$$-\Delta\lambda \propto (T-T_c)^{1/4}. \qquad (3.30)$$

The two examples considered show how fluctuations in the critical range, during phase transitions of the second kind in ferroelectric materials and liquid crystals, affect the characteristics of laser radiation produced by impurity atoms and molecules. A deeper investigation of this problem is naturally of interest. Of particular interest is a study of the mutual influence of equilibrium and nonequilibrium phase transitions.

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Large-amplitude spin waves and magnetic relaxation in the superfluid phases of ^{3}He

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Solutions in the form of large-amplitude spin waves are obtained in the long-wave limit for the spin-dynamics equations of superfluid ^{3}He. The dispersion laws of these waves in the A and B phases are obtained and their stability is investigated. The magnetic-relaxation diffusion mechanism due to the spatial nonuniformity of the magnetization distribution is considered. A method of measuring the spin-diffusion coefficient in the superfluid phases of ^{3}He is proposed.

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1. INTRODUCTION

Experiments[1-3] show that the relaxation of the magnetization in superfluid ^{3}He to the equilibrium value from the initial nonequilibrium state can apparently not be attributed in all cases to the action of one mechanism. The experiments are presently interpreted usually by the only theoretically sufficiently well developed "intrinsic" mechanism of Legget and Takagi,[4] whose applicability is restricted by the requirement that the magnetization be uniformly distributed in space. This restriction is strong in those cases when the initial state is prepared by tipping the magnetization by a finite angle from the direction of the external magnetic field H_0. Even a small inhomogeneity of H_0, because of the difference between the Larmor frequencies at various points of space, leads after a sufficiently long time to a considerable nonuniformity in the distribution of the magnetization. Even more important is the fact that in the A phase the spatially uniform precession of the magnetization is unstable and a nonuniform distribution of the magnetization sets-in spontaneously even in an ideally uniform external field.[5] The onset of the nonuniformity, together with spin diffusion, leads to a rather effective mechanism of longitudinal relaxation in the superfluid ^{3}He.

A theoretical investigation of the spatially inhomogenous states calls in the general case for the solution of a rather complicated nonlinear system of partial differential equations (see, e.g., Ref. 6). The problem is simpler for weakly inhomogeneous states, where the influence of the spatial nonuniformity on the precession of the magnetization can be regarded as a small perturbation. This approach is applicable if the energy of the inhomogeneity of the condensate of the Cooper pairs is small compared with the magnetic energy $\sim\chi H_0^2$, where χ is the magnetic susceptibility of the ^{3}He. In connection with the investigation of the relaxation, we shall be interested in states in which the energy inhomogeneity is comparable in order of magnitude with the spin-orbit energy. For the customarily employed fields $H_0 \approx 200$–300 Oe, the condition of smallness of the spin-orbit energy is satisfied with good accuracy at all temperatures.

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Near T_c, where both the spin-orbit energy and the inhomogeneity energy tend to zero, this condition is satisfied also for weaker fields.

In the present paper we obtain weakly inhomogeneous, in the sense indicated above, solutions of the system of spin-dynamics equations of superfluid ^{3}He. These solutions are a natural generalization of the spatially uniform magnetization precession on the one hand, and of small-amplitude spin waves on the other. The obtained solution make it possible to consider the action of a new, diffusion mechanism of the relaxation of magnetization and to obtain for simple cases predictions of the time dependence of the relaxation and of the dependence of the relaxation time on the external parameters. We emphasize that the considered mechanism, although connected with the flow of spin currents, is a volume phenomenon. Consequently, we are not dealing here with the influx of magnetization into the measuring volume from the outside, as is the case in the relaxation mechanism proposed by Anderson (see the reference in the article by Corruccini and Osheroff[2]) and Vuorio.[7]

A brief exposition of the principal results of the present paper was published in Refs. 5 and 8.

2. EQUATIONS OF MOTION

Our problem involves the following characteristic lengths: the coherence length ξ_0, the magnetic length $l_H \sim c/\omega_L$, and the dipole length $l_Q \sim c/\Omega$. Here $\omega_L = -gH_0$ is the Larmor frequency, g is the gyromagnetic ratio for the ^{3}He nuclei, Ω is the frequency of the longitudinal oscillations of the magnetization, and c is the spin-wave velocity. We shall consider, as is customary in spin dynamics, a magnetic-field region for which $l_H \gg \xi_0$. For such fields, the order parameter can be regarded as "rigid," i.e., its spatial and temporal variations reduce to rotations in spin space (the orbital coordinates are assumed fixed). These rotations are conveniently parametrized with the aid of the Euler angles α, β, γ in accordance with the equation

$$A_{i\xi}(\mathbf{r}, t) = R_{ii}(\alpha, \beta, \gamma) A_{i\xi}^{(0)} = (\hat{R} A_\xi^{(0)})_i, \tag{1}$$

where $\hat{R}$ is the rotation matrix and $A_{i\xi}^{(0)} = (\mathbf{A}_\xi^{(0)})$ is the initial form, independent of the coordinates, of the order parameter.

The Hamiltonian that generates the equations of motion of the inhomogeneous case is obtained by adding to the Leggett Hamiltonian[9] the inhomogeneity energy of the condensate. The volume density of the inhomogeneity energy F_∇ depends on the spatial derivatives $A_{i\xi}$, which can be conveniently expressed in terms of the "angular velocities" $\omega_{k\eta}$ in accordance with the definition

$$\partial A_{i\xi}/\partial x_\eta = e_{ikl}\omega_{k\eta} A_{l\xi}. \tag{2}$$

For substitution in F_∇ it is convenient to change over in this formula to a coordinate system that rotates together with $A_{i\xi}$; in this system, in the lowest order in the spatial derivatives, we have

$$F_\nabla = {}^1/_2 (\hbar/m)^2 \rho_{ik\xi\eta} \tilde{\omega}_{i\xi} \tilde{\omega}_{k\eta}, \tag{3}$$

where the tensor of the superfluid spin densities $\rho_{ik\xi\eta}$ is a certain combination of $A_i^{(Q)}{}_\xi$ and $(A_i^{(Q)}{}_\xi)$; m is the mass of the ^{3}He atom, and the "angular velocity" in the moving system of coordinates $\tilde{\omega}_{i\xi}$ is expressed in the following form in terms of the angles α, β, and γ and their derivatives with respect to the coordinates $\alpha_{,\xi} = \partial\alpha/\partial x_\xi$ etc.:

$$\begin{aligned} \tilde{\omega}_{1\xi} &= -\alpha_{,\xi} \sin\beta \cos\gamma + \beta_{,\xi} \sin\gamma, \\ \tilde{\omega}_{2\xi} &= \alpha_{,\xi} \sin\beta \sin\gamma + \beta_{,\xi} \cos\gamma, \\ \tilde{\omega}_{3\xi} &= \alpha_{,\xi} \cos\beta + \gamma_{,\xi}. \end{aligned} \tag{4}$$

Let the inhomogeneity be characterized by a wave vector q. The requirement that the inhomogeneity be weak then means that $ql_H \ll 1$. We assume also that $l_H/l_Q \sim \Omega/\omega_L \ll 1$. The procedure described in our preceding paper[10] allows us to obtain for the magnetization equations of motion that are accurate to $(ql_H)^2$ and $(\Omega/\omega_L)^2$ inclusive. Just as before,[10] the unit of measurements of the spin S is taken to be its equilibrium value, but we continue to use a dimensional time, to be able to observe directly the dependence on H_0 in the final formulas. The density of the Hamiltonian must in this transition be divided by $\chi H_0/g$. The independent variables are chosen to be α and $\Phi = \alpha + \gamma$ and their canonically conjugate momenta $P = S_z - S_\zeta$ and S_ζ, where S_z and S_ζ are the projections of S on the axis z and ζ of the immobile and moving coordinate systems, respectively. The z axis is oriented along the equilibrium direction of the spin, i.e., opposite to $\mathbf{H}_0$. By virtue of the previously proved[11] conservation of the orientation of S relative to the moving system of coordinates we have $S_\zeta = |\mathbf{S}| = S$. The angle β between the magnetization $-\chi H_0 \mathbf{S}$ and $\mathbf{H}_0$ is connected with P and S by the relation $\cos\beta = 1 + P/S$.

The density of the Hamiltonian must be averaged over the "fast" variables that vary with a frequency $\sim\omega_L$. In the "resonant" case, $S - 1 \leq \Omega/\omega_L$, there is only one fast variable α, and in the "nonresonant" case, $S - 1 \sim 1$, both angles α and Φ are fast. Putting $G = \overline{F}_\nabla/\chi H_0{}^2$ and $V = \overline{U}/\chi H_0{}^2$, where the bar denotes the corresponding averaging, we arrive at the following density of the Hamiltonian $\mathscr{H}$

$$\mathscr{H}/\omega_L = {}^1/_2 (S-1)^2 - P + V + G. \tag{5}$$

It differs from the density of the Hamiltonian of the preceding paper[10] in that V is replaced here by $V + G$.

In both the resonant and nonresonant cases G, in accordance with its definition and with Eqs. (3) and (4), is a function of the angle β and of the derivatives $\alpha_{,\xi}, \beta_{,\xi}, \Phi_{,\xi}$. The derivatives are assumed to be slowly varying and are not averaged. The equations of motion are obtained in accordance with the general procedure (see, e.g., Ref. 12) by making the following substitution in the equations of Ref. 10:

$$\frac{\partial V}{\partial \alpha} \to \frac{\partial (V+G)}{\partial \alpha} - \frac{\partial}{\partial x_\xi}\left(\frac{\partial G}{\partial \alpha_{,\xi}}\right),$$

and similar ones for the angles β and Φ.

To describe the relaxation we must include in the equations of motion also the dissipative terms. We consider here the hydrodynamic limit $\omega\tau_0 \ll 1$, where τ_0 is the time between the quasiparticle collisions,

when the disequilibrium can be described with the aid of the dissipation function. It suffices to retain in the dissipation function only the terms with the spatial derivatives of S. The dissipation due to the temporal derivatives of S were considered in Ref. 13 and is of no importance here. We then have for the change in the energy density:

$$\frac{dE}{dt}=-\chi H_0^2 D_{\xi\eta}\frac{\partial S_i}{\partial x_\xi}\frac{\partial S_i}{\partial x_\eta}. \tag{6}$$

Here $D_{\xi\eta}$ is the tensor of the spin-diffusion coefficients.

The equations of motion that agree in form with the Hamiltonian density (5) and the dissipation function (6) take in the resonant case the form

$$\frac{1}{\omega_L}\frac{\partial S}{\partial t}=-\frac{\partial V}{\partial\Phi}+\frac{\partial}{\partial x_\xi}\left(\frac{\partial G}{\partial\Phi_{,\xi}}\right)+\frac{D_{\xi\eta}}{\omega_L}[S_{,\xi,\eta}-S\sin^2\beta\alpha_{,\xi}\alpha_{,\eta}-S\beta_{,\xi}\beta_{,\eta}], \tag{7}$$

$$\frac{1}{\omega_L}\frac{\partial\Phi}{\partial t}=S-1+(1-\cos\beta)\left(\frac{1}{\omega_L}\frac{\partial\alpha}{\partial t}+1\right), \tag{8}$$

$$\frac{1}{\omega_L}\frac{\partial P}{\partial t}=\frac{\partial}{\partial x_\xi}\left(\frac{\partial G}{\partial\alpha_{,\xi}}\right)+\frac{D_{\xi\eta}}{\omega_L}[P_{,\xi,\eta}+S\sin^2\beta\alpha_{,\xi}\alpha_{,\eta}+S\beta_{,\xi}\beta_{,\eta}], \tag{9}$$

$$\frac{1}{\omega_L}\frac{\partial\alpha}{\partial t}+1=-\frac{1}{S\sin\beta}\left[\frac{\partial(V+G)}{\partial\beta}-\frac{\partial}{\partial x_\xi}\left(\frac{\partial G}{\partial\beta_{,\xi}}\right)\right]$$
$$+\frac{D_{\xi\eta}}{\omega_L S\sin\beta}[S\sin\beta\alpha_{,\xi,\eta}+2\sin\beta S_{,\xi}\alpha_{,\eta}+2S\cos\beta\alpha_{,\xi}\beta_{,\eta}]. \tag{10}$$

In the nonresonant case, the obtained system must also be averaged with respect to Φ, as a result of which $\partial V/\partial\Phi$ vanishes and $\partial V/\partial\beta$ in (10) should be replaced by $\partial\overline{V}/\partial\beta$. In all other respects the equations remain unchanged.

3. SPIN WAVES

The system of equations derived in the preceding section contains the angle only in the derivatives with respect to the time and the coordinates; it is therefore convenient to introduce new variables $\mathbf{h}=\nabla\alpha$ and $v=\omega_L+\partial\alpha/\partial t$, which are connected by the relation

$$\partial\mathbf{h}/\partial t=\nabla v. \tag{11}$$

Using the condition $\omega\tau_0\ll 1$, we neglect for the time being the dissipative terms in Eqs. (6)–(10). The system has then stationary solutions of the form $\mathbf{h}=\mathbf{h}^{(0)}$, $\beta=\beta_0$, $\Phi=\Phi_0$, $S=S_0$ and

$$v=\frac{\partial(V+G)}{\partial\cos\beta}\omega_L. \tag{12}$$

The constants $h_0^{(0)}$ and β_0 are determined by the initial conditions, Φ_0 is the root of the equation $\partial V/\partial\Phi=0$, and $S_0=1+\nu(\cos\beta-1)/\omega_L$. These solutions describe the precession of magnetization turned into a spiral, with wave vector $\mathbf{h}$, and it is natural to regard them as large-amplitude spin waves. The precession frequency $\omega=-\omega_L+\nu$ is determined by Eq. (12).

As $\beta\to 0$, the dependence of ω on $\mathbf{h}$ goes over into the dispersion law of ordinary spin waves in the strong-field limit. At $\mathbf{h}=0$ we have also $G=0$ and Eq. (12) determines the homogeneous-precession frequency shift (cf. Ref. 10). Solutions with $\mathbf{h}\neq 0$ correspond to states with a nonzero average flux of the quantity P determined by the derivative $\partial G/\partial h_\eta$.

In the nonresonant case the angle also becomes a cyclic variable. The result are stationary solutions characterized by two constant vectors $\mathbf{h}^{(0)}=\nabla\alpha$ and $\mathbf{g}^{(0)}=\nabla\Phi$. The deviation of the precession frequency from the Larmor frequency is also determined for such solutions by Eq. (12), and the rate of change of the angle Φ is determined by Eq. (8). In this case a flux of S, equal to $\partial G/\partial g_\eta$ is present in addition to the flux P.

If we now take also the dissipative terms into account in the equation of motion, then the obtained solutions become damped. In the resonant case the condition $\partial S/\partial t=0$ takes the form

$$\frac{\partial V}{\partial\Phi}+\frac{1}{\omega_L}\sin^2\beta(\mathbf{h}|D|\mathbf{h})=0, \tag{13}$$

where $(\mathbf{h}|D|\mathbf{h})=D_{\xi\eta}h_\xi h_\eta$. Equations (13) has a solution at not too large h, and determines Φ_0, while S_0 is obtained from the condition $\partial\Phi/\partial t=0$. The value of S is close to unity and changes little, therefore $\partial P/\partial t=\partial(\cos\beta)/\partial t$, and then Eq. (9) determines the time variation of the longitudinal magnetization component

$$\partial(\cos\beta)/\partial t=\sin^2\beta(\mathbf{h}|D|\mathbf{h}).$$

Integration of this equation leads to the angle-relaxation law

$$\operatorname{tg}\frac{\beta}{2}=\operatorname{tg}\frac{\beta_0}{2}\exp[-(\mathbf{h}|D|\mathbf{h})t]. \tag{14}$$

We see thus that in the resonant case states with uniform spin current relax within a time of the order of $(Dh^2)^{-1}$.

In the nonresonant case the diffusion ensures relaxation of only the transverse spin components. At constant $S_\parallel$ the angle β varies in accord with

$$\operatorname{tg}\beta=\operatorname{tg}\beta_0\exp[-(\mathbf{h}|D|\mathbf{h})t], \tag{15}$$

while the quantity $S_\parallel$ can be restored only via other mechanisms, for example the mechanism of Leggett and Tagaki. In contrast to the resonant case, states with spin current are possible here, and when the magnetization is uniformly distributed these states correspond to $\mathbf{h}=0$ but $\mathbf{g}\neq 0$. These states can also relax via the mechanism of Leggett and Tagaki, but in the next higher order in $(\Omega/\omega_L)^2$.

The obtained solutions with constant $\mathbf{h}$ in the resonant state and with constant $\mathbf{h}$ and $\mathbf{g}$ in the nonresonant state constitute a rather special class of solutions. This raises the question of their stability to violation of homogeneity. The stability of the solutions depends on the properties of the given concrete phase and must be investigated separately for each phase.

4. SPIN-WAVE STABILITY IN THE *A* PHASE

We first write down an expression for the inhomogeneity energy density F_∇ in the A phase. The order parameter in the A phase is of the form

$$A_{k\xi}=(^3/_2)^{1/2}\Delta d_k(f_\xi^{(1)}+if_\xi^{(2)}),$$

where $\mathbf{d}$, $\mathbf{f}^{(1)}$, and $\mathbf{f}^{(2)}$ are real unit vectors, and $\mathbf{f}^{(1)}\cdot\mathbf{f}^{(2)}=0$. In place of $\mathbf{f}^{(1)}$ and $\mathbf{f}^{(2)}$ it is frequently sufficient to specify the vector $\mathbf{l}=\mathbf{f}^{(1)}\times\mathbf{f}^{(2)}$. It is important that $A_{k\xi}$ can be factored into spin and orbital parts. Since F_∇ is

invariant to separate rotations in spin and orbit spaces, the spin part of $\rho_{ik\xi\eta}$ can depend only on d, and the orbital part only on l. The spin part should be proportional to $\delta_{ik} - d_i d_k$, in order that the rotation that leaves d in place not lead to a change in energy. For the orbital part there are two independent combinations, and as a result

$$F_v = \frac{\chi H_0^2}{2\omega_L^2}(\delta_{ik}-d_id_k)[c_\perp^2(\delta_{\xi\eta}-l_\xi l_\eta)+c_\parallel^2 l_\xi l_\eta]\bar{\omega}_{i\xi}\bar{\omega}_{k\eta}. \quad (16)$$

Here $c_\parallel$ and $c_\perp$ are the velocities of the spin waves propagating respectively along and across l. Near T_c

$$c_\parallel^2 = \frac{1}{2}c_\perp^2 = \frac{\chi_n}{3\varkappa'\chi}\left(1+\frac{Z_0}{4}\right)\left(1-\frac{T}{T_c}\right)v_F^2, \quad (17)$$

where χ_n and χ are respectively the magnetic susceptibilities of the normal ^{3}He and of the A phase, Z_0 is the zeroth harmonic of the spin part of the Fermi-liquid interaction, v_F is the Fermi velocity, and $\varkappa'$ is a coefficient close to unity. It takes into account the strong-coupling effects and is connected with the jump Δc_v of the specific heat on going from the normal phase to the A phase of ^{3}He by the relation $\Delta c_v/c_v = 1.42\cdot{}^5/_6\varkappa'$. Far from T_c allowance for the strong coupling effect is complicated, and the velocities $c_\parallel$ and $c_\perp$ should be regarded as phenomenological constants of the order of v_F. Details on the temperature dependences of $c_\parallel$ and $c_\perp$ can be found in the review of Leggett.[14]

We shall assume that at the initial instant of time $\mathbf{d}\perp\mathbf{H}_0$. Substitution of (4) in (16) and subsequent averaging over the angle α lead then to the following expression for G:

$$G = \frac{1}{2\omega_L^2}C_{\xi\eta}\left[\frac{1}{2}(1-u)(3-u)\alpha_{,\xi}\alpha_{,\eta}+\frac{1}{2}\beta_{,\xi}\beta_{,\eta} - (1-u)(\alpha_{,\xi}\Phi_{,\eta}+\alpha_{,\eta}\Phi_{,\xi})+\Phi_{,\xi}\Phi_{,\eta}\right], \quad (18)$$

where, to abbreviate the notation, we have introduced $u = \cos\beta$ and

$$C_{\xi\eta} = c_\perp^2(\delta_{\xi\eta}-l_\xi l_\eta)+c_\parallel^2 l_\xi l_\eta. \quad (19)$$

Substituting $\alpha_{,\xi} = h_\xi$, and $\beta_{,\xi} = \Phi_{,\xi} = 0$ in (18) and then in (12) we obtain an explicit expression for the frequency of the spin wave in the A phase

$$\frac{\omega}{\omega_L} = -1-\frac{\Omega_A^2}{8\omega_L^2}(1+3u)-\frac{(2-u)}{2\omega_L^2}(\mathbf{h}|C|\mathbf{h}). \quad (20)$$

To investigate the stability of the homogeneous spiral solutions it is necessary to examine the behavior of small inhomogeneous perturbations of the wave vector h, putting $\mathbf{h} = \mathbf{h}^{(0)}+\varepsilon(r,t)$ and substituting h in the latter form in the system (5.14) and (5.17), with G defined by Eq. (18). The other variables also deviate from their stationary values, $\Phi = \Phi_0+\varphi$, etc. The system linearized with respect to the perturbation has solutions of the form $\exp[i(\mathbf{k}\cdot\mathbf{r}-\omega t)]$, where ω and k are connected by the dispersion equation:

$$\begin{vmatrix} 1 & i\omega/\omega_L & -(V+G)_{,u} & i(1-u)\omega/\omega_L \\ -i\omega/\omega_L & \nabla_{,\Phi,\Phi} & \nabla^*_{,u,\Phi} & (1-u)(\mathbf{k}|C|\mathbf{k})/\omega_L^2 \\ -(V+G)_{,u} & \nabla_{,u,\Phi} & \nabla_{,u,u} & -i\tilde{\omega}/\omega_L \\ -i(1-u)\frac{\omega}{\omega_L} & (1-u)\frac{(\mathbf{k}|C|\mathbf{k})}{\omega_L^2} & i\frac{\tilde{\omega}}{\omega_L} & \frac{(1-u)(3-u)(\mathbf{k}|C|\mathbf{k})}{2\omega_L^2} \end{vmatrix} = 0. \quad (21)$$

We have introduced here the abbreviated notation:

$$\nabla_{,\Phi,\Phi} = V_{,\Phi,\Phi}+(\mathbf{k}|C|\mathbf{k})/\omega_L^2 = \partial^2 V/\partial\Phi^2+(\mathbf{k}|C|\mathbf{k})/\omega_L^2,$$
$$\nabla_{,u,u} = \partial^2(V+G)/\partial u^2+(\mathbf{k}|C|\mathbf{k})/2\omega_L^2(1-u^2),$$
$$\nabla_{,u,\Phi} = \partial^2 V/\partial u\partial\Phi+i(\mathbf{h}|C|\mathbf{k})/\omega_L^2,$$
$$\tilde{\omega} = \omega+(u-2)(\mathbf{h}|C|\mathbf{k})/\omega_L.$$

In the A phase

$$V = -\frac{\Omega_A^2}{8\omega_L^2}\left[u^2+\frac{1}{2}(1+u)^2\cos 2\Phi\right], \quad (22)$$

and at the minimum $\Phi = 0$ of the potential

$$V_{,u,u} = \frac{-3\Omega_A^2}{8\omega_L^2}; \quad V_{,\Phi,\Phi} = \frac{\Omega_A^2}{4\omega_L^2}(1+u)^2; \quad V_{,u,\Phi} = 0.$$

At $(\mathbf{k}|C|\mathbf{k}), (\mathbf{h}|C|\mathbf{k}), (\mathbf{h}|C|\mathbf{h}) \sim \Omega_A^2$, Eq. (21) has two pairs of roots. The first pair $\omega_{1,2} \sim \Omega_A$ corresponds to waves that go over as $k \to 0$ into longitudinal oscillations of the magnetization. In the principal order in Ω_A/ω_L these waves have the following dispersion law:

$$\omega_{1,2} = {}^1/_4\Omega_A(1+u)^2+(\mathbf{k}|C|\mathbf{k}). \quad (23)$$

This branch is obviously stable.

The second pair of roots $\omega_{3,4}$ is of the order of Ω_A^2/ω_L. At these ω all the elements of the determinant in the left-hand side of (21) are of the order of Ω_A^2/ω_L^2, with the exception of one element of the order of unity. Equation (21) is satisfied in the prinicpal order in Ω_A/ω_L, if the minor complementing the unity element vanishes. Thus, the dispersion equation for the second branch takes the form

$$\nabla_{,\Phi,\Phi}\tilde{\omega}^2+2\tilde{\omega}\frac{1-u}{\omega_L^3}(\mathbf{h}|C|\mathbf{k})(\mathbf{k}|C|\mathbf{k})-\frac{1}{2}(1-u)(3-u)(\mathbf{k}|C|\mathbf{k})\left[\nabla_{,u,u}\nabla_{,\Phi,\Phi} - \frac{(\mathbf{h}|C|\mathbf{k})^2}{\omega_L^4}\right]+(1-u)^2\nabla_{,u,u}\frac{(\mathbf{k}|C|\mathbf{k})^2}{\omega_L^2} = 0. \quad (24)$$

At $h = 0$ we have therefore

$$\omega_{3,4}^2 = \frac{1}{4}(\mathbf{k}|C|\mathbf{k})\frac{(3-u)(1+u)\Omega_A^2+4(\mathbf{k}|C|\mathbf{k})}{(1+u)^2\Omega_A^2+4)(\mathbf{k}|C|\mathbf{k})}\left[(\mathbf{k}|C|\mathbf{k})-\frac{3}{4}(1-u^2)\Omega_A^2\right], \quad (25)$$

and for k satisfying the condition

$$(\mathbf{k}|C|\mathbf{k}) < \frac{3}{4}\Omega_A^2(1-u^2). \quad (26)$$

The frequency $\omega_{3,4}^2 < 0$, i.e., the homogeneous precession of the magnetization is unstable. Condition (26) shows that the increasing perturbations are those with $k^2 \sim (1-u^2)/l_\Omega^2$. According to (25), the instability development time for typical values of k is

$$\tau_{\text{inst}} \sim \omega_L/\Omega_A^2(1-u^2),$$

and increases as $u \to \pm 1$.

At $h \neq 0$ the condition that the roots of (24) be real is of the form

$$\left[-\frac{3}{8}\Omega_A^2+\frac{1}{2}(\mathbf{h}|C|\mathbf{h})+\frac{(\mathbf{k}|C|\mathbf{k})}{2(1-u^2)}\right]\left[\frac{1}{4}\Omega_A^2(1+u)^2+(\mathbf{k}|C|\mathbf{k})\right] - (\mathbf{h}|C|\mathbf{k})^2 \geqslant 0. \quad (27)$$

Let the vector h be directed along one of the principal directions of the tensor $C_{\xi\eta}$, e.g., $\mathbf{h}\parallel\mathbf{l}$. Then the condition (27) can be satisfied for all k if h lies in the interval defined by the inequalities

$$v^{1/2}-(2(v-3))^{1/2} \leqslant 2(c_\parallel^2 h^2/\Omega_A^{2-3/4})^{1/2} \leqslant v^{1/2}+(2(v-3))^{1/2}, \quad (28)$$

where $v=(1+u)/(1-u)$. It is also necessary to have $h^2 \geqslant 3\Omega_A^2/4c_\parallel^2$ and $u>\frac{1}{2}$; under these conditions all the expressions under the radical signs are positive. At $u>5/7$, the difference $v^{1/2}-2(v-3)^{1/2}$ becomes negative and it must be replaced by zero in the inequalities (28). The conditions (28) determine the region of the stability of the large-amplitude spin waves for $\mathbf{h}\parallel\mathbf{l}$. At $\mathbf{h}\perp\mathbf{l}$ the stability region is determined by the same inequalities, with $c_\parallel^2$ replaced by $c_\perp^2$. At arbitrary orientation of h the analysis becomes more cumbersome. It is clear, however, that there exists a region $h\sim 1/l_Q$ in which the large-amplitude spin waves are stable at not too large tipping angles of the magnetization.

We ascertain now how the instability development is influenced by spin diffusion. At $h=0$ the dispersion equation for small perturbations takes, with allowance for diffusion, the form

$$\omega=-i(\mathbf{k}|D|\mathbf{k})\pm\left\{(\mathbf{k}|C|\mathbf{k})\frac{V_{,u,u}}{V_{,\Phi,\Phi}}\left[\frac{1}{2}(1-u)(3-u)V_{,\Phi,\Phi}-(1-u)^2\frac{(\mathbf{k}|C|\mathbf{k})}{\omega_L^2}\right]\right\}^{1/2}. \tag{29}$$

At $(\mathbf{k}|C|\mathbf{k})\gg\Omega_A^2$ the principal term of the expression in the curly brackets is equal to $(k|C|\mathbf{k})^2/4$. The ratio of the first term to the second is then independent of k and it is convenient to characterize it by the parameter $\Lambda=aD\omega_L/c^2$, where D and c^2 are certain mean values of $D_{\xi\eta}$ and $C_{\xi\eta}$, and a is a number of the order of unity. If $\Lambda\ll 1$, then the effect of diffusion is small, and such a situation is possible far from the temperature T_c of the transition into the superfluid state, where $D\sim v_F^2\tau_0$ and $\Lambda\sim\omega_L\tau_0\ll 1$.

As $T\to T_c$ we have

$$c^2\sim v_F^2(1-T/T_c)\to 0,\quad \Lambda\sim\omega_L\tau_0/(1-T/T_c)\to\infty$$

and the oscillations in question acquire a diffusion character. In the region $(\mathbf{k}|C|\mathbf{k})\leqslant\Omega_A^2$, where instability exists if no account is taken of diffusion, the relative value of the diffusion term is of the order of Λkl_Ω, where $l_\Omega\sim c/\Omega_A$. At $\Lambda\ll 1$ the value of Λkl_Ω is also small, but even at $\Lambda\gtrsim 1$ there exists a region $k\ll 1/\Lambda l_Q$ where the diffusion is insignificant and instability is present as before. As seen from (29), the perturbations that are now increasing are those with $k\leqslant 1/\Lambda l_Q$ and, accordingly, the instability development time at $(1-u^2)\sim 1$ is

$$\tau_{\text{inst}}\sim\Lambda\omega_L/\Omega_A^2.$$

When account is taken of diffusion, the solutions with $\mathbf{h}\neq 0$ are nonstationary, but so long as the variation of $\mathbf{h}$ can be regarded as slow we can use the same instability-investigation procedure as for the stationary solutions. This procedure is suitable for estimates also when the rate of change of h is comparable with the characteristic frequencies of the perturbations. Just as in the case $\mathbf{h}=0$, the diffusion is insignificant at $\Lambda\ll 1$. At $\Lambda\gg 1$ there can exist stable solutions with $h\gtrsim 1/\Delta l_\Omega$.

In the nonresonant case, substituting $\nabla\alpha=\mathbf{h}$ and $\nabla\Phi=\mathbf{g}$ in (18), and inserting the obtained expression in (12), we obtain the precession frequency

$$\frac{\omega}{\omega_L}=-1-\frac{1}{4}\frac{\Omega_A^2}{\omega_L^2}u-\frac{1}{2\omega_L^2}[(2-u)(\mathbf{h}|C|\mathbf{h})-2(\mathbf{h}|C|\mathbf{g})]. \tag{30}$$

The stability analysis is similar to that of the resonant case. The equation for the frequencies of the small perturbations has here, too, two pairs of roots.

$$\omega_{1,2}^2=(\mathbf{k}|C|\mathbf{k}), \tag{31}$$

$$\tilde{\omega}_{3,4}=(1-u)\frac{(h|C|k)}{\omega_L}\pm\left\{\frac{1-u^2}{2}\left[\frac{(\mathbf{k}|C|\mathbf{k})}{2\omega_L^2}\left((\mathbf{h}|C|\mathbf{h})-\Omega_A^2+\frac{(\mathbf{k}|C|\mathbf{k})}{1-u^2}\right)-\frac{(\mathbf{h}|C|\mathbf{k})^2}{\omega_L^2}\right]\right\}^{1/2}, \tag{32}$$

where

$$\tilde{\omega}=S\omega+[(u-2)(\mathbf{h}|C|\mathbf{k})+(\mathbf{g}|C|\mathbf{k})]/\omega_L. \tag{33}$$

It is easily seen that at all h there exist k that satisfy the condition

$$(\mathbf{k}|C|\mathbf{k})\left[(\mathbf{h}|C|\mathbf{h})-\Omega_A^2+\frac{(\mathbf{k}|C|\mathbf{k})}{1-u^2}\right]-(\mathbf{h}|C|\mathbf{k})^2<0, \tag{34}$$

and in the nonresonant case there exist no regions of stability of large-amplitude spin waves.

5. MAGNETIZATION RELAXATION IN THE *A* PHASE

It is customary to describe magnetic relaxation in terms of the so-called longitudinal relaxation time τ_1, i.e., the time required to restrore the magnetization projection parallel to $\mathbf{H}_0$. The equation for the longitudinal component of S is obtained by adding Eqs. (7) and (9)

$$\frac{1}{\omega_L}\frac{\partial S_z}{\partial t}=-\frac{\partial V}{\partial\Phi}+\frac{\partial}{\partial x_\xi}\left(\frac{\partial G}{\partial\Phi_{,\xi}}+\frac{\partial G}{\partial\alpha_{,\xi}}\right)+\frac{D_{\xi\eta}}{\omega_L}\frac{\partial^2 S_z}{\partial x_\xi\partial x_\eta}. \tag{35}$$

In the experiments is measured the magnetization of a certain finite volume, and Eq. (35) must be integrated over this "measurement" volume. The last two terms in the right-hand side are the respective divergences of the superfluid and diffusion spin currents and lead to surface contributions to the magnetization change due to its influx into the measurement volume from the outside We shall assume here that there is no such influx and take into account only the volume magnetization source

$$\frac{\partial V}{\partial\Phi}=\frac{1}{8}\frac{\Omega_A^2}{\omega_L^2}(1+\cos\beta)^2\sin 2\Phi. \tag{36}$$

The previosuly discussed stationary solutions correspond to an absence of a magnetization outflow ($\sin 2\Phi_0=0$). The maximum possible value of the outflow $\Omega_A^2(1+\cos\beta)^2 8\omega_L^2$ is reached at $\sin 2\Phi=1$. If this value of Φ is maintained in the entire measurement volume, then the relaxation will take place within the minimum possible time, of the order of $\tau_{1\,\min}=8\omega_L/\Omega_A^2$. For a pressure 20 bar we have $\tau_{1\,\min}\approx 10^{-7}H_0/(1-T/T_c)$ [sec/Oe]. The actually observed times are usually one or two orders of magnitude higher, meaning that $\sin 2\Phi$ or its mean value are usually small.

In the discussed solutions of the spin-wave type, the deviation of Φ from Φ_0 is ensured by the influence of the spin diffusion [see Eq. (13)]. We ascertain now the conditions under which this relaxation mechanism becomes important. We consider to this end the relaxation in a slightly inhomogeneous magnetic field, and

then ω_L in Eqs. (7)–(10) depends on the coordinates. The field directions are also inevitably slightly different at various points, but this matters little if the inhomogeneity is small. Taking the gradients of both sides of (10), we get

$$\partial \mathbf{h}/\partial t = -\nabla \omega_L = -\omega_L \mathbf{b}. \tag{37}$$

We have left out here the terms of order $\Omega^2/\omega_L L$, $c^2/\omega_L L^3$, and $D\omega_L t/L^3$, which remain small compared with $\nabla\omega_L$, under the conditions that will be imposed below on b, during times on the order of the relaxation time. L is the characteristic length over which the field varies; in particular, $|b| \sim 1/L$.

According to (37), h increases in proportion to the time at $\mathbf{b} \neq 0$. Retaining in (7)–(10) terms of higher order in h, we arrive at a system of equations that determines the asymptotic solutions of the system (7)–(10) at large t:

$$\frac{1}{\omega_L}\frac{\partial S}{\partial t} = \frac{1}{8}\frac{\Omega_A^2}{\omega_L^2}(1+\cos\beta)^2 \sin 2\Phi - \frac{(\mathbf{h}|D|\mathbf{h})}{\omega_L} S \sin^2\beta, \tag{38}$$

$$\frac{1}{\omega_L}\frac{\partial \Phi}{\partial t} = S - 1 + (1-\cos\beta)\left(\frac{1}{\omega_L}\frac{\partial \alpha}{\partial t} + 1\right), \tag{39}$$

$$\frac{1}{\omega_L}\frac{\partial P}{\partial t} = \frac{S}{\omega_L}(\mathbf{h}|D|\mathbf{h})\sin^2\beta. \tag{40}$$

The asymptotic values are reached after times $t \gtrsim 1/\omega_L \Lambda$, i.e., short even compared with $\tau_{1\min}$.

An important property of the system (37)–(40) is that it does not contain spatial derivatives, so that in this approximation the magnetization behaves independently at different points of space. If h varies slows (a criterion will be formulated later), then the system (37)–(40) has solutions close to damped large-amplitude spin waves.

Assuming $\partial S/\partial t = 0$, we obtain the value of $\sin 2\Phi$ which determines in this case the relaxation rate

$$\sin 2\Phi_0 = -\frac{8\omega_L S(\mathbf{h}|D|\mathbf{h})}{\Omega_A^2}\operatorname{tg}^2\frac{\beta}{2}. \tag{41}$$

So long as its right-hand side is less than unity, Eq. (41) has solutions and the system is close to resonance. The relaxation is determined in this case by formula (14) in which, however, owing to the time dependence of h, it is necessary to replace $(\mathbf{h}|D|\mathbf{h})t$ and $(\mathbf{h}|D|\mathbf{h})dt$:

$$\operatorname{tg}\frac{\beta}{2} = \operatorname{tg}\frac{\beta_0}{2}\exp\left[-\frac{1}{3}(\mathbf{b}|D|\mathbf{b})\omega_L^2 t^3\right]. \tag{42}$$

Only the principal term was retained in the argument of the exponential. The corrections are of relative magnitude $1/\omega_L t\Lambda$ and it must be ensured that their absolute values remain small compared with unity, i.e., formula (42) cannot be used for times significantly longer than the relaxation time

$$\tau_D = [3/(\mathbf{b}|D|\mathbf{b})\omega_L^2]^{1/3}.$$

The obtained relaxation law is particularly useful in the case when b can be regarded as constant in the entire measurement volume. Formula (42) then describes the change of a directly observable quantity. For the relaxation to proceed in accord with (42) it is necessary that the condition that follows from $\sin 2\Phi \leqslant 1$ not be violated during the entire process. Substitution of (42) in (41) shows that $\sin 2\Phi$ has as a function of the time a maximum whose value, at the specified external conditions, is determined by the tipping angle β_0. $\sin 2\Phi$ reaches unity first at

$$\left(\operatorname{tg}\frac{\beta_0}{2}\right)^2 = \frac{\Omega_A^2}{8\omega_L}\left[\frac{e^2}{(\mathbf{b}|D|\mathbf{b})\omega_L^2}\right]^{1/3} = \left(\frac{e^2}{3}\right)^{1/3}\frac{\tau_D}{\tau_{1\min}}. \tag{43}$$

At larger tipping angles, the character of the relaxation should vary in the course of the magnetization relaxation 1). The dependence of the course of the relaxation on the tipping angle, observed experimentally by Webb,[3] is apparently explained precisely by this fact. A quantitative comparison of Webb's results with the formulas obtained here is made difficult by the fact that in his experiments the field did not have the simple form $H_z = H_0(1 + \mathbf{b}\cdot\mathbf{r})$, at which the relaxation proceeds in the same manner at all points of the measurement volume. A quantitative check on formulas (42) and (43) calls for experiments with a controllable magnetic field gradient.

Under the conditions when the right-hand side of (41) exceeds unity we have $\partial S/\partial t < 0$ and the system goes off resonance. The produced transition is a transition from the "intrinsic" stationary Josephson effect[9] to the nonstationary one. In the resonant case, i.e., at $|S-1| \gg \Omega_A/\omega_L$, we can neglect $\partial V/\partial\Phi$ in (38) and the subsequent relaxation will proceed according to (38)–(40) at constant S_z. The restoration of S_z is due to the action of other mechanisms, for example, the mechanism of Leggett and Takagi.

We formulate now the restrictions that must be imposed on the field inhomogeneity b in order that the approximations made in the discussion of the relaxation process be valid. Above all, the resultant inhomogeneity must be weak in the sense formulated in the introduction, i.e., we must have $ch \ll \omega_L$. If we stipulate that this condition be satisfied during the relaxation time τ_D, a limitation is imposed on $\mathbf{b} - |\mathbf{b}|$,

$$bl_Q \ll \Lambda\omega_L/\Omega_A. \tag{44}$$

The requirement that h vary slowly, $\Omega_A\tau_D \ll 1$, imposes a stronger restriction on b:

$$bl_Q \ll \Omega_A/\Lambda\omega_L. \tag{45}$$

If we forego this condition, then the system (38)–(40) remains valid, but we can no longer use the quasistationary solutions for its analysis, and must seriously take into account the time dependences of S and Φ.

We have so far disregarded the fact that at small h the spiral solutions are unstable. To be able to neglect the instability it is necessary that the inhomogeneity due to the instability development be small compared with the inhomogeneity produced by the external field during the relaxation time. At $\Lambda \ll 1$ the perturbations with $k \sim 1/l_Q$ increase, and it is therefore natural to assume that the scale of the produced inhomogeneity is l_Q. The requirement $b\omega_L\tau_D \gtrsim 1/l_Q$ leads to a lower bound on b

$$bl_Q \gtrsim \Lambda/\omega_L\tau_{1\min}. \tag{46}$$

In the case $\Lambda \gg 1$ the corresponding condition is

$$bl_Q \gtrsim 1/\Lambda^2\omega_L\tau_{1\min}. \tag{47}$$

If the inequality signs in (46) and (47) are reversed, then the relaxation due to the instability and inhomogeneity of the field $\mathbf{H}_0$ can be neglected. The development of the instability leads to a state with a complicated inhomogeneous distribution of the magnetization. The characteristic dimension of the measurement volume is usually large compared with the scale of the inhomogeneity produced in this manner, and to describe the relaxation in this case it is necessary to average Eqs. (7) and (35) over the magnetization fluctuations

$$\frac{1}{\omega_L}\frac{\partial S}{\partial t}=-\frac{\overline{\partial V}}{\partial \Phi}+\frac{D_{\eta\xi}}{\omega_L}\overline{[S_{,\xi,\eta}-S\sin^2\beta\alpha_{,\xi}\alpha_{,\eta}-S\beta_{,\xi}\beta_{,\eta}]}, \tag{48}$$

$$\frac{1}{\omega_L}\frac{\partial S_z}{\partial t}=-\frac{\overline{\partial V}}{\partial \Phi}. \tag{49}$$

The spectrum of the fluctuations is unknown and it is impossible to calculate the mean values explicitly. Some conclusions concerning the dependence of the relaxation time on the quanities that enter in the problem can be drawn on the basis of dimensionality analysis. It is useful to consider together with the time also the procession-dephasing time τ_2^*, which is determined in this case by instability-development time. The dimensionless relaxation time $\omega_L\tau_1$ should be a function of the parameters Λ and $(\Omega_A/\omega_L)^2$ as well as, generally speaking, of the amplitudes $\alpha_k^{(0)}$ of the increasing perturbations. Since $(\Omega_A/\omega_L)^2$ is small, we can expand $\omega_L\tau_1$ in powers of this parameter. In accordance with the statements made at the beginning of the section, the expansion should start with $(\Omega_A/\omega_L)^2$, i.e.,

$$\tau_1=\tau_{1\min}f(\Lambda,\alpha_k^{(0)}). \tag{50}$$

In the limit $\Lambda\gg 1$ the characteristic scale of the inhomogeneity is l_Ω and the diffusion term in (48) is of the order of $\Lambda/\omega_L\tau_{1\min}\ll(\partial V/\partial\Phi)_{\max}$. As a result, $(\overline{\partial V/\partial\Phi})$ can be constructed in such a way that the right-hand side of (48) vanishes. The spin varies in this case about the resonant value $S\simeq 1$.

Obtaining $(\overline{\partial V/\partial\Phi})$ from (48) and substituting it in (49), we obtain $\partial S_z/\partial t\sim\Lambda/\tau_{1\min}$ and $\tau_1\sim\tau_{1\min}f_0/\Lambda$. In this case $\tau_2^*\sim\tau_{1\min}\psi(\alpha_k^{(0)})$, i.e., $\tau_2^*\ll\tau_1$, therefore f_0 can be regarded as a constant independent of the initial conditions. In estimates, the dependence of τ^*_2 on $\alpha_k^{(0)}$ can be regarded as logarithmic, $\psi\sim\ln(1/\alpha_k^{(0)})$. In the opposite limiting case similar reasoning leads to the estimate $\tau_1\sim\Lambda\tau_{1\min}f_1(\alpha_k^{(0)})\sim\tau_2^*$. In this case we cannot neglect the dependence of τ_1 on the initial conditions. The characteristic scale of the inhomogeneities at large Λ is $\lambda\sim\Lambda l_\Omega\sim l_\Omega\omega_L\tau_0/(1-T/T_c)$. As $T\to T_c$ the parameter λ can become large and the instability is suppressed by the geometrical dimensions of the vessel.. For example, $\lambda\sim 0.1$ cm on the melting curve at $H_0=100$ Oe and $(1-T/T_c)\sim 10^{-3}$.

According to experiments[1,2] performed mainly in the region $\Lambda\sim 1$, the dependence of τ_1 on the magnetic field and on the temperature is of the form $H_0/(1-T/T_c)$. This indicates that the function f in (50) depends little on Λ in this region, which is natural. The numerical value of f turns out to be ~20 to 30. The cause of such an increase of the relaxation time can be the smallness of the initial perturbations, as well as the fact that the outflow of the magnetization is due to the value of $(\partial V/\partial\Phi)$ averaged over the fluctuations, which is certainly lower than the maximum value. The linear or near-linear time dependence of S_z observed in these experiments means that the rate of energy dissipation depends little on the angle β. Estimates based on dimensionality cannot predict the time dependence of the relaxation. It is clear, however, that if the relaxation is due to instability development the rate of energy dissipation should not depend strongly on β, and the relaxation picture considered here does not contradict the existing experiments.

6. SPIN WAVES IN THE *B* PHASE

In the B phase the equilibrium form of the order parameter $A_i^{(0)}_\xi$ is a matrix of rotation about the direction of $\mathbf{n}$ through an angle $\theta_0=\arccos(\frac{1}{4})$. Far from the walls of the vessel we have at equilibrium $\mathbf{n}\parallel\mathbf{H}_0$, the so-called Leggett configuration, which is in fact investigated in most experiments. The effect of the walls on the orientation of the vector n can, however, extend over distances ~1 mm, and using a system of plane-parallel plates it is possible to orient $\mathbf{n}$ at various angles relative to $\mathbf{H}_0$.[15] It makes sense therefore to regard in the analysis the angle ψ between $\mathbf{n}$ and $\mathbf{H}_0$ as arbitrary, and assume $\psi=0$ for the transition to the Leggett configuration.

From symmetry considerations the condensate inhomogeneity energy in the B phase is expressed in the following form in terms of $A_i^{(0)}_\xi$ and the angular velocities $\tilde\omega_{i\xi}$:

$$F=\frac{\chi H_0^2}{2\omega_L^2}[c_\parallel^2\delta_{ik}\delta_{\xi\eta}-(c_\parallel^2-c_\perp^2)(A^{(0)}_{i\xi}A^{(0)}_{k\eta}+A^{(0)}_{i\eta}A^{(0)}_{k\xi})]\tilde\omega_{i\xi}\tilde\omega_{k\eta}. \tag{51}$$

Here $c_\parallel$ and $c_\perp$ are the velocities of the ordinary spin waves propagating respectively along and across the magnetic field $\mathbf{H}_0$ in the Leggett configuration.

Near T_c we have

$$c_\parallel^2=\frac{4}{3}c_\perp^2=\frac{2}{5\varkappa'}\frac{\chi_n}{\chi}\left(1+\frac{Z_0}{4}\right)\left(1-\frac{T}{T_c}\right)v_F^2. \tag{52}$$

Substitution of (4) in (51) and averaging over the fast variable lead to the following expression for $G=\overline{F}/\chi H_0^2$ in the B phase:

$$G=\frac{c_\parallel^2}{2\omega_L^2}[2(1-u)(\alpha_{,\xi}\alpha_{,\xi}-\alpha_{,\xi}\Phi_{,\xi})+\Phi_{,\xi}\Phi_{,\xi}+\beta_{,\xi}\beta_{,\xi}]$$
$$-\frac{c_\parallel^2-c_\perp^2}{2\omega_L^2}[(1-u^2)\alpha_{,\xi}\alpha_{,\xi}+\beta_{,\xi}\beta_{,\xi}+(1-u)(1-3u)\alpha_{,A}^2-\beta_{,A}^2$$
$$+2\Phi_{,A}^2-4(1-u)\Phi_{,A}\alpha_{,A}]. \tag{53}$$

Here $u=\cos\beta$, $\Phi=\alpha+\gamma+\theta_0$, $\alpha_{,\xi}=\partial\alpha/\partial x_\xi$, etc.

$$\alpha_{,A}=A^{(0)}_{z\xi}\alpha_{,\xi}=-\frac{1}{4}(\hat{\mathbf{z}}\nabla\alpha)+\frac{5}{4}(\hat{\mathbf{zn}})(\mathbf{n}\nabla\alpha)+\frac{\sqrt{15}}{4}(\nabla\alpha[\hat{\mathbf{zn}}]),$$

$\hat{\mathbf{z}}$ is a unit vector in the direction of the z axis. In the Leggett configuration $\mathbf{n}\parallel\hat{\mathbf{z}}$ and $\alpha_{,A}=\alpha_{,3}$.

The energy of the spin-orbit interaction in the B phase at arbitrary ψ is of the form[10]

$$V=\frac{2\Omega_B^2}{15\omega_L^2}[w(1+w)+N], \tag{54}$$

where

$$w=\tfrac{1}{2}(2-\mu)(1+u)\cos\Phi+(1-\mu)u-1,$$
$$N=\mu(1-u)[2(2+w)-\tfrac{7}{8}\mu(1-u)],$$
$$\mu=\tfrac{5}{4}\sin^2\psi.$$

Substitution of expressions (53) and (54) in (12) leads to the following dispersion law for the large-amplitude spin waves in the B phase:

$$\frac{\omega_{\mathbf{h}}}{\omega_L}=-1+\frac{\partial V}{\partial u}+\frac{c_\parallel^2-c_\perp^2}{\omega_L^2}[u\mathbf{h}^2+(2-3u)(\mathbf{hA}_3^{(0)})^2]-\frac{c_\parallel^2}{\omega_L^2}\mathbf{h}^2. \qquad (55)$$

An investigation of the stability of the spin waves is complicated in this case,[2)] since the expressions for V and G are more complicated than in the A phase, and we confine ourselves therefore to a thermodynamic analysis, which makes it possible to establish the stability of the considered states with respect to perturbations with $k\to 0$. As seen from a comparison with the A phase, such an analysis is sufficient for the investigation of the stability of uniform precession. At $\mathbf{h}\neq 0$, the thermodynamic analysis yields only the necessary criteria for the stability of the spin-wave solutions.

Since the variable α is cyclic, its canonically conjugate momentum P is conserved, i.e., the variation of P in a certain region of space can be due only to its influx from neighboring regions. This allows us to regard P as an additive thermodynamic variable. In accordance with the general procedure,[16] in the stable stationary state the quantity

$$\tilde{\mathcal{H}}=\mathcal{H}-\omega_{\mathbf{h}}P \qquad (56)$$

should have a minimum, where $\mathcal{H}$ is described by formula (5), and $\omega_{\mathbf{h}}$ is the precession angular velocity defined by Eq. (55).

The conditions that the second differential of $\tilde{\mathcal{H}}$ be positive with respect to the variables S, u, and Φ are readily seen to be

$$V_{,\Phi,\Phi}>0, \qquad (57)$$
$$R\equiv(V_{,u,u}+G_{,u,u})V_{,\Phi,\Phi}-V_{,u,\Phi}>0. \qquad (58)$$

To calculate R it is convenient to use the fact that

$$R=V_{,\Phi,\Phi}\frac{d\omega_{\mathbf{h}}}{du}. \qquad (59)$$

In fact

$$\frac{d\omega_{\mathbf{h}}}{du}=V_{,u,u}+G_{,u,u}+V_{,u,\Phi}\frac{d\Phi}{du},$$

where Φ is regarded as a function of u, defined by the stationarity condition $\partial V/\partial\Phi=0$, i.e., $\partial\Phi/\partial u=-V_{,u,\Phi}/V_{,\Phi,\Phi}$, from which we in fact obtain (59).

The equation $\partial V/\partial\Phi=0$ has several roots, from which we must choose those satisfying the condition (57). The change of the character of the stationary values of Φ takes place on the lines $V_{,\Phi,\Phi}=0$:

$$1+\mu+(4-5\mu)u=0,\quad 1+\mu(1-u)=0,$$

which break up the physical region of the values of μ and u, $0\leq\mu\leq\frac{5}{4}$ and $-1\leq u\leq 1$, into three regions (see Fig. 1) with different dependences[3)] of $\omega_{\mathbf{h}}$ on u. Substituting Eqs. (55) and (59) in (58) we obtain the necessary stability conditions for the spin-wave solutions.

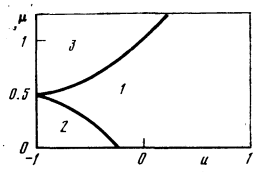

FIG. 1.

In region 1

$$-\tfrac{1}{2}\Omega_B^2\mu^2+(c_\parallel^2-c_\perp^2)[\mathbf{h}^2-3(\mathbf{hA}_3^{(0)})^2]>0, \qquad (60)$$

in region 2:

$$\tfrac{2}{15}\Omega_B^2[8-20\mu+\tfrac{35}{4}\mu^2]+(c_\parallel^2-c_\perp^2)[\mathbf{h}^2-3(\mathbf{hA}_3^{(0)})^2]>0, \qquad (61)$$

and in region 3:

$$\tfrac{1}{10}\Omega_B^2\mu^2+(c_\parallel^2-c_\perp^2)[\mathbf{h}^2-3(\mathbf{hA}_3^{(0)})^2]>0. \qquad (62)$$

The homogeneous precession is unstable in region 1 and stable in regions 2 and 3. The Leggett configuration corresponds to the line $\mu=0$. At $u>-\frac{1}{4}$ this line is the homogeneous-precession stability boundary, since $R=0$ on it. The stability of the homogeneous precession on this segment is ensured by the terms of higher order in the wave vector $\mathbf{k}$ in the dispersion equation for the small perturbations. Depending on the direction of $\mathbf{h}$, it can stabilize as well as destabilize the precession. In particular, in the Leggett configuration the unstable solutions are those with $\mathbf{h}\parallel\hat{\mathbf{z}}$.

In the nonresonant case the spin waves are characterized by two wave vectors $\mathbf{h}=\nabla\alpha$ and $\mathbf{g}=\nabla\Phi$. Accordingly, the spin-wave frequency is then determined by the expression

$$\frac{\omega_{\mathbf{hg}}}{\omega_L}=-1+\frac{1}{5}\Omega_B^2u[3(\mu-1)^2-1]+\frac{c_\parallel^2}{\omega_L^2}[(\mathbf{gh})-\mathbf{h}^2]$$
$$+\frac{c_\parallel^2-c_\perp^2}{\omega_L^2}[u\mathbf{h}^2+(2-3u)(\mathbf{hA}_3^{(0)})^2-2(\mathbf{hA}_3^{(0)})(\mathbf{gA}_3^{(0)})]. \qquad (63)$$

The necessary condition for the stability of the spin-wave solutions in this case is the inequality

$$\tfrac{1}{5}\Omega_B^2[3(\mu-1)^2-1]+(c_\parallel^2-c_\perp^2)[\mathbf{h}^2-3(\mathbf{hA}_3^{(0)})^2]>0. \qquad (64)$$

The uniform precession is stable at $\mu<1-1/\sqrt{3}$.

7. MAGNETIZATION RELAXATION IN THE B PHASE

Magnetization relaxation in the B phase was experimentally investigated only in the Leggett configuration. According to Corruccini and Osheroff[2] the longitudinal component of the magnetization relaxes exponentially in this case, and the reciprocal relaxation time is a linear function of the magnetic field gradient. Even at not very large gradients, $\nabla\omega_L/\omega_L\sim 10^{-2}$ to 10^{-3} cm^{-1}, the relaxation due to the field inhomogeneity plays the principal role.

The system of equations describing the motion of the magnetization in the B phase in a weakly inhomogeneous magnetic field at long times ($t\gg 1/\Lambda\omega_L$) differs from the corresponding system (37)–(40) for the A phase only in that $\partial V_B/\partial\Phi$ in the right-hand side of (38) should be replaced by $\partial V_A/\partial\Phi$, so that this equation takes the form

$$\frac{1}{\omega_L}\frac{\partial S}{\partial t}=\frac{4\Omega_B^2}{15\omega_L^2}(1+u)\left[\left(u-\frac{1}{2}\right)+(u+1)\cos\Phi\right]\sin\Phi-\frac{D\mathbf{h}^2}{\omega_L}(1-u^2). \qquad (65)$$

We have also taken here into account the fact that the spin-diffusion coefficient in the B phase should be isotropic and that $(\mathbf{h}|D|\mathbf{h})=D\mathbf{h}^2$, while the remaining equations remain unchanged.

Arguments similar to those advanced for the A phase show that in a weakly inhomogeneous field the relaxation should follow the law

$$\operatorname{tg}\frac{\beta}{2}=\operatorname{tg}\frac{\beta_0}{2}\exp\left[-\left(\frac{t}{\tau}\right)^3\right], \tag{66}$$

where $\tau^3=3/D(\nabla\omega_L)^2$. In the A phase, owing to the competition with the instability development, this law can be observed only if the $\nabla\omega_L$ satisfy the condition (46). In the B phase, the uniform precession is stable and the indicated lower bound of the field gradient does not occur. The instability at finite h takes place at not all the directions of h, and is furthermore suppressed by the diffusion at $\Lambda\gtrsim 1$. Inasmuch as all the existing experiments were performed at just such values of Λ, we shall disregard hereafter this instability. At $u>-\frac{1}{4}$ the B phase is also not acted upon by the "intrinsic" relaxation mechanism.[4,13] As a result, the applicability of the law (66) in the B phase is restricted only by the uncontrollable field gradient and by the inhomogeneity of the initial texture.

Neither the time dependence of the relaxation nor the dependence of τ on $\nabla\omega_L$, predicted by Eq. (66), agrees with the observations of Corruccini and Osheroff. It should be noted that their results are presented in a form already analysed in accord with an exponential law. Bearing it in mind that at $t\sim\tau$ a cubic parabola can be approximated in a certain interval by a straight line, it seems possible that the law (66) is not in direct contradiction with experiment.[2]

In Webb's experiment[3] the relaxation was slower on the initial section, in qualitative agreement with (66). Here, too, it is impossible to make a quantitative comparison, since the field gradient was not controlled in this experiment. An argument favoring the relaxation mechanism considered here is Webb's observation[3] that in those cases when the relaxation had an "exponential" character in both superfluid phases the relaxation time remained practically unchanged on going from one phase to the other. This is to be expected if the relaxation is due to the diffusion mechanism, since the spin-diffusion coefficeint, which determines the relaxation time in this case, should not change greatly on going from the A to the B phase.

Observation of a relaxation that follows the law (66) can serve as a method of measuring the spin-diffusion coefficient in the superfluid phases of ^{3}He. An attempt to use for this purpose a spin-echo method of the type $90°-T-180°-2T-180°$ encountered difficulties[2] due apparently to the influence of the spin-orbit interaction on the motion of the magnetization. Less sensitive to such an influence is another type of echo, due to the change of the magnetic-field gradient.[18] Observation of this echo can also be used to measure the spin-diffusion coefficient.

If b is varied periodically in (37) in such a way that $\int_0^T \mathbf{b}dt=0$, then h will vanish with a period T, and induction signals will be produced. The relaxation of the angle β is determined by $\int \mathbf{h}^2dt$ and is monotonic. For example, if

$$|\mathbf{b}|=b_0\sin(2\pi t/T+\varphi_0),$$

then at the instant of the m-th echo we have

$$\operatorname{tg}\frac{\beta}{2}=\operatorname{tg}\frac{\beta_0}{2}\exp\left[-D\left(\frac{b_0\omega_L T}{2\pi}\right)^2\left(\frac{1}{2}+\cos^2\varphi_0\right)mT\right]. \tag{67}$$

The period of variation of b should satisfy the adiabatically condition $\Omega_B T\gg 1$. The induction signal is proportional to $\sin\beta$. The situation is analogous in many respects to that in ordinary magnets.[19] There is, however, a substantial difference, namely, the decrease of the induction signal in superfluid ^{3}He is due to longitudinal relaxation.

In the B phase, $\partial V/\partial\Phi$ is also bounded. By the same token, the maximum possible outflow of the longitudinal component of the magnetization is limited. If the field is sufficiently strongly inhomogeneous, the second term in the right-hand side of (65) can become larger than the maximum value of the first term in the course of the relaxation, and the stationary condition $\partial S/\partial t=0$ cannot be satisfied. Then the character of the relaxation will change in the same manner as in the one considered for the A phase.

It is convenient to express the condition that the right-hand side of (65) vanish in the form

$$\frac{45}{4\omega_L\tau}\left(\frac{\omega_L}{\Omega_B}\right)^2\left(\frac{t}{\tau}\right)^2\operatorname{tg}^2\frac{\beta}{2}=\sin\Phi\left[\cos\Phi+\frac{1}{4}\left(1-3\operatorname{tg}^2\frac{\beta}{2}\right)\right]. \tag{68}$$

This is the equation for the determination of Φ, and its left-hand side, when (66) is taken into account, has a maximum at $t=t_{\max}=\tau/3^{1/3}$. Equation (68) first ceases to have solutions at a certain t close to $t_{\max}$. Substituting in the right-hand side of (6.20) the value of $\tan(\beta/2)$ corresponding to $t_{\max}$ we obtain an approximate equation that determines, for angles that are not too close to 180°, the smallest initial deviation angle β_0 at which the relaxation changes its character:

$$\frac{45}{4(3e)^{2/3}\omega_L\tau}\left(\frac{\omega_L}{\Omega_B}\right)^2\operatorname{tg}^2\frac{\beta_0}{2}=F\left(\operatorname{tg}^2\frac{\beta_0}{2}\right). \tag{69}$$

Here F is the maximum value (with respect to Φ) of the right-hand side of (6.20). $F\sim 1$ and varies slowly at not too large $\tan(\beta_0/2)$. Its plot is shown in Fig. 2. For the conditions of the experiment of Corruccini and Osheroff,[2] the left-hand side of (22) is of the order of 10^{-2}, i.e., no changes were to be expected in the character of the relaxation in these experiments, whereas for the conditions of Webb's experiment,[2] where such a

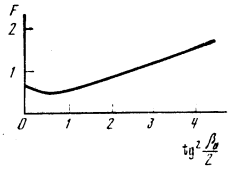

FIG. 2.

change was observed, the left-hand side of (22) was of the order of unity. We used for the estimates the experimentally observed values of τ.

We have considered so far only the situation in the Leggett configuration. We now make a few remarks concerning the relaxation at arbitrary μ. By virtue of the stability of the uniform precession in regions 2 and 3 of Fig. 1, the magnetization inhomogeneity is due here to the inhomogeneity of the magnetic field, and the relaxation should obey the law (66). In region 1, the uniform precession is unstable and one should expect the relaxation to be of the same type as in the A phase.

It must be borne in mind, however, that to obtain states with $\mu \neq 0$ the helium is placed in a gap with a width of only several tenths of a millimeter and the effect of the walls on the relaxation process can become significant. This effect increases with decreasing gap, first because of the larger role of the surface relaxation, and second, as noted in the article of Corruccini and Osheroff (1978), the texture effects become more substantial.

CONCLUSIONS

We see thus that the dephasing of the magnetization precession does indeed exert a substantial influence on the magnetic relaxation in the superfluid phases of ^{3}He. This influence is effected via spin diffusion. The diffusion tends to equalize the gradients of the transverse component of the magnetization, which are due to the dephasing, but the spin-orbit interaction maintains the spin near the resonant value $S \approx 1$. Equalization of the transverse gradients with conservation of the value of S leads inevitably to relaxation of the longitudinal component of S.

The cause of the dephasing can be the instability of the uniform precession of the magnetization, or the difference between the Larmor frequencies at different points, and accordingly the character of the relaxation will also be different. In particular, the relaxation should proceed differently in the A and B phases in the Leggett configuration. In the A phase, the dephasing is due to the instability of the uniform precession, whereas in the B phase there is no instability and the inhomogeneity of the magnetic field is essential.

The considered diffusion mechanism of the relaxation appears even in first order in $(\Omega/\omega_L)^2$ and is generally speaking more effective than the intrinsic mechanism that leads to longitudinal relaxation only in second order in the indicated parameter. Conditions are possible, however, in which the diffusion mechanism does not operate. Such, e.g., is the case with relaxation when the magnetization remains at all time parallel to the external homogeneous field. The relaxation due to the intrinsic mechanism can be observed also immediately after the initial deflection of the magnetization within times shorter than the dephasing time. The existence of two precession regimes, resonant and nonresonant, and the possibility of the transition from one regime to another, make the picture of the relaxation in the superfluid phases of ^{3}He even more varied. As a result, the combination of the two relaxation mechanisms, diffusion and internal, explains, at least qualitatively, the experimentally observed variety of the laws of magnetization relaxation to equilibrium in the superfluid phases of ^{3}He.

Unfortunately, the available experimental data cannot be quantitatively compared with the theoretical predictions for the diffusion mechanism, so that the question whether these two mechanisms are sufficient to explain the experimentally observed relaxation remains unanswered. To answer this question, further experimental studies must be made of magnetic relaxation in superfluid He.[3] These studies will yield also additional information on the properties of the superfluid phases, and will make it possible in particular to measure the spin-diffusion coefficient in these phases.

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[1)] The critical angle determined by formula (43) differs from that previously obtained,[8] since the criterion stipulated there for the change in the relaxation time was that $\sin 2\Phi$ become equal to unity at the very end of the tipping process. This is too strong a requirement and does not conform to the experimental conditions of Ref. 3.

[2)] Such an investigation was carried out earlier[17] for uniform precession.

[3)] Expressions for $\partial V_B/\partial u$ in various regions of the (u, μ) plane were obtained earlier.[10] In formula (B.11) of this paper there is an error. The last term in the square brackets should be half as large, i.e., take the form $\frac{5}{8}(\frac{6}{4}\sin^2\chi)^2(1+7\cos\beta)$.

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