of linear algebraic equations:

$$\sum_m A_{nm}\psi_m=f_n,\quad A_{nm}=\delta_{nm}-\int_x^\infty f_n(s,t)\varphi_m(s,t)\,ds. \tag{29}$$

Solving this set and then using Eq. (22) we can obtain the following expression for $u(x,t)$ (see, e.g., Ref. 12):

$$u=-2\frac{\partial^2}{\partial x^2}\ln\Lambda, \tag{30}$$

where

$$\Lambda=\det A_{nm}. \tag{31}$$

In the region where it is regular, the function $u(x,t)$ determined by Eqs. (30) and (31) necessarily satisfies Eq. (19).

We let the imaginary parts of the numbers k_{nx} and k_{ny} tend to zero, assuming additionally that none of the quantities $k_{nx}+k_{ny}$ vanishes. The function Λ which is obtained as the result of taking this a limit turns out to be regular on the whole of the x-axis and not to be decreasing as $x\to\pm\infty$. In the simplest case when the sum (25) consists of two complex conjugate terms, Λ is given by the following formula:

$$\Lambda=1-2\frac{a}{k}\exp\left(\frac{3^{1/2}}{2}k\varkappa t\right)\cos(kx+\theta)+\frac{a^2}{k^2}\left(1-\frac{k^2}{\varkappa^2}\right)\exp(3^{1/2}k\varkappa t). \tag{32}$$

The constants a, θ, k, and $\varkappa$ are real, while a and θ are arbitrary, and k and $\varkappa$ connected by the relation

$$k^2+3\varkappa^2=4.$$

Substituting $\Lambda(x,t)$ in Eq. (30) we get

$$\frac{u}{2k^2}=\frac{1-\rho\cos(kx+\theta)}{[\rho-\cos(kx+\theta)]^2}, \tag{33}$$

where

$$\rho(t)=\frac{k}{2a}\exp\left(-\frac{3^{1/2}}{2}k\varkappa t\right)+\frac{a}{2k}\left(1-\frac{k^2}{\varkappa^2}\right)\exp\left(\frac{3^{1/2}}{2}k\varkappa t\right).$$

We trace the evolution of the solution (33) assuming that initially there is an unstable small-amplitude perturbation (see Fig. 2a). In that case $\rho(0)\gg1$, and k and $\varkappa$ have the same sign. For not too long times $[t \ll \ln(\rho(0)/k\varkappa u)]$ Eq. (33) describes the linear stage of the instability:

$$u\approx-4ak\exp\left(\frac{3^{1/2}}{2}k\varkappa t\right)\cos(kx+\theta).$$

When the perturbation grows its shape starts to deviate from sinusoidal: the maxima become flatter and the minimum steeper (see Fig. 2b). When $\rho=2$ the first three derivatives of the function $u(x,t)$ vanish at its maxima. After that in the position of each previous maximum there occurs a minimum and in its vicinity two symmetrically positioned maxima appear which grow and move away from each another (see Fig. 2c). As $\rho\to1$ the new maxima approach the points $x=x_s=(2\pi n-\theta)/k$, where the function $u(x)$ has absolute minima, and the solution becomes singular.

The singularity occurs at the time

$$t=t_s=\frac{2}{3^{1/2}k\varkappa}\ln\frac{\varkappa k}{(k+\varkappa)a}$$

when the minimum value of Λ vanishes. In the vicinity of the singularity, $u(x,t)$ varies in a self-similar manner:

$$u(x,t)=-\frac{4}{3^{1/2}(t_s-t)}\frac{1-(x-x_s)^2/3^{1/2}(t_s-t)}{[1+(x-x_s)^2/3^{1/2}(t_s-t)]^2}. \tag{34}$$

We see that this function is universal: its form does not depend on the parameters of the initial wave. The initial conditions determine solely the location x_s and the time t_s where the singularity appears. It is noteworthy that the singularity [if it occurs at all in the solution (30), (31)] has the form (34) not only for an initial condition in the form of a single wave, but also for any other initial condition. Indeed, an expansion of the function $\Lambda(x,t)$ in power series in $x-x_s$ and t_s-t in the vicinity of its zero must have the form

$$\Lambda=g[(x-x_s)^2+\beta^2(t_s-t)].$$

The coefficient g does not affect $u(x,t)$ while β^2 is uniquely determined from Eq. (19): $\beta^2=3^{1/2}$. Hence it follows at once that $u(x,t)$ is given by Eq. (34). We note also that the function (34) is an exact solution of the equation

$$\frac{\partial^2u}{\partial t^2}+\frac{1}{4}\frac{\partial^4u}{\partial x^4}-\frac{3}{4}\frac{\partial^2u^2}{\partial x^2}=0.$$

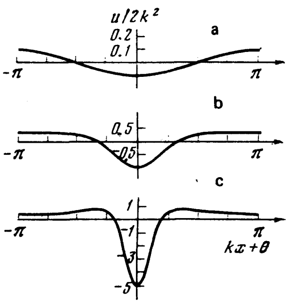

FIG. 2. Evolution of an unstable small-amplitude plasma density perturbation [see Eq. (33)]. The figures correspond to the following values of the parameter ρ: a) $\rho=10$, b) $\rho=2$, c) $\rho=1.2$.

Knowing the plasma density profile $u(x,t)$, it is easy to find the energy distribution of the Langmuir waves. To do this we remember that at small excess above threshold, and this is the case to which Eq. (19) refers, the perturbation of the gas-kinetic pressure is almost completely compensated by the plasmon density. Therefore, up to small corrections, the perturbation of the energy density of the waves is proportional to the function $u(x,t)$ with the opposite sign.

Kernels F of the form (25) generate not only the solution (33) corresponding to a single wave, but also solutions that initially are arbitrary superpositions of unstable sinusoidal small-amplitude waves. The number of such waves is equal to the number of complex conjugate pairs in the sum (25). One can show that under very lax limitations on the initial conditions, the solutions of such a form become singular at some time. The conclusion that a singularity appears refers, in particular, to that case where the initial spectrum of the plasma density perturbations is a noise spectrum.

The solutions described above are an example of a hard transition of the modulational instability into the non-linear regime. They show that in the region where Eq. (19) is applicable the instability is not suppressed. Regardless of the excess above threshold, the perturbations of the plasma density reach a level

$$n/n_0\sim k_0^2r_D^2,$$

corresponding to the limit of applicability of our approach. The spatial scale of the perturbations is then comparable with the characteristic wavelength k_0^{-1} of the Langmuir oscillations.

We note that the picture considered here of the evolution of the modulational instability reminds us qualitatively of the results obtained in numerical modeling of one-dimensional Langmuir turbulence (see Ref. 13). It is well known that the calculations lead to appearance of localized density perturbations which diminish in size with time. The shape of these perturbations (solitons) is similar to the self-similar solution (34). It is, unfortunately, impossible to pursue their comparison here quantitatively, however, since the calculations of Ref. 13 include pumping and dissipation of Langmuir waves.

3. Sub-threshold instability of finite-amplitude perturbations

Since, as we saw, a small non-linearity does not lead to suppression of the instability, it is natural to assume that in the given case (as in the case of the hard excitation of hydrodynamic turbulence[14]) there must occur an instability of finite-amplitude perturbations in the sub-threshold regime. We can verify the validity of this assumption as follows. We introduce a function $\xi(x,t)$ which is the displacement of the ions from their equilibrium position. In that case

$$u(x,t)=-\frac{\partial}{\partial x}\xi(x,t).$$

The function $\xi(x,t)$ satisfies the following equation [see Eq. (19)]:

$$\frac{\partial^2\xi}{\partial t^2}\pm\frac{\partial^2\xi}{\partial x^2}+\frac{1}{4}\frac{\partial^4\xi}{\partial x^4}+\frac{3}{4}\frac{\partial}{\partial x}\left(\frac{\partial\xi}{\partial x}\right)^2=0. \quad (35)$$

The upper sign of ξ_{xx} corresponds here to the above-threshold regime, and the lower one to the sub-threshold regime. We can write Eq. (35) in Hamiltonian form:

$$\partial\xi/\partial t=\delta H/\delta\eta, \quad \partial\eta/\partial t=-\delta H/\delta\xi, \quad (36)$$

where

$$H={}^1/_2\int dx(\eta^2\mp\xi_x^2-{}^1/_2\xi_x^3+{}^1/_4\xi_{xx}^2). \quad (37)$$

The displacement ξ and velocity $\eta=\xi_t$ of the ions are here the canonically conjugate variables. As the Hamiltonian H does not depend explicitly on the time, Eq. (35) has an energy integral: $H=E=\text{const}$.

One can, using Eqs. (36), easily verify that the following relation holds:

$$\frac{d^2}{dt^2}\int dx\,\xi^2=-6E+\int dx\left(5\xi_t^2\mp\xi_x^2+\frac{1}{4}\xi_{xx}^2\right). \quad (38)$$

From this it clear that in the sub-threshold regime, when the second term on the right-hand side of (38) is positive, any perturbation with a negative energy grows without limit with time. The negative contribution to the energy is connected with the cubic terms in the Hamiltonian (37). Therefore, the corresponding perturbation must necessarily have an amplitude u of order unity. In terms of dimensional variables this critical amplitude is the smaller the closer the system is to the instability boundary.

[1)]From the degeneracy of the dispersion law for electromagnetic waves it follows that, strictly speaking, these waves are not characterized by the vectors S_2 and S_3, but by a polarization tensor. In introducing here the vectors S_2 and S_3 we imply, in fact, that there is some small perturbation which lifts the degeneracy.

[1]A. A. Vedenov and L. I. Rudakov, Dokl. Akad. Nauk SSSR **159**, 767 (1964) [Sov. Phys. Dokl. **9**, 1073 (1965)].
[2]A. A. Vedenov, A. V. Gordeev, and L. I. Rudakov, Plasma Phys. **9**, 719 (1967).
[3]A. A. Galeev, R. Z. Sagdeev, V. D. Shapiro, and V. I. Shevchenko, Zh. Eksp. Teor. Fiz. **73**, 1352 (1977) [Sov. Phys. JETP **46**, 711 (1977)].
[4]A. V. Gaponov and M. A. Miller, Zh. Eksp. Teor. Fiz. **34**, 242 (1958) [Sov. Phys. JETP **7**, 168 (1958)].
[5]V. E. Zakharov, Zh. Eksp. Teor. Fiz. **62**, 1745 (1972) [Sov. Phys. JETP **35**, 908 (1972)].
[6]E. A. Kuznetsov, Zh. Eksp. Teor. Fiz. **66**, 2037 (1974) [Sov. Phys. JETP **39**, 1003 (1974)].
[7]V. V. Krasnosel'skikh and V. I. Sotnikov, Fiz. Plazmy **3**, 872 (1977) [Sov. J. Plasma Phys. **3**, 491 (1977)].
[8]V. N. Tsytovich, Nelineĭnye éffekty v plazme (Non-linear effects in plasmas) Nauka, 1967 [English translation published by Plenum Press, New York].
[9]R. Pozzoli and D. D. Ryutov, Phys. Fluids **22**, 1782 (1979).
[10]V. E. Zakharov and A. B. Shabat, Funkts. Analiz **8**, 54 (1974) [Funct. Anal. Appl. **8**, 43 (1974)].
[11]A. B. Shabat, Dokl. Akad. Nauk SSSR **211**, 1310 (1973).
[12]I. A. Kunin, Teoriya uprugikh sred s mikrostrukturoĭ (Theory of elestic media with micro-structure) Nauka, 1975, Ch. V.
[13]T. A. Gorbushina, L. M. Degtyarev, R. Z. Sagdeev, V. D. Shapiro, and V. I. Shevchenko, Preprint No 17, Inst. Appl. Math. Acad. Sc. USSR, Moscow, 1978.
[14]L. D. Landau and E. M. Lifshitz, Mekhanika sploshnykh sred (Mechanics of Continuous Media) Gostekhizdat, 1954 [English translation published by Pergamon Press, Oxford].

Translated by D. ter Haar

Expansion of collisionless plasma in a vacuum

V. G. Eselevich and V. G. Faĭnshteĭn

Institute of Terrestrial Magnetism and Radiowave Propagation, Siberian Division, USSR Academy of Sciences

(Submitted 1 February 1980)
Zh. Eksp. Teor. Fiz. **79**, 870–882 (September 1980)

The expansion, in a vacuum, of a collisionless plasma with a front of width $\Delta<10r_D$ and $\Delta\gtrsim 10r_D$ is investigated experimentally. [$r_D=(T_e/4\pi\,ne^2)^{1/2}$ is the local Debye radius determined by the plasma parameters at the crest of the front; $n\approx 10^7$–10^{10} cm^{-3}, $T_e\approx$ 2–10 eV]. It is shown that as the front moves away from the source, the action of the force of the electronic pressure gradient produces on the front a continuous acceleration of the ions to velocities much higher than the velocity of ion sound. The electronic heat conduction, which supplies energy to the electrons that accelerate the ions, turn out to be much less than in the case of a collisionless plasma. The physical aspect of these processes is investigated in detail.

PACS numbers: 52.30. + r

I. INTRODUCTION

Expansion of a collisionless plasma in a vacuum is one of the important phenomena in plasma physics. Its effects play a substantial role in the acceleration of charged particles in laboratory and cosmic plasma, in the flow of plasma out of stars and in laser-mediated thermonuclear fusion. Many aspects of this phenomena, however, remain unclear to this day.

Ion acceleration in the expansion of the plasma in a vacuum was observed in many experiments, starting with Tanberg's 1930 work[1] (see, e.g., Refs. 2–4). An explanation for this effect, on the basis of the mechanism of ambipolar ion acceleration by electrons, was proposed by Plyutto.[4] A more rigorous treatment of the problem of expansion of a collisionless plasma in a vacuum, assuming a constant electron temperature T_{e0}, was carried out by A.V. Gurevich *et al.*[5] One of

0038-5646/80/090441-08$02.40