Melting of adsorbed monolayer films

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A theoretical study is made of the melting of a monolayer film of atoms adsorbed on a crystal substrate. When the periodic structure of a film is commensurable with the external periodic field, the partition function of such a system has a dual symmetry. This makes it possible to determine the temperatures of phase transitions in an adsorbed film which occur at certain concentrations.

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1. INTRODUCTION

A theory of phase transitions in two-dimensional systems was proposed by Berezinskiĭ[1] and by Kosterlitz and Thouless.[2] In the low-temperature phase a gas of bound pairs of positive and negative vortices is considered and it is assumed that some of these vortices dissociate on transition to a disordered phase. José *et al.*[3] carried out more detailed calculations for such a system in the presence of anisotropy. They found that the model has a certain thermodynamic symmetry associated with dual transformations of the partition function. This symmetry has made it possible to study an additional phase transition in systems with anistropy of order higher than fourth.

Kosterlitz and Thouless[2] also suggested that the melting of a two-dimensional crystal can be considered in a similar manner. Here, instead of vortices there are dislocations and the phase transition is attributed to dissociation of pairs of dislocations which have equal but opposite Burgers vectors. These ideas were developed by Halperin and Nelson[4,5] who discovered that the transition transforms a crystal into an intermediate liquid-crystal phase which retains the long-range orientational order. The next transition from the intermediate phase to a disordered liquid phase involves dissociation of disclination pairs. Halperin and Nelson also allowed for the influence of the periodic potential of the substrate on these phase transitions.

The presence of the periodic potential of the substrate makes it necessary to consider, in the ordered phase, the states of adatoms which are commensurable and incommensurable with the substrate. These states are separated by a phase transition curve in the phase diagram. As pointed out in Ref. 4, this curve may or may not intersect the curve representing the transition to the liquid-crystal phase, depending on the ratio of the lattice periods of the substrate and of the commensurable structure of adatoms.

We shall study in greater detail the phase transitions from commensurable to incommensurable phases and from incommensurable to liquid (or liquid-crystal) phases. Therefore, we shall not ignore the presence of dislocations in the first case or the periodic potential in the second, contrary to the earlier treatments. This can be done in the case when the periodic structure of a surface film is commensurable with the external periodic field (i.e., it can be done at the points *B* and *A* in the phase diagram of Fig. 1) because in this case the system has an internal symmetry.

The presence of an internal symmetry in various physical systems is always an important property because it makes it possible to study more thoroughly the behavior of such systems and their properties. Therefore, determination of this symmetry (associated with the dual transformation of the partition sum of a system of absorbed atoms) is the main task of the present study.

2. DUAL TRANSFORMATIONS

Following the notation adopted by Halperin and Nelson,[4] we shall describe the energy of an elastic film in the form

$$\mathscr{H}=\frac{1}{2}\int d^2r\left[2\mu u_{ij}^2+\lambda u_{kk}^2+\gamma\left(\frac{\partial u_x}{\partial y}-\frac{\partial u_y}{\partial x}\right)^2\right], \tag{1}$$

where the stress tensor is

$$u_{ij}=\frac{1}{2}\left[\frac{\partial u_i(\mathbf{r})}{\partial r_j}+\frac{\partial u_j(\mathbf{r})}{\partial r_i}\right]$$

λ and μ are the Lame constants, $\theta(\mathbf{r})=\frac{1}{2}(\partial u_y(\mathbf{r})/\partial x-\partial u_x(\mathbf{r})/\partial y)$ is the orientation angle of the side of the lattice joining two neighboring atoms relative to a selected direction,

$$\frac{1}{2}\int d^2r\gamma\left(\frac{\partial u_x}{\partial y}-\frac{\partial u_y}{\partial x}\right)^2$$

is the energy associated with deviation from the energetically preferred orientation of the adatom lattice, and a_0 is the lattice period.

A dislocation located at a point $\mathbf{r}$ has the property that

$$\oint d\mathbf{u}=a_0\mathbf{b}(\mathbf{r})=n(\mathbf{r})a_0\mathbf{e}_1+m(\mathbf{r})a_0\mathbf{e}_2,$$

where $\oint d\mathbf{u}$ is an integral over a contour going round the

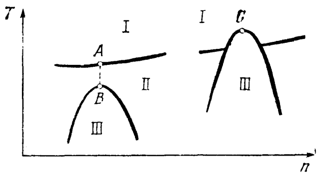

FIG. 1. Phase diagram: I) liquid-crystal phase; II) incommensurable phase; III) commensurable phase.

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point **r**; **b**(**r**) is the Burgers vector; $\mathbf{e}_1$ and $\mathbf{e}_2$ are the basis vectors of the lattice. In the case of a square lattice, we have

$$\mathbf{e}_1=\begin{pmatrix}1\\0\end{pmatrix},\quad \mathbf{e}_2=\begin{pmatrix}0\\1\end{pmatrix},$$

whereas for a triangular lattice, we obtain

$$\mathbf{e}_1=\begin{pmatrix}1\\0\end{pmatrix},\quad \mathbf{e}_2=\begin{pmatrix}1/2\\ \sqrt{3}/2\end{pmatrix}.$$

We shall assume subsequently that $a_0=1$ in all cases.

We shall rewrite Eq. (1) in the form

$$\tilde{\mathscr{H}}=-\frac{\mathscr{H}}{T}=-\frac{1}{2}\int d^2r\left\{(2\bar{\mu}+\bar{\lambda})\left[\left(\frac{\partial u_x}{\partial x}\right)^2+\left(\frac{\partial u_y}{\partial y}\right)^2\right]\right.$$
$$+(\bar{\mu}+\bar{\gamma})\left[\left(\frac{\partial u_x}{\partial y}\right)^2+\left(\frac{\partial u_y}{\partial x}\right)^2\right]+2(\bar{\mu}-\bar{\gamma})\frac{\partial u_x}{\partial y}\frac{\partial u_y}{\partial x}$$
$$\left.+2\bar{\lambda}\frac{\partial u_x}{\partial x}\frac{\partial u_y}{\partial y}\right\},\qquad(2)$$

where $\bar{\mu}=\mu/T$, $\bar{\lambda}=\lambda/T$, $\bar{\gamma}=\gamma/T$ and we shall assume that $\mathbf{u}=u\mathbf{e}_1+v\mathbf{e}_2$.

We shall begin derivation of the partition function by representing Eq. (2) as a difference. For this purpose we shall introduce a square lattice in the space **r**. This lattice has no relation to the adatom lattice but is associated with the cutoff in the space of **r**. Then,

$$\tilde{\mathscr{H}}=\sum_{\langle\mathbf{r},\mathbf{r}'\rangle}V[\mathbf{u}(\mathbf{r})-\mathbf{u}(\mathbf{r}')],$$

where the summation is carried out over the nearest neighbors. It should be noted that in the case of integration in a partition function with large values of **u**(**r**) and $V(\mathbf{u})$ we have to continue periodically in the range $|\mathbf{u}|\geqslant|\mathbf{b}|$ and assume that $V(\mathbf{u}+\mathbf{b})=V(\mathbf{u})$ because of the translational symmetry of the lattice. Therefore, the integration with respect to u and v has to be carried out within one lattice period in order to avoid inclusion of the degenerate states in the partition function.

The partition function then becomes

$$Z=\left(\prod_{\mathbf{r}}\int_0^1\int_0^1 du(\mathbf{r})dv(\mathbf{r})\right)\exp\{V[\mathbf{u}(\mathbf{r})-\mathbf{u}(\mathbf{r}')]\},\qquad(3)$$

or, following the usual procedure of dual transformations (see, for example, Ref. 3),

$$Z=\sum_{\{\mathbf{b}(\mathbf{R})\}}\left(\prod_{\mathbf{R}}\int_{-\infty}^{\infty}\int d\varphi(\mathbf{R})d\psi(\mathbf{R})\right)\exp\left\{\sum_{\langle\mathbf{R},\mathbf{R}'\rangle}\tilde{V}[\varphi(\mathbf{R})-\varphi(\mathbf{R}')]\right.$$
$$\left.+\sum_{\mathbf{R}}i\mathbf{b}(\mathbf{R})\varphi(\mathbf{R})\right\}.$$

We shall simplify subsequent steps by considering initially a square lattice of adsorbed atoms. Then,

$$\tilde{V}[\varphi(\mathbf{R})-\varphi(\mathbf{R}')]=-2\pi^2\left\{\frac{(2\bar{\mu}+\bar{\lambda})}{4\bar{\mu}(\bar{\mu}+\bar{\lambda})}\left[\left(\frac{\partial\varphi}{\partial y}\right)^2+\left(\frac{\partial\psi}{\partial x}\right)^2\right]\right.$$
$$\left.-\frac{2\bar{\lambda}}{4\bar{\mu}(\bar{\mu}+\bar{\lambda})}\frac{\partial\varphi}{\partial y}\frac{\partial\psi}{\partial x}+\frac{(\bar{\mu}+\bar{\gamma})}{4\bar{\mu}\bar{\gamma}}\left[\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2\right]-\frac{2(\bar{\mu}-\bar{\gamma})}{4\bar{\mu}\bar{\gamma}}\frac{\partial\varphi}{\partial x}\frac{\partial\psi}{\partial y}\right\}.$$

In this expression the finite differences are replaced with derivatives for the sake of compactness.

The partition function is not affected if we make the substitution $\psi\to-\psi$. Therefore, we can rewrite the partition function in the form

$$Z=\sum_{\{\mathbf{b}(\mathbf{R})\}}\left(\prod_{\mathbf{R}}\iint_{-\infty}^{\infty}d\varphi(\mathbf{R})d\psi(\mathbf{R})\right)\exp\left\{-2\pi^2\int d^2R\left[\frac{(2\bar{\mu}+\bar{\lambda})}{4\bar{\mu}(\bar{\mu}+\bar{\lambda})}(\nabla\varphi)^2\right.\right.$$
$$\left.\left.+\frac{\bar{\mu}+\bar{\lambda}-\bar{\gamma}}{4\bar{\gamma}(\bar{\mu}+\bar{\lambda})}(\nabla\cdot\varphi)^2\right]+\int d^2Ri\mathbf{b}(\mathbf{R})\varphi(\mathbf{R})\right\}.\qquad(4)$$

Following the method of integration of such expressions considered by Nelson,[5] we obtain

$$Z=\sum_{\{\mathbf{b}(\mathbf{R})\}}{}'\exp\left\{\int d^2R_1\int_{|\mathbf{R}_1-\mathbf{R}_2|>a}d^2R_2\cdot\frac{1}{8\pi}\left[K_1\mathbf{b}(\mathbf{R}_1)\mathbf{b}(\mathbf{R}_2)\ln\left(\frac{|\mathbf{R}_1-\mathbf{R}_2|}{a}\right)\right.\right.$$
$$\left.\left.-K_2\frac{(\mathbf{b}(\mathbf{R}_1)(\mathbf{R}_1-\mathbf{R}_2))(\mathbf{b}(\mathbf{R}_2)(\mathbf{R}_1-\mathbf{R}_2))}{|\mathbf{R}_1-\mathbf{R}_2|^2}\right]+\ln y_0\int_{|\mathbf{R}|>a}d^2R|\mathbf{b}(\mathbf{R})|^2\right\},$$

where

$$K_1=\frac{4\bar{\mu}(\bar{\mu}+\bar{\lambda})}{2\bar{\mu}+\bar{\lambda}}+\frac{4\bar{\mu}\bar{\gamma}}{\bar{\mu}+\bar{\gamma}},\quad K_2=\frac{4\bar{\mu}(\mu+\bar{\lambda})}{2\bar{\mu}+\bar{\lambda}}-\frac{4\bar{\mu}\bar{\gamma}}{\bar{\mu}+\bar{\gamma}}.$$

More cumbersome procedures applied to a triangular lattice give exactly the same result.

We can thus see that dual transformations of the partition function result, as expected, in the same expression as calculations carried out using the theory of elasticity.[4] However, in the subsequent discussion it will be convenient to describe the partition function in the form given by Eq. (4).

3. SYMMETRY OF PARTITION FUNCTIONS

In this section we shall show that a system which allows for the periodic potential of the substrate commensurable with the film structure has the dual symmetry. We shall now write Eq. (3) for a triangular lattice:

$$Z=\left(\prod_{r}\int_0^1\int_0^1 dudv\right)\exp\left\{-\frac{1}{2}\int d^2r\left[(2\bar{\mu}+\bar{\lambda})\left[\left(\frac{\partial u}{\partial x}\right)^2\right.\right.\right.$$
$$\left.+\frac{1}{4}\left(\frac{\partial v}{\partial x}\right)^2+\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{3}{4}\left(\frac{\partial v}{\partial y}\right)^2\right]+(\bar{\mu}+\bar{\gamma})\left[\left(\frac{\partial u}{\partial y}\right)^2\right.$$
$$\left.+\frac{1}{4}\left(\frac{\partial v}{\partial y}\right)^2+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}+\frac{3}{4}\left(\frac{\partial v}{\partial x}\right)^2\right]+2(\bar{\mu}-\bar{\gamma})\left[\frac{3^{1/2}}{2}\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}+\frac{3^{1/2}}{4}\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right]$$
$$\left.\left.+2\lambda\left[\frac{3^{1/2}}{2}\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}+\frac{3^{1/2}}{4}\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right]\right]\right\}$$

or after dual transformations:

$$Z=\sum_{\{\mathbf{b}(\mathbf{R})\}}\left(\prod_{\mathbf{R}}\iint_{-\infty}^{+\infty}d\varphi(\mathbf{R})d\psi(\mathbf{R})\right)\exp\left\{-\frac{1}{2}\int d^2R\left[(2\bar{\mu}_1+\bar{\lambda}_1)\left[\left(\frac{\partial\psi}{\partial y}\right)^2\right.\right.\right.$$
$$\left.+\frac{1}{4}\left(\frac{\partial\varphi}{\partial y}\right)^2-\frac{\partial\psi}{\partial y}\frac{\partial\varphi}{\partial y}+\frac{3}{4}\left(\frac{\partial\varphi}{\partial x}\right)^2\right]+(\bar{\mu}_1+\bar{\gamma}_1)\left[\left(\frac{\partial\psi}{\partial x}\right)^2\right.$$
$$\left.+\frac{1}{4}\left(\frac{\partial\varphi}{\partial x}\right)^2-\frac{\partial\psi}{\partial x}\frac{\partial\varphi}{\partial x}+\frac{3}{4}\left(\frac{\partial\psi}{\partial y}\right)^2\right]+2(\bar{\mu}_1-\bar{\gamma}_1)\left[-\frac{3^{1/2}}{2}\frac{\partial\psi}{\partial x}\frac{\partial\varphi}{\partial y}\right.$$
$$\left.\left.+\frac{3^{1/2}}{4}\frac{\partial\varphi}{\partial y}\frac{\partial\varphi}{\partial x}\right]+2\bar{\lambda}_1\left[-\frac{3^{1/2}}{2}\frac{\partial\psi}{\partial y}\frac{\partial\varphi}{\partial x}+\frac{3^{1/2}}{4}\frac{\partial\varphi}{\partial y}\frac{\partial\varphi}{\partial x}\right]\right]$$
$$\left.+\int d^2Ri\mathbf{b}(\mathbf{R})\varphi(\mathbf{R})+(\ln y_0)\int d^2R|\mathbf{b}(\mathbf{R})|^2\right\},\qquad(5)$$

where

$$\bar{\mu}_1=\frac{4\pi^2}{3}\frac{1}{\bar{\mu}},\quad\bar{\gamma}_1=\frac{4\pi^2}{3}\frac{1}{\bar{\mu}+\bar{\lambda}},\quad\bar{\lambda}_1=\frac{4\pi^2}{3}\frac{\bar{\mu}-\bar{\gamma}}{\bar{\mu}\bar{\gamma}}.$$

We shall now allow for the contribution made to the partition function by the periodic potential of the substrate and we shall do this for a concentration corresponding to the AB line in the phase diagram (Fig. 1).

At this concentration the incommensurable lattice period is identical with the commensurable lattice period, but—in contrast to the commensurable case—the adatom lattice in the incommensurable phase may move freely over the substrate surface.

Let us assume that the potential of the periodic field of the substrate is

$$V[\mathbf{u}(\mathbf{r})]=\sum_{\{M\}} h_M \exp\{i\mathbf{M}\mathbf{u}(\mathbf{r})\},$$

where

$$\exp\{V[\mathbf{u}(\mathbf{r})]\}=\sum_{\{\mathbf{M}\}} A_M \exp(i\mathbf{M}\mathbf{u}(\mathbf{r})),$$

$$A_M=A_0\exp\left\{(\ln y_M)\frac{|\mathbf{M}|^2}{{}^{16}/_3\pi^2p^2}\right\},$$

where in the limit $y_M\to 0$ we have $h_M=y_M$

$$\mathbf{M}=p(M_1\mathbf{m}_1+M_2\mathbf{m}_2),$$

M_1 and M_2 integers, and **M** is the smallest (in respect of the modulus) vector which is the common reciprocal vector for the film and substrate lattices so that p is the ratio of the modulus of the vector **M** to the modulus of the reciprocal vector of the adatom lattice (see Ref. 4).

After inclusion of a new term representing the commensurable substrate potential, we find that the term with γ as a partition function represents the contribution of the potential of the incommensurable harmonics.[4] We shall carry out a series of dual transformations following the scheme of Ref. 3. The calculations given in the Appendix show that the partition function has the dual symmetry in full analogy with Ref. 3:

$$Z[\bar\mu,\bar\lambda,\bar\gamma;y_0,y_M]\sim Z\left[\frac{4\pi^2p^2}{3\bar\mu},\frac{4\pi^2p^2(\bar\mu-\bar\gamma)}{3\bar\mu\bar\gamma},\frac{4\pi^2p^2}{3(\bar\mu+\bar\lambda)};y_M,y_0\right]. \tag{6a}$$

The same relationship can be presented in a more convenient form

$$Z[K_1,K_2;y_0,y_M]\sim Z[K_1',K_2';y_M,y_0], \tag{6b}$$

where

$$K_1=\frac{4\bar\mu(\bar\mu+\bar\lambda)}{2\bar\mu+\bar\lambda}+\frac{4\bar\mu\bar\gamma}{\bar\mu+\bar\gamma},\quad K_2=\frac{4\bar\mu(\bar\mu+\bar\lambda)}{2\bar\mu+\bar\lambda}-\frac{4\bar\mu\bar\gamma}{\bar\mu+\bar\gamma},$$

$$K_1'=\frac{16\pi^2p^2}{3}\frac{3\bar\mu+\bar\lambda+\bar\gamma}{(\bar\mu+\bar\gamma)(2\bar\mu+\bar\lambda)},\quad K_2'=\frac{16\pi^2p^2}{3}\frac{\bar\mu+\bar\lambda-\bar\gamma}{(\bar\mu+\bar\gamma)(2\bar\mu+\bar\lambda)}.$$

It follows from Eq. (6), representing the internal symmetry of the system, that we can consider a phase transition from a commensurable to an incommensurable phase and from an incommensurable to a liquid (or liquid-crystal) phase without ignoring the presence of dislocations in the first case or the influence of the periodic potential of the substrate in the second case.

It is shown in Ref. 4 that a phase transition from an ordered to a disordered phase, considered ignoring the periodic potential of the substrate (i.e., assuming that $y_M=0$) occurs for $K_{1R}=16\pi$. In view of the symmetry of Eq. (6), another singularity of the partition function occurs at $K_{1R'}=16\pi$, when $y_0=0$. For nonzero values of y_M and y_0 we can show, using Eq. (6) and the symmetry of the partition function relative to the substitution $y_M\to -y_M$ (see Ref. 3), that the temperatures of phase transitions are found from the relationships

$$\frac{4\bar\mu_A(\bar\mu_A+\bar\lambda_A)}{2\bar\mu_A+\bar\lambda_A}+\frac{4\bar\mu_A\bar\gamma_A}{\bar\mu_A+\bar\gamma_A}=16\pi,$$

$$\frac{16\pi^2p^2}{3}\frac{3\bar\mu_B+\bar\lambda_B+\bar\gamma_B}{(\bar\mu_B+\bar\gamma_B)(2\bar\mu_B+\bar\lambda_B)}=16\pi.$$

Hence,

$$T_A=\frac{\mu_A(\mu_A{}^2+\mu_A\lambda_A+2\lambda_A\gamma_A+3\mu_A\gamma_A)}{4\pi(2\mu_A+\lambda_A)(\mu_A+\gamma_A)},$$

$$T_B=\frac{3(\mu_B+\gamma_B)(2\mu_B+\lambda_B)}{\pi p^2(3\mu_B+\lambda_B+\gamma_B)}, \tag{7}$$

where $\mu_A=\mu_R(T_A)$ and $\mu_B=\mu_R(T_B)$ are the renormalized constants.

These transition temperatures are also obtained in Refs. 4 and 6 but on the assumption that $y_M=0$ in the former case and $y_0=0$ in the second. Our approach provides a more rigorous derivation of the relationships for T_A and T_B.

It follows from these relationships that if

$$p^2>\frac{12(\mu_R+\gamma_R)^2(2\mu_R+\lambda_R)^2}{(3\mu_R+\lambda_R+\gamma_R)(\mu_R{}^2+\lambda_R\mu_R+2\lambda_R\gamma_R+3\mu_R\gamma_R)\mu_R} \tag{8}$$

the phase transitions curves do not intersect and the melting passes through an intermediate incommensurable phase.[4] In the opposite case the transition occurs from a commensurable phase directly to a liquid phase (if the temperature is sufficiently high). We shall study this transition by considering the case of a deep potential relief of the substrate when the adatom system is described by a discrete model. We can easily see that this corresponds to the $y_M=1$ case. It is easiest to discuss the situation when $\gamma=\lambda+\mu$. We then have

$$K_1=\frac{8\bar\mu(\bar\mu+\bar\lambda)}{2\bar\mu+\bar\lambda},\quad K_2=0,$$

$$K_1'=\frac{32\pi^2p^2}{3}\frac{1}{(2\bar\mu+\bar\lambda)},\quad K_2'=0$$

and the symmetry of the partition function of Eq. (6) allows us to determine accurately the temperature of the transition at the critical melting point of the commensurable phase (i.e., at the point C in Fig. 1).

The expression (6b) can be rewritten in the form

$$Z\left[\frac{8\bar\mu(\bar\mu+\bar\lambda)}{2\bar\mu+\bar\lambda};\ y_0=1,\ y_M=1\right]\sim Z\left[\frac{32\pi^2p^2}{3}\frac{1}{2\bar\mu+\bar\lambda};\ y_M=1,\ y_0=1\right]$$

[it should be noted that the term with y_0 is introduced artificially in Eq. (5) and, therefore, we have $y_0=1$ although $y_{0R}\neq 1$] and then we can find directly the phase transition temperature

$$T^2=3\mu(\mu+\lambda)/4\pi^2p^2. \tag{9}$$

We have given some results which follow from the dual symmetry and we hope that further studies of this symmetry will give a better understanding of the properties of adsorbed films.

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APPENDIX

We shall now show that the investigated systems have the dual symmetry described by Eq. (6). We shall carry out a series of dual transformations following a scheme proposed in Ref. 3. Dual transformation of Eq. (5) subject to the substrate potential and integral values of p gives

$$Z=\left(\prod_{\mathbf{r}}\int_0^1\int_0^1 du\,dv\right)\sum_{\{\mathbf{b}(\mathbf{r},\mathbf{r}')\}}\sum_{\{\mathbf{M}(\mathbf{r})\}}\exp\Big\{\sum_{\langle\mathbf{r},\mathbf{r}'\rangle}V[\mathbf{u}(\mathbf{r})-\mathbf{u}(\mathbf{r}')$$
$$-\mathbf{b}(\mathbf{r},\mathbf{r}')]+\sum_{\mathbf{R}}(\ln y_0)|S_{\mathbf{R}}(\tilde{\mathbf{b}}(\mathbf{R}))|^2+\sum_{\mathbf{r}}(\ln y_M)\frac{|\mathbf{M}(\mathbf{r})|^2}{{}^{16}/_3\pi^2p^2}$$
$$+\sum_{\mathbf{r}}i\mathbf{M}(\mathbf{r})\mathbf{u}(\mathbf{r})\Big\}, \tag{A.1}$$

where

$$|S_{\mathbf{R}}(\tilde{\mathbf{b}}(\mathbf{R}))|^2=|\mathbf{b}(\mathbf{r}_4,\mathbf{r}_1)+\mathbf{b}(\mathbf{r}_1,\mathbf{r}_2)-\mathbf{b}(\mathbf{r}_3,\mathbf{r}_2)-\mathbf{b}(\mathbf{r}_4,\mathbf{r}_3)|^2$$
$$=|\tilde{\mathbf{b}}(\mathbf{R})|^2=\tilde{b}_1^2(\mathbf{R})+\tilde{b}_2^2(\mathbf{R})+\tilde{b}_1(\mathbf{R})\tilde{b}_2(\mathbf{R}),$$
$$\mathbf{b}=b_1\mathbf{e}_1+b_2\mathbf{e}_2,\quad \mathbf{M}=(M_1\mathbf{m}_1+M_2\mathbf{m}_2)p,$$
$$|\mathbf{M}(\mathbf{r})|^2=\frac{16\pi^2p^2}{3}[M_1^2(\mathbf{r})+M_2^2(\mathbf{r})-M_1(\mathbf{r})M_2(\mathbf{r})],$$
$$i\mathbf{M}(\mathbf{r})\mathbf{u}(\mathbf{r})=(2\pi iM_1u+2\pi iM_2v)p.$$

Inverse dual transformation of this expression gives

$$Z=\left(\prod_{\mathbf{R}}\int_0^p\int_0^p d\tilde{\varphi}(\mathbf{R})\,d\tilde{\psi}(\mathbf{R})\right)\sum_{\{\tilde{\mathbf{M}}(\mathbf{R},\mathbf{R}')\}}\sum_{\{\tilde{\mathbf{b}}(\mathbf{R})\}}\exp\Big\{\tilde{V}[\tilde{\boldsymbol{\varphi}}(\mathbf{R})-\tilde{\boldsymbol{\varphi}}(\mathbf{R}')-\tilde{\mathbf{M}}(\mathbf{R},\mathbf{R}')]$$
$$+\sum_{\mathbf{R}}(\ln y_0)|\tilde{\mathbf{b}}(\mathbf{R})|^2+\sum(\ln y_M)\frac{|S_{\mathbf{r}}(\mathbf{M}(\mathbf{r}))|^2}{{}^{16}/_3\pi^2p^2}+\sum_{\mathbf{R}}i\tilde{\boldsymbol{\varphi}}(\mathbf{R})\tilde{\mathbf{b}}(\mathbf{R})\Big\}, \tag{A.2}$$

where $\tilde{V}[\varphi(\mathbf{R})]$ is defined by Eq. (5),

$$|S_r(\mathbf{M}(\mathbf{r}))|^2=|\mathbf{M}(\mathbf{r})|^2={}^{16}/_3\pi^2p^2[M_1(\mathbf{r})^2+M_2(\mathbf{r})^2-M_1(\mathbf{r})M_2(\mathbf{r})],$$
$$|\tilde{\mathbf{b}}(\mathbf{R})|^2=\tilde{b}_1(\mathbf{R})^2+\tilde{b}_2(\mathbf{R})^2+\tilde{b}_1(\mathbf{R})\tilde{b}_2(\mathbf{R}),$$
$$i\tilde{\varphi}(\mathbf{R})\tilde{b}(\mathbf{R})=2\pi i\tilde{\varphi}(\mathbf{R})\tilde{b}_1(\mathbf{R})+2\pi i\tilde{\psi}(\mathbf{R})\tilde{b}_2(\mathbf{R}).$$

In order to compare the expressions for the partition functions given by Eqs. (A.1) and (A.2), we shall carry out the following operations on Eq. (A.2):

1) the substitution

$$\tilde{\varphi}(\mathbf{R})\to\tilde{\varphi}(\mathbf{R})+\tilde{\mathbf{M}}(\mathbf{R}),\quad \tilde{\mathbf{M}}(\mathbf{R},\mathbf{R}')\to\tilde{\mathbf{M}}(\mathbf{R},\mathbf{R}')+\tilde{\mathbf{M}}(\mathbf{R})-\tilde{\mathbf{M}}(\mathbf{R}')$$

is used to go over from integration with respect to $\tilde{\varphi}$ and $\tilde{\psi}$ between 0 and p to integration between $-\infty$ and $+\infty$;

2) the integration and summation variables are replaced as follows:

$$\tilde{\psi}(\mathbf{R})\to-\tilde{\psi}(\mathbf{R}),\quad \tilde{M}_2(\mathbf{R},\mathbf{R}')\to-\tilde{M}^2(\mathbf{R},\mathbf{R}'),\quad \tilde{b}_2(\mathbf{R})\to-\tilde{b}_2(\mathbf{R}):$$

3) the inverse substitution brings us back to integration with respect to $\tilde{\varphi}$ and $\tilde{\psi}$ between 0 and p.

The partition function of Eq. (A.2) now becomes

$$Z=\left(\prod_{\mathbf{R}}\int_0^p\int_0^p d\tilde{\varphi}(\mathbf{R})\,d\tilde{\psi}(\mathbf{R})\right)\sum_{\{\tilde{\mathbf{M}}(\mathbf{R},\mathbf{R}')\}}\sum_{\{\tilde{\mathbf{b}}(\mathbf{R})\}}\exp\Big\{V_1[\tilde{\boldsymbol{\varphi}}(\mathbf{R})$$
$$-\tilde{\boldsymbol{\varphi}}(\mathbf{R}')-\tilde{\mathbf{M}}(\mathbf{R},\mathbf{R}')]+\sum_{\mathbf{R}}(\ln y_0)[\tilde{b}_1^2(\mathbf{R})+\tilde{b}_2^2(\mathbf{R})-\tilde{b}_1(\mathbf{R})\tilde{b}_2(\mathbf{R})]$$
$$+\sum_{\mathbf{r}}(\ln y_M)[\tilde{M}_1(\mathbf{r})^2+\tilde{M}_2(\mathbf{r})^2+\tilde{M}_1(\mathbf{r})\tilde{M}_2(\mathbf{r})]$$
$$+\sum_{\mathbf{R}}2\pi i\tilde{\varphi}(\mathbf{R})\tilde{b}_1(\mathbf{R})+2\pi i\tilde{\psi}(\mathbf{R})\tilde{b}_2(\mathbf{R})\Big\}, \tag{A.3}$$

where $V_1[\mathbf{u}(\mathbf{r})]$ differs from $V[\mathbf{u}(\mathbf{r})]$ only by the coefficients $\bar{\mu}_1$, and $\bar{\gamma}_1$, and $\bar{\lambda}_1$ which replace $\bar{\mu}$, $\bar{\gamma}$, and $\bar{\lambda}$.

The expressions (A.1) and (A.3) are easily compared and we then have a situation fully analogous to that considered in Ref. 3 for the model with a scalar field $\theta(\mathbf{r})$ and an anisotropy $h_p\cos[p\theta(\mathbf{r})]$. In our case the substrate potential is anisotropic. We then have

$$Z[\bar{\mu},\bar{\lambda},\bar{\gamma};\ y_0,y_M]\sim Z\left[\frac{4\pi^2p^2}{3}\frac{1}{\mu},\ \frac{4\pi^2p^2}{3}\frac{\bar{\mu}-\bar{\gamma}}{\bar{\mu}\bar{\gamma}},\ \frac{4\pi^2p^2}{3}\frac{1}{\bar{\mu}+\bar{\lambda}};\ y_M,y_0\right].$$

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