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Contribution to the theory of the two-dimensional mixed state in type-I superconductors

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We consider the nature of the two-dimensional mixed state produced on the inner surface of a hollow cylinder when the superconductivity is destroyed by current. The two-dimensional mixed-state layer constitutes a structure periodic along the cylinder axis, consisting of alternating annular superconducting regions and regions in which the macroscopic phase coherence is disturbed and the order-parameter phase undergoes at certain instants of time 2π jumps at a frequency satisfying the Josephson condition, while the order parameter oscillates between zero and a certain finite value. This picture is analogous to the phase slippage centers in the resistive state of a narrow superconducting channel. The current-voltage characteristic of the sample is calculated, and one of its peculiarities is the presence of an excess current that depends little on the sample voltage.

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1. INTRODUCTION

In the study of the properties of current-carrying super-conductors a situation frequently arises wherein, despite of the presence of a constant electric field in the sample, purely thermodynamic factors favor the formation of a superconducting state either in the entire sample or in definite sections of the sample (if the temperature of the superconductor and the magnetic field in these sections are lower than the critical values). Thus, the coexistence of a constant electric field and superconductivity is observed in narrow (quasi-one-dimensional) superconducting channels in a certain range of current (the so-called resistive state; see, e.g., Refs. 1 and 2). One more example is connected with the destruction of the superconductivity by current in solid type-I superconductors, when the sample becomes stratified into alternating normal and superconducting domains (the intermediate state). The superconducting domains cannot be in touch with one another on the macroscopic sections, for otherwise the sample becomes short-circuited. It is clear nevertheless that near the cylinder axis, where the magnetic field is weak, the formation of the superconducting state should be favored.

A peculiar situation takes place when the superconductivity is destroyed by current in hollow type-I cylindrical samples. As noted by L. Landau,[3] when the current through the sample exceeds $\mathscr{T}_c (r_1^2+r_2^2)/2r_1r_2$ (where $\mathscr{T}_c=cH_cr_2/2$ is the critical current, and r_1 and r_2 are the radii of the inner and outer surfaces of the cylinder), the intermediate state in the interior of the sample vanishes and goes over into the normal state. At the same time, on the inner surface the field is weak, therefore the normal state is unstable there. The state produced near the inner surface of the cylinder is, however, not purely superconducting, since a constant electric field is present in the sample. Such a state is called two-dimensional mixed (TM), and was experimentally observed by I. Landau and Sharvin.[4] A qualitatively similar picture appears on the surface of a bulky superconducting sample when an external magnetic field exceeding the critical value is turned off. When turned off, the magnetic field in space vanished rapidly, whereas in the sample volume, on account of the induced eddy currents, it retains a large value for a rather long time. As a result, the formation of the TM state turns out to be convenient on the surface. This situation was investigated experimentally in detail by Dorozhkin and Dolgopolov.[5]

In all the listed examples, in some sections of the superconductor there exists simultaneously a constant electric field and superconductivity. The primary reason is that the constant electric field penetrates into the superconductor to a finite depth l_E. It is known that this depth as a rule greatly exceeds the coherence length $\xi(T)$ as well as the penetration depth $\lambda(T)$ of a constant magnetic field (for alloys without paramagnetic impurities we have near the critical temperature $l_E=l_\varepsilon(4T/\pi\Delta)^{1/2}$,[6] where l_E is the diffusion length of the quasiparticles $l_\varepsilon=(\mathscr{D}\tau_{ph})^{1/2}$).

We are dealing thus with a situation in which the established superconductivity exists against the background of a constant electric field. If the conditions of the problem are such that the field differs from zero in macroscopic sections of the sample, then the scalar potential φ can assume large values. It is clear that when the latter increases the superconductivity should become destroyed in the entire volume. This, however,

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does not take place for the following reason. The behavior of the superconductor is determined by the gauge invariant scalar and vector potentials

$$\Phi=\varphi+\frac{1}{2e}\frac{\partial\chi}{\partial t}; \quad \mathbf{Q}=\mathbf{A}-\frac{c}{2e}\nabla\chi,$$

where χ is the phase of the order parameter. At some points of the sample and at definite instants of time, the phase χ undergoes jumps in order to ensure by the same token a finite, in the mean, gauge-invariant scalar potential Φ. In these space-time points (which are called the phase slippage centers[7]—PSC) the order parameter, naturally, vanishes. Thus, at the expense of destruction of the superconductivity in relatively narrow vicinities of the PSC, the existence of superconductivity is ensured in macroscopic sections of the sample.

The PSC concept turned out to be quite fruitful as applied to the resistive state of quasi-one-dimensional superconductors.[1,2,7-10] In the present paper we attempt to use a previously developed approach[2] to construct a model of the TM state that is produced on the inner surface of a hollow cylinder.

According to the premises of the earlier studies,[2] the PSC which are essentially dynamic objects, occupy rather narrow regions in the superconducting structure, whereas in the bulk of the volume the picture is static, with the electric field E and all the quantities that describe the superconductor evolving slowly over scales of the order of the penetration depth l_E of the electric field, which exceeds the coherence length and the penetration depth of the magnetic field. Of very great use in this connection are the ideas advanced by Andreev and Tekel' (AT),[11] that for a weak electric field[1)] the structure of the TM layer in the main volume is determined by the same factors that take place in the absence of an electric field.

The picture proposed below can be briefly described in the following manner. In view of the weakness of the electric field, the thickness d of the TM layer, which in our case turns out to be less than l_E, as well as the behavior of the order parameter in the layer and the value of the magnetic field on the boundary with the normal region are determined locally (i.e., at a given point along the cylinder axis) are determined by the AT solutions[11]; these quantities, in turn, vary slowly along the cylinder axis over scales of the order of l_E. When that section of the TM layer where the macroscopic phase coherence is approached, [we shall call this section the phase slippage region (PSR)], the gauge-invariant potential Φ increases in absolute value.

In the PSR, in accordance with the premises of the preceding studies,[2] the order parameter executes strong oscillations, and at the instant when the order parameter turns to zero its phase experiences a jump of 2π. The frequency at which the order parameter oscillates and at which its phase experiences jumps is determined by the Josephson relation $2\ eV=\omega$, so that a poetntial Φ that is finite in the measn is ensured.[7,9] This picture repeats periodically along the cylinder axis, with V in the relation above representing the potential difference between two neighboring superconducting sections. The question of the structure of the PSR itself remains as yet open.

The proposed picture differs from the model of Gor'kov and Dorokov[12] primarily because the electric field in the superconducting TM layer is in our case different from zero on account of the large penetration depth, $l_E \gg \xi(T), d$, and has a component tangential to the layer.

2. MODEL DYNAMIC EQUATIONS OF SUPERCONDUCTIVITY

The complete system of dynamic equations that describe the behavior of a real superconductor with a gap in the energy spectrum is quite complicated. In addition for the equations for the modulus of the order parameter Δ and for the potentials $\mathbf{Q}$ and Φ, it contains also the kinetic equations for the distribution function. A closed system containing only the superconducting parameters Δ, $\mathbf{Q}$, and Φ can be obtained only near the critical temperature and under most stringent restrictions on the characteristic spatial and temporal scales of variation of all the quantities. It is therefore quite useful to consider first the simplest model dynamic equations, which constitute a time-dependent generalization of the Ginzburg-Landau equations. In dimensionless variables, they take the form

$$u\partial\Delta/\partial t+(\Delta^2+\mathbf{Q}^2-1)\Delta-\nabla^2\Delta=0, \quad (1)$$

$$\mathbf{j}=\varkappa^2 \operatorname{rot}\operatorname{rot}\mathbf{Q}=-\partial\mathbf{Q}/\partial t-\nabla\Phi-\Delta^2\mathbf{Q}, \quad (2)$$

$$u\Delta^2\Phi+\operatorname{div}(\Delta^2\mathbf{Q})=0. \quad (3)$$

The unit length is here the coherence length $\xi(T)$, the unit of Δ is its equilibrium value in the absence of a magnetic field, and the magnetic field is measured in units of $c2e\xi^2$. The critical field H_c corresponds to $H_c=\frac{1}{2}^{1/2}\varkappa$, where $\varkappa$ is the Ginzburg-Landau parameter. The potentials $\mathbf{Q}$ and Φ are given by

$$\mathbf{Q}=A-\nabla\chi, \quad \Phi=\varphi+\partial\chi/\partial t.$$

It follows from (2) and (3) that

$$u\Delta^2\Phi-\nabla^2\Phi-\frac{\partial}{\partial t}\operatorname{div}\mathbf{Q}=0. \quad (4)$$

We see therefore that the parameter u is connected with the depth of penetration of the constant electric field

$$u=\xi^2(T)/l_E^2.$$

At $u=12$, the system (1)–(4) corresponds to a superconducting alloy with a large concentration of paramagnetic impurities.[13] In this paper, just as before,[2] we shall, however, regard u as a free parameter. As noted in the introduction, for ordinary superconductors (with a gap in the energy spectrum), as a rule $l_E \gg \xi(T)$. We therefore consider below the case $u \ll 1$.

Concerning Eq. (2) we must make one more remark. The local connection between the field

$$\mathbf{E}=-\nabla\Phi-\partial\mathbf{Q}/\partial t \quad (5)$$

and the current, which was proposed in (2), takes place only in rather dirty superconductors, whereas in the experiments of Refs. 4 and 5 the samples used were quite pure. Nonetheless, when considering the simple situation, we shall use this equation.

We assume that the sample is a type-I superconductor with $\varkappa \ll 1$. The wall thickness of the hollow cylinder is assumed to be small compared with its radius $D \ll r$. By virtue of the axial symmetry of the problem, we direct the z axis along the generator of the inner surface of the cylinder, and the x axis along the normal to the inner surface, so that $x=0$ corresponds to the inner surface and $x=D$ to the outer one. We assume, of course, that the wall thickness D is much larger than the depth of penetration of the field $l_E = u^{-1/2}$.

As indicated in the introduction, it is assumed in the considered picture that the PSR, which constitute in essence dynamic formations occupy rather narrow sections compared with the regions where the static situation holds. The conditions needed for this purpose will be discussed later. We consider first the behavior of all the quantities in the static region.

The thickness d of the TM layer is generally speaking of the order of unity (i.e., of the order of ξ in ordinary units) and is small compared with the depth of penetration of the electric field, $d \ll u^{-1/2}$. Therefore Φ changes little over the thickness of the TM layer. Integrating the z-component of Eq. (2) with respect to dx from 0 to d, we obtain

$$\varkappa^2 H_0 = -d\frac{d\Phi}{dz} + I_s, \tag{6}$$

where H_0 is the value of the magnetic field on the boundary between the TM layer and the normal region, and

$$I_s = \int_0^d j_s\,dx = -\int_0^d \Delta^2 Q_z\,dx \tag{7}$$

is the total superconducting current flowing over the TM layer (per unit length of the inner periphery of the cylinder base).

Since the density of the superconducting layer along the x axis vanished on the inner surface of the cylinder and on the boundary of the TM layer with the normal region, the integration of Eq. (3) with respect to dx yields

$$u\eta\Phi = dI_s/dz, \qquad \eta = \int_0^d \Delta^2\,dx. \tag{8}$$

In the derivation (8) we used the fact that $\Delta = 0$ at $x=d$.

From (4) we obtain in the same manner

$$u\eta\Phi + j_{nx} - d\frac{d^2\Phi}{dz^2} = 0, \tag{9}$$

where

$$j_{nx} = -\left.\frac{\partial\Phi}{dx}\right|_{x=d}$$

is the density of the normal current that flows out of the TM layer along the x axis. In Eqs. (6), (8) and (9) all the quantities are functions of the coordinate z and vary over scales of the order of $u^{-1/2}$.

In the static region we have $\mathbf{E} = -\nabla\Phi = \nabla\varphi$, therefore the chemical potential of the pairs

$$\mu_p = \frac{1}{2e}\frac{\partial\chi}{\partial t}$$

does not depend on the coordinates. The values of μ_p for neighboring static regions differ from each other by an amount $\Phi(-L) - \Phi(L) = V$. What is of importance to us at present is the circumstance that, by virtue of the continuity of the potential φ and of the electric field we can thus assume that in each region with static conditions the potential Φ in the TM is connected, accurate to the constant μ_p, with the potential $\varphi_n(x,z)$ in the normal region by the relations

$$\Phi = \varphi_n(x=d;z), \qquad -\left.\frac{d\varphi_n(x,z)}{\partial x}\right|_{x=d} = j_{nx}. \tag{10}$$

The potential $\varphi_n(x,z)$ satisfies the equation

$$\nabla^2\varphi_n(x,z) = 0. \tag{11}$$

The electric field is periodic, with a structure period $2L \sim u^{-1/2}$, so that the solution of (11) takes the form

$$\varphi_n(x,z) = -\bar{E}z + \sum_{k\neq 0} a_k \frac{\mathrm{ch}[\pi k L^{-1}(D-x)]}{\mathrm{ch}[\pi k L^{-1} D]} \sin\left(\frac{\pi k}{L}z\right). \tag{12}$$

(We have chosen a solution that is odd with respect to the plane $z=0$.) The average intensity E is defined as $\bar{E} = V/2L$. Using the conditions (10) we obtain

$$j_{nx} = \sum_{k\neq 0} \frac{\pi k}{L}\,\mathrm{th}\left[\frac{\pi k D}{L}\right] \int_{-L}^{L} \frac{dz'}{L}[\Phi(z') + \bar{E}z'] \sin\left(\frac{\pi k}{L}z'\right) \sin\left(\frac{\pi k}{L}z\right). \tag{13}$$

The quantity j_{nx} is of the order of $\Phi/L \sim u^{1/2}\Phi$. Estimating the remaining terms in Eq. (9), we see that they are of the order of $\Phi d L^{-2} \sim ud\Phi$. Since $d \ll L \sim u^{-1/2}$, the term with j_{nx} in this equation is the largest. Thus, Eq. (9) for the potential Φ can be solved by successive approximations

$$\Phi = -\bar{E}z + \Phi'.$$

For Φ' we obtain in turn

$$\Phi'(z) = u \sum_{k\neq 0} \int_{-L}^{L} \eta \bar{E}z' \sin\left(\frac{\pi k}{L}z'\right) \sin\left(\frac{\pi k}{L}z\right) \mathrm{th}^{-1}\left[\frac{\pi k D}{L}\right] \frac{dz'}{\pi k}.$$

The quantity $\Phi'(z) \sim d\bar{E}z/L \ll \bar{E}z$.

For the potential in the normal region we obtain with the aid of (12)

$$\varphi_n(x,z) = -\bar{E}z + u \sum_{k\neq 0} \frac{\mathrm{ch}[\pi k L^{-1}(D-x)]}{\mathrm{sh}[\pi k L^{-1} D]} \times \int_{-L}^{L} \eta(z')\bar{E}z' \sin\left(\frac{\pi k}{L}z'\right) \sin\left(\frac{\pi k}{L}z\right) \frac{dz'}{\pi k}.$$

From this we readily get the distribution of the field $\mathbf{E} = -\nabla\varphi_n$ in the normal region. The fact that $\Phi \simeq -\bar{E}z$ is evidence that the problem of the structure of the TM layer should be solved for a specified constant electric field, which is formed by the macroscopic normal region adjacent to the TM layer.

Substituting $\Phi = -\bar{E}z$ in Eqs. (6) and (8), we get

$$\varkappa^2 H_0 - d\bar{E} = I_s, \tag{14}$$

$$-u\eta\bar{E}z = dI_s/dz. \tag{15}$$

The quantities $H_0(z)$, $d(z)$ and $\eta(z)$ can be connected with one another with the aid of the AT solution. Let us recall in this connection the principal result of that reference. Since the TM layer is narrow, $d \ll L$, we need retain in (1) only the terms Q_z^2 and $\partial^2\Delta/\partial x^2$, since $Q_x^2 \ll Q_z^2$ and $\partial^2\Delta/\partial z^2 \ll \partial^2\Delta/\partial x^2$. Writing down (1) together with the z-component of Eq. (2) we have (here and elsewhere we designate the average electric field intensity by the letter E):

$$(\Delta^2+Q_z^2-1)\Delta-\frac{\partial^2\Delta}{\partial x^2}=0, \tag{16}$$

$$-\varkappa^2\frac{\partial^2 Q_z}{\partial x^2}=E-\Delta^2 Q_z. \tag{17}$$

These equations have a first integral

$$\frac{1}{2}\Delta^4-\Delta^2+Q_z^2\Delta^2-\left(\frac{\partial\Delta}{\partial x}\right)^2-\varkappa^2\left(\frac{\partial Q_z}{\partial x}\right)^2-2EQ_z=C. \tag{18}$$

In the interior of the superconducting region (see Fig. 1) the total current is screened, so that $Q_z=E/\Delta^2$. On the inner surface of the cylinder we have $\Delta=\Delta_0$, $\partial Q_z/\partial x=-H=0$ and $\partial\Delta/\partial x=0$. Hence $C=\Delta_0^4/2-\Delta_0^2-E^2/\Delta_0^2$. In the normal region $\Delta=\partial\Delta/\partial x=0$, $\partial Q_z/\partial x=-H_0$, so that

$$\varkappa^2 H_0^2=\Delta_0^2-\frac{1}{2}\Delta_0^4+E^2/\Delta_0^2-2EQ_z.$$

The width of the transition section between the normal and superconducting regions coincides with the depth of the screening of the strong magnetic field, which, according to Ref. 14, is of the order of $\varkappa^{1/2}$. In this region $\Delta\sim\varkappa^{1/2}$ and $Q\sim\varkappa^{2-1/2}$, so that at $E\ll\varkappa^{1/2}$ the magnetic field on the boundary of the TM layer is

$$H_0=\frac{1}{\varkappa}\left(\Delta_0^2-\frac{1}{2}\Delta_0^4\right)^{1/2},$$

and coincides with the AT result.[11]

The inequality $E\ll\varkappa^{1/2}$ is the limit of applicability of the AT solution. Actually the electric field is much we weaker. The critical current that produces on the outer surface of the cylinder a field $\frac{1}{2}^{1/2}\varkappa$ is equal to $\mathcal{I}_c=2^{1/2}\pi r\varkappa$, and its density is $j_c=\varkappa/2^{1/2}D$. Thus, at a current of the order of critical we have $E\sim\varkappa/D$. The upper limit of E, according to AT, is $E_{max}\sim\varkappa$, which is also much less than $\varkappa^{1/2}$.

The spatial behavior of $\Delta(x)$ can be determined from (18) by leaving out of it the terms containing Q, which are significant only over distances on the order of $\varkappa^{1/2}$ from the boundary. According to AT (Ref. 11), $\Delta(x)$ is expressed in terms of the elliptic sine

$$\Delta(x)=\Delta_0\,\mathrm{sn}\left[\frac{d-x}{(1+k^2)^{1/2}},k\right],$$

where the parameter k, $0\leqslant k<1$, is connected with Δ_0 by the relation

$$\Delta_0=2^{1/2}k/(1+k^2)^{1/2}. \tag{19}$$

The thickness of the layer in a magnetic field H_0 is expressed in terms of the same parameter:

$$d=(1+k^2)^{1/2}K(k);\quad H_0=\frac{2^{1/2}k}{\varkappa(1+k^2)}. \tag{20}$$

Here $K(k)$ is a complete elliptic integral of the first kind.

When k changes from 0 to 1, the order parameter Δ_0 on the inner surface increases from 0 to 1, the field H_0 increases from 0 to H_c, and the thickness d of the layer changes from $\pi/2$ at $k=0$ to logarithmically large values $d=2^{-1/2}\ln[1/(1-k)]$ as $k\to 1$.

In contrast to the AT model, in our case the parameter k in Eqs. (19) and (20) depends adiabatically on z. With the aid of (14) we obtain in turn

$$I_s=\varkappa\left\{\frac{2^{1/2}k}{1+k^2}-\frac{E}{\varkappa}(1+k^2)^{1/2}K(k)\right\}. \tag{21}$$

The dependence of I_s on k is shown schematically in Fig. 2.

A specified value I_s in the interval $-\pi E/2<I_s<I_{s\,max}$ corresponds, generally speaking, to two values of k. It is shown in the Appendix that only the descending branch of the $I_s(k)$ plot is stable. This result is in a certain sense analogous to the known situation in the one-dimensional case, where the superconducting current and the order parameter Δ are connected by the relation $j_s=\Delta^2(1-\Delta^2)^{1/2}$, and again only the descending branch $(\frac{2}{3})^{1/2}\leqslant\Delta\leqslant 1$ is stable. The instability of the ascending branch of $I_s(k)$ can be seen also from the following example. Assume that the total current $\mathcal{I}$ is less than the critical current, so that $E=0$. If the branch $0\leqslant k<1$ were stable, then a possible solution would be one in which the superconducting region occupies only part of the cylinder, with thickness $d(k)$ determined by the condition $I=I_c 2k/(1+k^2)$, and the entire current would flow through the superconducting region. In the actual case this is not so, and the superconducting region extends at $\mathcal{I}<\mathcal{I}_c$ over the entire sample.

We express η in (8) in terms of k with the aid of the AT solution:

$$\eta=\frac{2}{(1+k^2)^{1/2}}[K(k)-E(k)], \tag{22}$$

where $E(k)$ is a complete elliptic integral of the second kind. It is now easy to integrate Eq. (15):

$$\frac{uz^2}{2}=-\frac{1}{E}\int_{k_0}^{k}\eta^{-1}(k)\frac{dI_s}{dk}dk. \tag{23}$$

Here k_0 is an integration constant located on the descending branch of the $I_s(k)$ defined by Eq. (21), and k

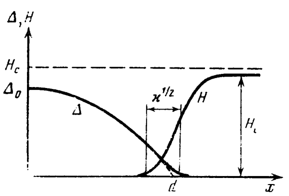

FIG. 1. Schematic dependence of the order parameter Δ and of the magnetic field H in the TM layer on the coordinate x directed into the interior of the sample in the AT model (local dependence of these quantities in our model). The value $x=0$ corresponds to the inner surface of the cylinder.

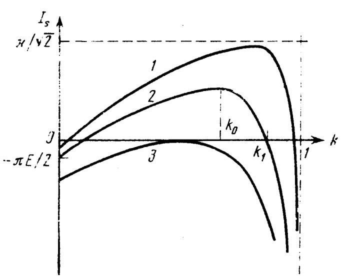

FIG. 2. Schematic plot of $I_s(k)$. Curves 1, 2, and 3 correspond to different values of $E/\varkappa$: $E_1<E_2<E_3$.

$> k_0$. Expression (23) gives the dependence of the parameter k of a coordinate z along the cylinder axis, and determines by the same token the behavior of the layer thickness $d(z)$, of the magnetic field $H_0(z)$ on the boundary, and of the order parameter $\Delta_0(z)$ within the limits of one static region $-L < z < L$. These static regions form a periodic structure along the cylinder axis and are separated from one another by the PSR.

Figure 3 shows the force lines of the electric field and the boundary of the TM layer as functions of the coordinate z along the cylinder axis. The shaded sections correspond to the PSR, where the static approximation is violated and the order parameter undergoes strong oscillations from zero to a certain value of the order of unity. At the instant of vanishing of the order parameter, the superconducting current flowing through the TM layer, I_s $(z = L)$, also vanishes at the points $z = \pm L$. The amplitude $\delta I_s(L)$ of its oscillations can be estimated from Eq. (8), which is valid also in the nonstationary region.

Let the width of the nonstationarity region, i.e., of the PSR, by z_1, with $z_1 \ll l_E$. Then in the static region, when the distances δz from the PSR satisfy the condition $z_1 \ll \delta z \ll L$, the oscillations of I_s are small: $\delta I_s \ll I_s$. Integrating Eq. (8) with respect to dz from $L - \delta z$ to L, we find that the amplitude of the oscillations in the PSR itself are of the order of

$$\delta I_s(L) \sim u \int_{L-\delta z}^{L} \eta\Phi \, dz.$$

Since the product $\Delta^2\Phi$ is always finite (the gauge-invariant potential Φ itself, of course, becomes infinite at the phase-slippage instant), we have for $\delta I_s(L)$

$$\delta I_s(L) \sim dE\delta z/L \ll dE.$$

Let now the superconducting current near the PSR be equal to $I_s(L - \delta z)$. With the aid of exactly the same estimates we can show that $|I_s(L) - I_s(L - \delta z)| \ll Ed$. Inasmuch as at the instant of phase slippage $I_s(L) = 0$, and the amplitude of its oscillations is small, we always have $I_s(L) \ll Ed$. It follows therefore that also $I_s(L - \delta z) \ll Ed$. Since we have chosen $\delta z \ll L$, it follows that, considering the problem of scales $z \sim u^{-1/2} \sim L$, it can be shown that the point $z = L$ corresponds to the value $k = k_1$ determined by the condition $I_s(k_1) = 0$:

$$K(k_1)\frac{(1+k_1^2)^{3/2}}{2^{3/2}k_1} = \frac{\varkappa}{E}. \tag{24}$$

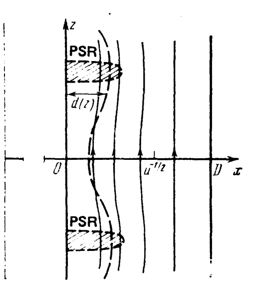

FIG. 3. Distribution of the field E and of the TM layer over the cylinder thickness. Solid line—force lines of E; dashed line—boundary of superconducting region; the shaded sections are annular PSR; the dash-dot line corresponds to the cylinder axis.

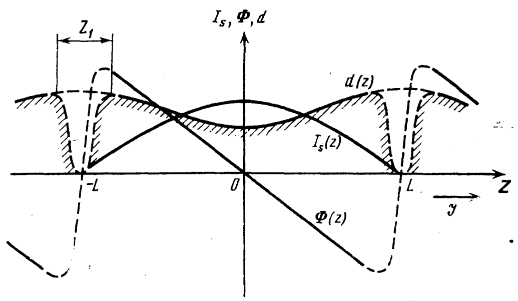

FIG. 4. Structure of TM layer. The solid lines show the quantities $d(z)$, $I_s(z)$, $\Phi(z)$ in the static region. The dashed lines show their behavior in the PSR. The dashed hatched line shows the position of the TM layer boundary at the instant of phase slippage.

The period $2L$ of the structure is obtained from (23), where we must put $k = k_1$, $z = L$. Expression (23) contains the free parameter k_0, and the question of its choice will be discussed in the next section.

As for the properties of the PSR itself, this question calls for further investigation. It can be assumed that in this region, at definite instants of times, an instability develops and leads to collapse of the order parameter and to propagation of a normal section into the interior of the superconducting region. When the normal section reaches the inner surface of the cylinder, a phase slippage by 2π takes place. The normal section then begins to vanish and the superconductivity in the PSR is restored. This process repeats periodically in time with a frequency satisfying the Josephson relation. Figure 4 shows schematically the boundary of the TM layer (the dashed line shows its position in the PSR at various instants of time), as well as the behavior of the potential Φ and of the superconducting current I_s as a function of the coordinate z along the cylinder axis.

The width of the PSR can be estimated by stipulating that outside this region the derivative with respect to time in (2) must be anomalously large. Then all the quantities are constant in time in first-order approximation, and the equations that determine them are obtained by averaging the initial equations over the time (cf. Ref. 2). The derivative is $\partial Q/\partial t \sim ELQ$, and the value of $\partial\Phi/\partial z$ can be estimated from (3):

$$\partial\Phi/\partial z \sim Q/uz_1^2,$$

where z_1 is the width of the PSR. Thus, the static approximation is violated in a neighborhood of the PSR with a width of the order of $z_1 \sim (uEL)^{-1/2}$. Since the applicability of our model requires $z_1 \ll l_E \sim u^{-1/2}$, the field E should satisfy the condition $EL \gg 1$, i.e., $E \gg u^{1/2}$.

3. CURRENT-VOLTAGE CHARACTERISTIC

We calculate now the current flowing through the normal region

$$I_n = -\int_d^D \frac{\partial\varphi_n(x,z)}{\partial z} dx.$$

With the aid of (12) we get

$$I_n = E(D-d) - \sum_{k\neq 0} a_k \operatorname{th}\left[\frac{\pi k D}{L}\right] \cos\left(\frac{\pi k}{L} z\right).$$

The total current through the sample is

$$I=\varkappa^2H_0+I_n=\varkappa^2H_0+E(D-d)-\sum_{k\neq0}a_k\,\text{th}\left[\frac{\pi kD}{L}\right]\cos\left(\frac{\pi k}{L}z\right).$$

If we average this expression over the period $2L$ of the structure, then the last term in the left-hand side drops out, and we obtain

$$j=E+\frac{1}{DL}\int_0^L I_s dz, \qquad (25)$$

where $j=I/D$ is the average density of the total current through the sample.

It is useful also to write down the free energy of the sample (in the given field H). As shown by AT, the main contribution is made here by the magnetic energy normal to the sample surface. The difference $\Delta F=F-F_N$ between the free energies, where F_N is the free energy of a purely normal sample, referred to unit length is of the form[11]

$$\frac{\Delta F}{F_0}=\frac{2^{1/2}}{\varkappa}\frac{\mathcal{I}}{\mathcal{I}_c}\frac{1}{L}\int_0^L(jd-\varkappa^2H_0)\,dz,$$

where $F_0=H_c^2rD/12$. (We use here the circumstance that at $E\gg L^{-1}\sim l_E^{-1}$ and $l_E\ll D$ the current density is $j\gg D^{-1}\gg j_c\approx\varkappa/D$. It is seen from (25) that at $j\gg j_c$ the field is $E\simeq j$, so that the free-energy difference is given by

$$\frac{\Delta F}{F_0}=-\frac{2^{1/2}}{\varkappa}\frac{\mathcal{I}}{\mathcal{I}_c}\frac{1}{L}\int_0^L I_s dz. \qquad (26)$$

From a comparison of (25) and (26) we see that the free energy of the sample and the dissipative function at a constant total current, whose density is

$$w=\overline{\mathbf{jE}}=j\left(j-\frac{1}{DL}\int_0^L I_s dz\right),$$

reach a minimum simultaneously, if the integration co constant k_0 in (23) is determined from the condition that the integral

$$\frac{1}{L}\int_0^L I_s\,dz=\max$$

is a maximum. It is clear from Fig. 2 that this corresponds to a choice of k_0 such that the function $I_s(k)$ reaches a maximum

$$I_s(k_0)=\max. \qquad (27)$$

We shall use henceforth precisely this choice of the integration constant k_0.

The maximum field E_{max} at which a TM layer still exists will be determined from the condition of vanishing of the maximum of $I_s(k)$. This coincides precisely with the condition used by Andreev and Tekel′ to determine the maximum current. According to their result we have

$$\mathcal{I}_{max}/\mathcal{I}_c\simeq0{,}21\,(2D/\xi).$$

In this case $j_{max}=E_{max}=0.30\varkappa$. Since the condition $E\gg u^{1/2}$ must be satisfied in order for our model to be valid, we must postulate the inequality $u^{1/2}\ll\varkappa$.

Expression (25) for the current-voltage characteristic (CVC) can be simplified in the case of sufficiently small E, i.e., $E\ll\varkappa$ and $\mathcal{I}\ll\mathcal{I}_{max}$. At these values of the field the parameter k is close to unity. Using the asymptotic expression $K(k)=\frac{1}{2}\ln[8/(1-k)]$, which is valid at $k\to1$, we obtain from (27) the following value for k_0 (Ref. 11)

$$k_0=1-(E/\varkappa)^{1/2},$$

and for k_1 we get from (24)

$$\ln[8/(1-k_1)]=\varkappa/E,$$

which corresponds to a thickness d of the TM layer at a distance L from the center of the static region, equal to $d=\varkappa/2^{1/2}E$, which is much less than $u^{-1/2}$ at $E\gg u^{1/2}$.

The function $k(z)$ is obtained from (23):

$$\ln[1/(1-k)]=\frac{1}{2}\ln(\varkappa/E)\exp(uz^2/2). \qquad (28)$$

This yields

$$L=\left[\frac{2}{u}\ln\left(\frac{2\varkappa}{E\ln(\varkappa/E)}\right)\right]^{1/2}.$$

It is now easy to find the CVC. Calculating the integral in (25), we have

$$j=E+j_c\left(1-\frac{1}{2\ln(\varkappa/E)}\right).$$

In ordinary units we obtain

$$U=R(\mathcal{I}-\mathcal{I}_{exc}), \qquad (29)$$

where $\mathcal{I}$ is the total current, U is the voltage on the sample, R is the sample resistance in the normal state, and

$$\mathcal{I}_{exc}=\mathcal{I}_c\left[1-\frac{1}{2\ln(\mathcal{I}_{max}/\mathcal{I})}\right]$$

is the excess current due to the presence of the superconducting TM layer. In this range of currents, $\mathcal{I}_{exc}$ depends little on the total current (see Fig. 5).

It follows also from the experimental data of Ref. 4 that there exists an excess current that depends little on the voltage. It should be noted, however, that the range of currents used to measure the CVC in the experiment of Ref. 4, $\mathcal{I}\gtrsim\mathcal{I}_c$, lies mainly below the region of applicability of the results of the present paper:

$$1\ll\frac{D}{\varkappa l_E}\ll\frac{\mathcal{I}}{\mathcal{I}_c}\leqslant0.42\frac{D}{\xi}.$$

The expression for the CVC can also be simplified near the transition of the sample into the fully normal state, i.e., at $\mathcal{I}_{max}-\mathcal{I}\ll\mathcal{I}_{max}$. In the case when $\max I_s$ is close to zero, I_s can be approximately represented in the form

$$I_s=\varkappa\left\{\frac{2^{1/2}k_m}{1+k_m^2}\left(1-\frac{E}{E_{max}}\right)-\frac{1}{2}\left|\frac{\partial^2 I_s}{\varkappa\partial k^2}\right|(k-k_m)^2\right\}, \qquad (30)$$

where $k_m\simeq0.58$ is the limiting value obtained by AT. With the aid of (23) and (30) we easily determine the half-period of the structure

$$L=\frac{C_1}{u^{1/2}}\left(1-\frac{E}{E_{max}}\right)^{1/2},$$

where the numerical constant C_1 is of the order of unity. It is seen therefore that the formulas obtained below are valid so long as $L\gg d$, i.e., at $u\ll1-E/E_{max}\ll1$. With the aid of (23) and (30), expression (25) takes the form

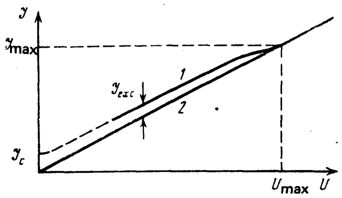

FIG. 5. Schematic current-voltage characteristic. Curve 1 corresponds to the existence of a TM layer, and the line 2 corresponds to the fully normal state. The dashed curve shows the section of the CVC (see Ref. 4) lying outside the region of applicability of the considered model.

$$j=E+j_c\frac{4k_m}{3(1+k_m^2)}\left(1-\frac{j}{j_{max}}\right).$$

In ordinary units this yields expression (28) with an excess current

$$\mathcal{J}_{exc}=\mathcal{J}_c\frac{4k_m}{3(1+k_m^2)}\left(1-\frac{\mathcal{J}}{\mathcal{J}_{max}}\right).$$

Thus, as $\mathcal{J}\to\mathcal{J}_{max}$ the CVC goes over smoothly into the characteristic of the normal sample. In this case, however the superconducting state in the TM layer vanishes jumpwise, and the value of the order parameter on the inner surface of the cylinder, Δ_0, changes jumpwise from $\Delta_0(k_m)$ to zero.

These singularities can be conveniently observed by measuring the impedance on the inner surface of the cylinder. The mean value of the impedances

$$\frac{Z}{Z_s}=\frac{1}{L}\int_0^L\frac{dz}{\Delta_0},$$

were Z_s is the impedance of the sample in the fully superconducting state. At not too strong currents, $\mathcal{J}\ll\mathcal{J}_{max}$, we obtain with the aid of (19) and (28)

$$\frac{Z-Z_s}{Z_s}=\frac{1}{4}\ln^{-1/2}\left(\frac{\varkappa}{E}\right)\int_1^b e^{-ay}\frac{dy}{y\ln^{1/2}y},$$

where $a=\frac{1}{2}\ln(\varkappa/E)$ and $b=\varkappa/aE$. An estimate of this expression yields

$$\frac{Z-Z_s}{Z_s}\sim\left(\frac{\mathcal{J}}{\mathcal{J}_{max}}\right)^{1/2}\ln^{-1/2}\left(\frac{\mathcal{J}_{max}}{\mathcal{J}}\right).$$

If the current is close to the maximum value $\mathcal{J}_{max}$, then the change of Δ_0 over the length of the structure is small, so that $\Delta_0=\Delta_0(k_m)$ and

$$\frac{Z}{Z_s}=\left(\frac{1+k_m^2}{2k_m^2}\right)^{1/2}.$$

When the current increases above $\mathcal{J}_{max}$, the impedance increases jumpwise to its value in the normal sample. This results agrees with the statement of AT.[11]

4. GENERALIZATION TO THE CASE OF A SUPERCONDUCTOR WITH A GAP

As noted above, dynamic equations containing only the superconducting parameters Δ, $\mathbf{Q}$, and Φ can be written for a superconductor with a gap only under very stringent restrictions on the characteristic frequency and wave vectors of the problem. Thus, if $\mathcal{D}k^2\ll\gamma$ and $\omega\ll\gamma$, where $\gamma=\tau_{ph}^{-1}$, then the sought equations take the form (cf. Ref. 2)

$$\frac{T_c-T}{T}\Delta-\frac{7\zeta(3)}{8\pi^2T^2}\Delta^3-\frac{\pi\mathcal{D}}{2T}\left(\frac{e}{c}\mathbf{Q}\right)^2\Delta+\frac{\pi\mathcal{D}}{8T}\nabla^2\Delta-\frac{\pi}{4T\gamma}\Delta\frac{\partial\Delta}{\partial t}=0, \quad (31)$$

$$\mathcal{D}\nabla^2\Phi-\frac{\pi\Delta\gamma}{4T}\Phi=-\frac{\mathcal{D}}{c}\frac{\partial}{\partial t}\operatorname{div}\mathbf{Q}, \quad (32)$$

$$\mathbf{j}=-\sigma\left(\nabla\Phi+\frac{1}{c}\frac{\partial\mathbf{Q}}{\partial t}\right)-\frac{\pi\sigma\Delta^2}{2cT}\mathbf{Q}. \quad (33)$$

Equation (31) is similar to that obtained by Gor'kov and Éliashberg,[15] while Eq. (32) is known from the paper of Schmid and Schön.[6]

We introduce the dimensionless quantities

$$r=\xi r';\quad \Delta=\bar\Delta\Delta';\quad \mathbf{Q}=\frac{c}{2e\xi}\mathbf{Q}';\quad \mathbf{H}=\frac{c}{2e\xi^2}\mathbf{H}';$$

$$\mathbf{j}=\frac{\pi\sigma\bar\Delta^2}{4eT\xi}\mathbf{j}';\quad \Phi=\frac{\pi\bar\Delta^2}{4eT}\Phi';\quad \omega=\frac{\pi\bar\Delta^2}{2T}\omega',$$

where

$$\xi^2=\frac{\pi\mathcal{D}}{8(T_c-T)};\quad \bar\Delta^2=\frac{8\pi^2T(T_c-T)}{7\zeta(3)}.$$

Now Eqs. (31)–(33) take the form

$$\frac{a}{u}\Delta\frac{\partial\Delta}{\partial t}+(\Delta^2+\mathbf{Q}^2-1)\Delta-\nabla^2\Delta=0, \quad (34)$$

$$u\Delta\Phi-\nabla^2\Phi-\frac{\partial}{\partial t}\operatorname{div}\mathbf{Q}=0, \quad (35)$$

$$j=-\nabla\Phi-\frac{\partial\mathbf{Q}}{\partial t}-\Delta^2\mathbf{Q}, \quad (36)$$

where

$$u=\frac{\pi^2\bar\Delta\gamma}{32T(T_c-T)}=\frac{\xi^2(T)}{l_E^2};\quad a=[\pi^4/14\zeta(3)]^2.$$

To satisfy the inequalities above together with the requirement $u\ll 1$ it is necessary to have

$$\left(\frac{\gamma}{T}\right)^2\ll\frac{T_c-T}{T}\ll\frac{\gamma}{T}.$$

Equations (34)–(36) differ from the system (1)–(4) only in the term with the time derivative in (34), and in the power of Δ in (35).

After averaging the system (34)–(36) over the time, we obtain

$$\nabla^2\Delta+\Delta(1-\Delta^2-\mathbf{Q}^2)=0,$$
$$u\Delta\Phi+\operatorname{div}(\Delta^2\mathbf{Q})=0,$$
$$\varkappa^2\operatorname{rot}\operatorname{rot}\mathbf{Q}=-\nabla\Phi-\Delta^2\mathbf{Q}.$$

For this system we can repeat completely all the arguments of the preceding sections. The only difference is that now η in (8) takes the form

$$\eta=\int_0^d\Delta\,dx.$$

Expressing $\Delta(x)$ with the aid of the AB solution, we obtain

$$\eta=2^{-1/2}\ln(1+k)/(1-k). \quad (37)$$

The remaining results of the preceding sections can be transferred here without change. Indeed, the CVC at $\mathcal{J}\ll\mathcal{J}_{max}$ is determined by near-unity values of the parameter k, at which expressions (22) and (37) take the same asymptotic form. As to the CVC in the region $\mathcal{J}_{max}-\mathcal{J}\ll\mathcal{J}_{max}$, the quantity $\eta(k_m)$ does not enter at all in this expression.

Equations (31)–(33), unfortunately, cannot be used to investigate the processes that take place inside the PSR, since the jumps of the phase should lead to a restructuring of the distribution function in a region of the order of the quasiparticle diffusion length $l_\varepsilon = (D/\gamma)^{1/2} \ll l_E$, for Eqs. (31)–(33) do not hold.

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APPENDIX

To answer the question of the stability of various roots of Eq. (21), we turn to Eq. (2). Integrating the z component of this equation over the thickness of the TM layer, we have

$$\varkappa^2 H_0 - Ed - I_s = -\int_0^d \frac{\partial Q_z}{\partial t} dx. \qquad (A.1)$$

Estimating the term in the right-hand side we obtain

$$-\int_0^d \frac{dQ_z}{\partial t} dx \sim \alpha \frac{\partial k}{\partial t},$$

where the positive constant $\alpha \sim 1$. The total current flowing through the cylinder is

$$\mathcal{I} = 2\pi r \left(I_s + \int_0^D j_{nz}\, dx \right).$$

When k is varied, the layer thickness changes by an amount of the order of δd, and the changes in the distribution of the normal current occur only over distances of the order of $l_E \sim u^{-1/2}$. Therefore the changes of the integral in this expression are of the order of $d\delta d / l_E \ll \delta d$. This means that when variation is carried out in (A.1) with respect to k, it must be assumed that I_s is constant. As a result, for a small deviation of δk from k, satisfying Eq. (21), we obtain

$$\frac{\partial}{\partial k}[\varkappa^2 H_0 - Ed]\delta k = \alpha \frac{\partial(\delta k)}{\partial t}.$$

It follows therefore that only the descending branch of the function $I_s(k)$ in (21) is stable.

[1)] A weak electric field is ensured by the fact that the critical current for a cylindrical sample is proportional to the radius of the cylinder, and its density, and hence also the electric field strength, is inversely proportional to the thickness D of the cylinder walls. Since it is expressed in the units customarily used in superconductivity, the electric field intensity turns out to be low.

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