# Singularities of the order parameter of a superconductor

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It is shown that in the general case the singularities of $\Delta(\mathbf{p},\omega)$ in $\mathbf{p},\omega$ space are due to changes in the topology of the line of intersection of the Fermi surface $\varepsilon(\mathbf{p})=\varepsilon_F$ with the equal-energy phonon surfaces $\omega_\lambda(\mathbf{q})=\omega$. The singularities of the averaged quantity $\Delta(\omega)=\langle\Delta(\mathbf{p},\omega)\rangle_{\mathbf{p}}$ are the consequences of the singularities in $\Delta(\mathbf{p},\omega)$. The function $\Delta(\omega)$ can have singularities of two types. The first is due to the fact that at a definite value $\mathbf{p}=\mathbf{p}^*$ the surfaces $\varepsilon(\mathbf{p}^*+\mathbf{q})=\varepsilon_F$ and $\omega_\lambda(\mathbf{q})=\omega$ are tangent at the point $\mathbf{q}=\mathbf{q}_c$, where the vector $\mathbf{q}_c$ satisfies the conditions of Kohn singularity for the surfaces $\varepsilon(\mathbf{p}^*)=\varepsilon_F$ and $\varepsilon(\mathbf{p}^*+\mathbf{q}_c)=\varepsilon_F$. The second type of singularity of $\Delta(\omega)$ exists if the surface $\omega_\lambda(\mathbf{q})=\omega$ changes its topology at some value $\mathbf{q}=\mathbf{q}_k$, while the surfaces $\varepsilon(\mathbf{p})=\varepsilon_F$ and $\varepsilon(\mathbf{p}+\mathbf{q}_k)=\varepsilon_F$ intersect or are tangent.

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The most complete information on the energy spectrum of a superconductor can be obtained by investigating the tunnel effect. From the tunnel characteristics obtained for polycrystalline samples one determines the averaged value of the order parameter in the superconductor, $\Delta\omega=\langle\Delta(\mathbf{p},\omega)\rangle_p$.[1] Here and below the symbol

$$\langle\ldots\rangle_p=\frac{1}{(2\pi)^3}\int\frac{dS_p}{|\mathbf{v}_p|}(\ldots)\frac{1}{\nu_0(\varepsilon_F)},$$
$$\nu_0(\varepsilon_F)=\frac{1}{(2\pi)^3}\int\frac{dS_p}{|\mathbf{v}_p|}$$

means that the integration is carried out over the Fermi surface $\varepsilon(\mathbf{p})=\varepsilon_F$; $dS_p$ is the surface-area element, $\mathbf{v}_p=\nabla_p\varepsilon(\mathbf{p})$ at $\varepsilon(\mathbf{p})=\varepsilon_F$ is the quasiparticle velocity and $\nu_0(\varepsilon_F)$ is the electron-state density.

It was observed in experiment that the quantity $\Delta'(\omega)=\partial\Delta(\omega)/\partial\omega$ has singularities.[1] [The functions $f'(\omega)$ designate hereafter derivatives with respect to $\omega$.] Scalapino and Anderson[2] proposed that these singularities are due to the presence of a singularity in the derivative of the irregular increment to the density of the number of phonon states with respect to energy when the topology of the equal-energy surfaces of the phonons is changed.

The singularities of $\Delta'(\omega)$ are the consequence of singularities of the generalized order parameter $\Delta'(p,\omega)$, which depends on the momentum $\mathbf{p}$ and on the frequency $\omega$. The nature of the singularities in $\Delta'(\mathbf{p},\omega)$ in $\mathbf{p},\omega$ space was not established in Ref. 2. The conditions for the realization of the type of singularity chosen in Ref. 2, as well as the possibility of existence of other singularities in $\Delta'(\omega)$, were therefore left unexplained.

In this communication, following the previously developed[3] concepts, we consider from a unified point of view the possible singularities of $\Delta'(\mathbf{p},\omega)$ and of $\Delta'(\omega)$. It is shown that in the general case the singularities of $\Delta'(\mathbf{p},\omega)$ in $\mathbf{p},\omega$ space are due to a change in the topology of the line of intersection of the Fermi surface with the phonon equal-energy surfaces $\omega_\lambda(\mathbf{q})=\omega$. The function $\Delta'(\omega)$ can have singularities of two types. The first type of singularity in $\Delta'(\omega)$ is due to the fact that at a definite value of $\mathbf{p}=\mathbf{p}^*$ the surfaces $\varepsilon(\mathbf{p}^*+\mathbf{q})=\varepsilon_F$ and $\omega_\lambda(\mathbf{q})=\omega$ are tangent at the point $\mathbf{q}=\mathbf{q}_c$, where the vector $\mathbf{q}_c$ satisfies the conditions of the Kohn singularity for the surfaces $\varepsilon(\mathbf{p}^*)=\varepsilon_F$ and $\varepsilon(\mathbf{p}^*+\mathbf{q}_c)=\varepsilon_F$. The second type of singularity in $\Delta'(\omega)$ exists if the surface $\omega_\lambda(\mathbf{q})=\omega$ changes its topology at a certain value $\mathbf{q}=\mathbf{q}_k$, while the surfaces $\varepsilon(\mathbf{p})=\varepsilon_F$ and $\varepsilon(\mathbf{p}+\mathbf{q}_k)=\varepsilon_F$ intersect or are tangent.

## 1. DERIVATION OF THE INITIAL EQUATIONS

To investigate the singularities of $\Delta(\mathbf{p},\omega)=\psi(\mathbf{p},\omega)/Z(\mathbf{p},\omega)$ in $\mathbf{p},\omega$ space we start from the Eliashberg[4] system of equations for the functions $\psi(\mathbf{p},\omega)$ and $Z(\mathbf{p},\omega)$, which for our purposes are best written in the form

$$\begin{gathered}
\operatorname{Re}\psi(\mathbf{p},\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty}d\omega'\,\frac{\operatorname{Im}\psi(\mathbf{p},\omega')}{\omega-\omega'}+\psi_c,\\
\psi_c=\int_0^{\infty}\frac{d\omega'}{\pi}\int\frac{dp'}{(2\pi)^3}\operatorname{Im}\left\{\frac{\psi(\mathbf{p}',\omega')}{Q(\mathbf{p}',\omega')}\right\}U(\mathbf{p},\mathbf{p}')\operatorname{th}\frac{\omega'}{2T},\\
[1-\operatorname{Re}Z(\mathbf{p},\omega)]\,\omega=\frac{1}{\pi}\int_{-\infty}^{\infty}d\omega'\,\frac{\operatorname{Im}Z(\mathbf{p},\omega')\,\omega'}{\omega'-\omega},\\
\operatorname{Im}\psi(\mathbf{p},\omega)\,|_{\omega>0}=-\int_0^{\infty}d\omega'\operatorname{th}\frac{\omega'}{2T}\int_\Omega\frac{d\mathbf{q}}{(2\pi)^3}\operatorname{Im}\left\{\frac{\psi(\mathbf{p}+\mathbf{q},\omega')}{Q(\mathbf{p}+\mathbf{q},\omega')}\right\}\\
\times\sum_\lambda|g^\lambda_{\mathbf{p},\mathbf{p}+\mathbf{q}}|^2\delta(\omega_\lambda(\mathbf{q})+\omega'-\omega),\\
\omega\operatorname{Im}Z(\mathbf{p},\omega)\,|_{\omega>0}=\int_0^{\infty}d\omega'\operatorname{th}\frac{\omega'}{2T}\int_\Omega\frac{d\mathbf{q}}{(2\pi)^3}\operatorname{Im}\left\{\frac{\omega'Z(\mathbf{p}+\mathbf{q},\omega')}{Q(\mathbf{p}+\mathbf{q},\omega')}\right\}\\
\times\sum_\lambda|g^\lambda_{\mathbf{p},\mathbf{p}+\mathbf{q}}|^2\delta(\omega_\lambda(\mathbf{q})+\omega'-\omega),\\
Q(\mathbf{p},\omega)=(\omega Z(\mathbf{p},\omega))^2-\psi^2(\mathbf{p},\omega)-(\varepsilon(\mathbf{p})-\varepsilon_F)^2.
\end{gathered}\tag{1}$$

Here $\varepsilon(\mathbf{p})$ and $\omega_\lambda(\mathbf{q})$ are the electron and phonon dispersion laws, $g^\lambda_{\mathbf{p},\mathbf{p}'}$ is the parameter of the electron-phonon interaction, $\lambda$ is the number of the branch of the phonon spectrum, $\varepsilon_F$ is the Fermi energy, $\Omega$ is the volume of the unit cell in momentum space, $T$ is the temperature, and $U(\mathbf{p},\mathbf{p}')$ is the screened potential of the Coulomb interaction of the electrons.

It is seen from the system (1) that the singularities of the functions $\operatorname{Re}\psi(\mathbf{p},\omega)$ and $\operatorname{Re}Z(\mathbf{p},\omega)$ are connected with the singularities of $\operatorname{Im}\psi(\mathbf{p},\omega)$ and $\operatorname{Im}Z(\mathbf{p},\omega)$. To investigate the singularities in $\operatorname{Im}\psi(\mathbf{p},\omega)$ and in $\operatorname{Im}Z(\mathbf{p},\omega)$ it is convenient to change from integration with respect to $d\mathbf{q}$ to integration with respect to the variables $\omega_\lambda(\mathbf{q})$, $\varepsilon(\mathbf{p}+\mathbf{q})$, and $l_q^\lambda$–the length of the contour of the line of

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intersection of the surfaces $\varepsilon(\mathbf{p}+\mathbf{q})=\varepsilon$ and $\omega_\lambda(\mathbf{q})=\omega$ (henceforth referred to as the "$l_\mathbf{q}$ line" for brevity).[3] In the general case there can be several such $l_\mathbf{q}^\lambda$ lines, since the functions $\varepsilon(\mathbf{p})$ and $\omega_\lambda(\mathbf{q})$ are multiple-valued and periodic. To determine how many $l_\mathbf{q}^\lambda$ lines are realized, it is convenient to consider the functions $\varepsilon(\mathbf{p})$ and $\omega_\lambda(\mathbf{q})$ in the repeating-band scheme. In this scheme it is easy to take into account the umklapp processes in the electron-phonon interaction, and also to determine the $l_\mathbf{q}$ lines corresponding to these processes. The length of the $l_\mathbf{q}^\lambda$ line depends on the external parameters $\mathbf{p}$ and $\omega$. We assume hereafter that the summation is over all the possible $l_\mathbf{q}^\lambda$ lines.

Integrating with respect to the variables $\omega_\lambda(\mathbf{q})$ and $\varepsilon(\mathbf{p}+\mathbf{q})$, we obtain expressions for $\mathrm{Im}\psi(\mathbf{p},\omega)$ and $\mathrm{Im}Z(\mathbf{p},\omega)$ in the form

$$\mathrm{Im}\,\psi(\mathbf{p},\omega)=\pi\nu_0(\varepsilon_F)\int\limits_0^\infty d\omega'\,\mathrm{th}\frac{\omega'}{2T}$$
$$\times\sum_\lambda\int\frac{dl_\mathbf{q}^\lambda|\tilde{g}^\lambda_{\mathbf{p},\mathbf{p}+\mathbf{q}}|^2}{|\mathbf{v}_{\mathbf{p}+\mathbf{q}}||\mathbf{S}_\mathbf{q}^\lambda(\omega-\omega')|\sin\vartheta_\mathbf{q}}\mathrm{Re}\frac{\Delta(\mathbf{p}+\mathbf{q},\omega')}{[\omega'^2-\Delta^2(\mathbf{p}+\mathbf{q},\omega')]^{1/2}},$$
$$\omega\,\mathrm{Im}\,Z(\mathbf{p},\omega)=-\pi\nu_0(\varepsilon_F)\int\limits_0^\infty d\omega'\,\mathrm{th}\frac{\omega'}{2T} \tag{2}$$
$$\times\sum_\lambda\int\frac{dl_\mathbf{q}^\lambda|\tilde{g}^\lambda_{\mathbf{p},\mathbf{p}+\mathbf{q}}|^2}{|\mathbf{v}_{\mathbf{p}+\mathbf{q}}||\mathbf{S}_\mathbf{q}^\lambda(\omega-\omega')|\sin\vartheta_\mathbf{q}}\mathrm{Re}\frac{\omega'}{[\omega'^2-\Delta^2(\mathbf{p}+\mathbf{q},\omega')]^{1/2}},$$

where

$$|\tilde{g}^\lambda_{\mathbf{p},\mathbf{p}+\mathbf{q}}|^2=|g^\lambda_{\mathbf{p},\mathbf{p}+\mathbf{q}}|^2/(2\pi)^3\nu_0(\varepsilon_F);$$

$\vartheta_\mathbf{q}$ is the angle between the vectors $\mathbf{v}_\mathbf{p}$ and $\mathbf{S}_\mathbf{q}^\lambda(\omega)=\nabla_\mathbf{q}\omega_\lambda(\mathbf{q})$ at $\omega_\lambda(\mathbf{q})=\omega$, and depends on $\mathbf{p}$ and on $\omega$.

As seen from (2), when the parameters $\mathbf{p}$ and $\omega$ are varied the integral with respect to $dl_\mathbf{q}^\lambda$ can have singular parts at certain values $\mathbf{p}=\mathbf{p}^*$ and $\omega=\omega^*$, namely: 1) at the points $\mathbf{q}=\mathbf{q}_c$ at which $\sin\vartheta_{\mathbf{q}_c}=0$, and 2) at the points $\mathbf{q}=\mathbf{q}_k$ at which $\mathbf{S}^\lambda_{\mathbf{q}_k}(\omega^*)=0$.

The geometrical interpretation of these cases is the following. In the first case the surfaces $\varepsilon(\mathbf{p}^*+\mathbf{q})=\varepsilon_F$ and $\omega_\lambda(\mathbf{q})=\omega^*$ are tangent at the point $\mathbf{q}=\mathbf{q}_c$ and the vector $\mathbf{p}^*$ is determined from the conditions $\varepsilon(\mathbf{p}^*+\mathbf{q}_c)=\varepsilon_F$ and $\varepsilon(\mathbf{p}^*)=\varepsilon_F$. In the second case the surface $\varepsilon(\mathbf{p}^*+\mathbf{q})=\varepsilon_F$ intersects the surface $\omega_\lambda(\mathbf{q})=\omega$ near the direction $\mathbf{q}=\mathbf{q}_k$ where a change takes place in the topology of the equal-energy surfaces of the phonons, and the vector $\mathbf{p}^*$ satisfies the conditions $\varepsilon(\mathbf{p}^*)=\varepsilon_F$ and $\varepsilon(\mathbf{p}^*+\mathbf{q}_k)=\varepsilon_F$. Infinitesimally close to the points $\mathbf{q}=\mathbf{q}_c$ and $\mathbf{q}=\mathbf{q}_k$, the $l_\mathbf{q}$ line changes its topology. It will be shown later that the singular part of the integral with respect to $dl_\mathbf{q}$ is very sensitive to the type of topological changes of the $l_\mathbf{q}$ line. Under these conditions, following integration with respect to $dl_\mathbf{q}$, the integrand has a singularity that manifests itself in the functions $\mathrm{Im}\psi(\mathbf{p},\omega)$ and $\mathrm{Im}Z(\mathbf{p},\omega)$, owing to the square-root singularity of the integrand $[\omega^2-\Delta^2(\mathbf{p},\omega)]^{-1/2}$; this singularity is proportional to the density of the number of states of the quasiparticles in the superconductor.

To establish the analytic form of the singularities in $\mathrm{Im}\psi(\mathbf{p},\omega)$ and $\mathrm{Im}Z(\mathbf{p},\omega)$, we separate in the volume $\Omega$ the smaller volumes $\delta\Omega_{\mathbf{q}_i}$ near the points $\mathbf{q}_i=\mathbf{q}_c$ and $\mathbf{q}_i=\mathbf{q}_k$. The expressions (2) can then be written in the form

$$\mathrm{Im}\,\psi(\mathbf{p},\omega)=\mathrm{Im}\,\psi_0(\mathbf{p},\omega)+\sum_i\mathrm{Im}\,\delta\psi_{\mathbf{q}_i}(\mathbf{p},\omega),$$
$$\mathrm{Im}\,Z(\mathbf{p},\omega)=\mathrm{Im}\,Z_0(\mathbf{p},\omega)+\sum_i\mathrm{Im}\,\delta Z_{\mathbf{q}_i}(\mathbf{p},\omega), \tag{3}$$

where $\mathrm{Im}\psi_0(\mathbf{p},\omega)$ and $\mathrm{Im}Z_0(\mathbf{p},\omega)$ are the smooth parts and $\mathrm{Im}\delta\psi_{\mathbf{q}_i}(\mathbf{p},\omega)$ and $\mathrm{Im}\delta Z_{\mathbf{q}_i}(\mathbf{p},\omega)$ the singular parts of the considered functions, due to the change of the topology of the $l_\mathbf{q}^\lambda$ line near the points $\mathbf{q}_i=\mathbf{q}_c$ and $\mathbf{q}_i=\mathbf{q}_k$.

The functions $\mathrm{Im}\psi_0(\mathbf{p},\omega)$ and $\mathrm{Im}Z_0(\mathbf{p},\omega)$ are known.[2] We represent the functions $\mathrm{Im}\delta\psi_{\mathbf{q}_i}(\mathbf{p},\omega)$ and $\mathrm{Im}\delta Z_{\mathbf{q}_i}(\mathbf{p},\omega)$ in the form

$$\mathrm{Im}\,\delta\psi_{\mathbf{q}_i}(\mathbf{p},\omega)=\pi\nu_0(\varepsilon_F)\int\limits_0^\infty d\omega'\,\mathrm{th}\frac{\omega'}{2T}$$
$$\times\mathrm{Re}\left\{\frac{\Delta_0(\mathbf{p}+\mathbf{q}_i,\omega')}{[\omega'^2-\Delta_0^2(\mathbf{p}+\mathbf{q}_i,\omega')]^{1/2}}\right\}g_{\mathbf{q}_i}(\mathbf{p},\omega-\omega'),$$
$$\mathrm{Im}\,\delta Z_{\mathbf{q}_i}(\mathbf{p},\omega)=-\pi\nu_0(\varepsilon_F)\int\limits_0^\infty d\omega'\,\mathrm{th}\frac{\omega'}{2T} \tag{4}$$
$$\times\mathrm{Re}\left\{\frac{\omega'}{[\omega'^2-\Delta_0^2(\mathbf{p}+\mathbf{q}_i,\omega')]^{1/2}}\right\}g_{\mathbf{q}_i}(\mathbf{p},\omega-\omega'),$$

where

$$g_{\mathbf{q}_i}(\mathbf{p},\eta)=\sum_\lambda|\tilde{g}^\lambda_{\mathbf{p},\mathbf{p}+\mathbf{q}_i}|^2\int\limits_{\delta\Omega_{\mathbf{q}_i}}d\mathbf{q}\,\delta(\omega_\lambda(\mathbf{q}_i+\mathbf{q})-\eta)\,\delta(\varepsilon(\mathbf{p}+\mathbf{q}_i+\mathbf{q})-\varepsilon_F).$$

In the derivation of the expressions in (4) we took into account the fact that the surfaces $\varepsilon(\mathbf{p})=\varepsilon_F$ and $\omega_\lambda(\mathbf{q})=\omega$ have substantially differing gradients $\mathbf{v}_\mathbf{p}\gg\mathbf{S}_\mathbf{q}^\lambda(\omega)$ and effective masses $[\partial^2\varepsilon(\mathbf{p})/\partial\mathbf{p}^2\gg\partial^2\omega_\lambda(\mathbf{q})/\partial\mathbf{q}^2]$. We have also neglected the terms $|\mathbf{S}_\mathbf{q}^\lambda|\omega'/|\mathbf{v}_\mathbf{p}|$, since the characteristic interval of variation of the parameter $\omega$ and of the variable $\omega'$ is of the order of the end-point frequency of the phonon spectrum.

## 2. SINGULARITIES OF $\Delta'(\mathbf{p},\omega)$, DUE TO TANGENCY OF THE SURFACES $\varepsilon(\mathbf{p}+\mathbf{q})=\varepsilon_F$ AND $\omega_\lambda(\mathbf{q})=\omega$

In the calculation of the expressions $\mathrm{Im}\delta\psi_{\mathbf{q}_c}(\mathbf{p},\omega)$ and $\mathrm{Im}\delta Z_{\mathbf{q}_c}(\mathbf{p},\omega)$ it is convenient to introduce in $\mathbf{q}$-space rectangular Cartesian coordinates, taking the common tangency point $\mathbf{q}=\mathbf{q}_c$ of the two surfaces $\varepsilon(\mathbf{p}^*+\mathbf{q})=\varepsilon_F$ and $\omega_\lambda(\mathbf{q})=\omega^*$ to be the origin, and the tangent plane at the point $\mathbf{q}=\mathbf{q}_c$ to be the $xy$ plane, the normal to which (normal to the surface) is the $z$ axis. The expressions for $\varepsilon(\mathbf{p}+\mathbf{q})=\varepsilon_F$ and $\omega_\lambda(\mathbf{q})=\omega^*$, assuming the deviations of the vector $\mathbf{p}$ from $\mathbf{p}^*$ and of $\omega$ from $\omega^*$ to be small, can then be represented in the form

$$\varepsilon(\mathbf{p}+\mathbf{q})=\varepsilon(\mathbf{p}+\mathbf{q}_c)+|\mathbf{v}_{\mathbf{p}+\mathbf{q}_c}|z/(\mathbf{n}\mathbf{n}_0)+f_1(x,y);$$
$$\omega_\lambda(\mathbf{q})=\omega_c^*+|\mathbf{S}^\lambda_{\mathbf{q}_c}(\omega_c^*)|z+f_2(x,y),\quad \omega_c^*\equiv\omega_\lambda(\mathbf{q}_c), \tag{5}$$

where $f_{1,2}(x,y)$ are polynomial functions that are the first nonvanishing Taylor-series terms of the functions $\varepsilon(\mathbf{p})$ and $\omega_\lambda(\mathbf{q})$; $\mathbf{n}$ and $\mathbf{n}_0$ are unit vectors in the directions of $\mathbf{v}_{\mathbf{p}+\mathbf{q}_c}$ and $\mathbf{v}_{\mathbf{p}^*+\mathbf{q}_c}$, respectively.

Substituting (5) in (4) we get for $g_{\mathbf{q}_c}(\mathbf{p},\eta)$

$$g_{\mathbf{q}_c}(\mathbf{p},\eta)=\sum_\lambda a_\lambda^0\int dx\,dy\,\delta(f_\mathbf{p}(x,y,\eta)),$$
$$a_\lambda^0=|\tilde{g}^\lambda_{\mathbf{p},\mathbf{p}+\mathbf{q}_c}|^2(\mathbf{n}\mathbf{n}_0)/|\mathbf{v}_{\mathbf{p}+\mathbf{q}_c}||\mathbf{S}^\lambda_{\mathbf{q}_c}(\omega_c^*)|. \tag{6}$$

The equation $f_\mathbf{p}(x,y,\eta)=0$ defines the behavior of the

$l_{\mathbf{q}}^{\lambda}$ line in the vicinity of the tangency points. It follows from (6) that the analytic form of $g_{\mathbf{q}_c}(\mathbf{p},\eta)$ is very sensitive to the form of the function $f_{\mathbf{p}}(x,y,\eta)$, which reflects the distinguishing features of the tangency point. To obtain the singular part of the function $g_{\mathbf{q}_c}(\mathbf{p},\eta)$ for different topological changes of the $l_{\mathbf{q}}^{\lambda}$ line near the tangency point, one can use the results of Kaganov and Semenko.[5]

1. In the vicinity of the tangency point, the most probable description of the $l_{\mathbf{q}}^{\lambda}$ line is by the equation for a second-degree curve:

$$C_{xx}x^2+2C_{xy}xy+C_{yy}y^2+\delta_{\mathbf{p}}(\eta)=0, \tag{7}$$

$$\delta_{\mathbf{p}}(\eta)=(\eta-\omega_{\lambda}^{\pm}(\mathbf{q}_c,\mathbf{p}))/|S_{\mathbf{q}_c}^{\lambda}(\omega_c^*)|,$$

$$\omega_{\lambda}^{\pm}(\mathbf{q}_c,\mathbf{p})=\omega_c^*\mp\frac{|S_{\mathbf{q}_c}^{\lambda}(\omega_c^*)|}{|\mathbf{v}_{\mathbf{p}^*+\mathbf{q}_c}|}(\varepsilon(\mathbf{p}+\mathbf{q}_c)-\varepsilon_F),$$

$$C_{\alpha\beta}=\frac{1}{2}\left\{\frac{\mathbf{n}\mathbf{n}_0}{|\mathbf{v}_{\mathbf{p}^*+\mathbf{q}_c}|}\frac{\partial^2\varepsilon(\mathbf{p}+\mathbf{q})}{\partial q_\alpha\partial q_\beta}-\frac{1}{|S_{\mathbf{q}_c}^{\lambda}(\omega_c^*)|}\frac{\partial^2\omega_\lambda(\mathbf{q})}{\partial q_\alpha\partial q_\beta}\right\}_{\mathbf{q}=\mathbf{q}_c}.$$

The frequency $\omega_{\lambda}^{+}(\mathbf{q}_c,\mathbf{p})$ corresponds here to the case when the vectors $\mathbf{n}$ and $\mathbf{n}_0$ become parallel as $\mathbf{p}\to\mathbf{p}^*$, and $\omega_{\lambda}^{-}(\mathbf{q}_c,\mathbf{p})$ corresponds to the case when these vectors are antiparallel.

When the parameters $\mathbf{p}$ and $\omega$ are varied, two topological changes of the $l_{\mathbf{q}}^{\lambda}$ line are then possible: 1) the $l_{\mathbf{q}}^{\lambda}$ is an ellipse that degenerates into a point, $g_{\mathbf{q}_c}(\mathbf{p},\eta)\sim\theta[\delta_{\mathbf{p}}(\eta)]$ (elliptic tangency point), 2) the $l_{\mathbf{q}}^{\lambda}$ line is a hyperbola that degenerates into two straight lines, $g_{\mathbf{q}_c}(\mathbf{p},\eta)\sim\ln|\delta_{\mathbf{p}}(\eta)|$ (hyperbolic tangency point).

To establish the form of the singularities in $\delta\psi_{\mathbf{q}_c}(\mathbf{p},\omega)$ it is convenient to use the calculated results for the derivatives $\delta\psi_{\mathbf{q}_c}'(\mathbf{p},\omega)=\omega\delta Z_{\mathbf{q}_c}'(\mathbf{p},\omega)$ at $T=0$:

$$\delta\psi_{\mathbf{q}_c}'(\mathbf{p},\omega)=\sum_{\lambda}a_\lambda\nu_0(\varepsilon_F)\begin{cases}\pm(B^k(\mathbf{p},\omega)+iA^k(\mathbf{p},\omega)), & l_{\mathbf{q}}^{\lambda}\text{ — ellipse}\\ -(A^k(\mathbf{p},\omega)+iB^k(\mathbf{p},\omega)), & l_{\mathbf{q}}^{\lambda}\text{ — hyperbola}\end{cases},$$

$$a_\lambda=a_\lambda^0|C_{xx}C_{yy}-C_{xy}^2|^{-1/2}, \tag{8}$$

$$B^k(\mathbf{p},\omega)=\pi^2\Delta_0(\mathbf{p}+\mathbf{q}_c)\{\Delta_0^2(\mathbf{p}+\mathbf{q}_c)-(\omega-\omega_\lambda^{\pm}(\mathbf{q}_c,\mathbf{p}))^2\}^{-1/2}$$
$$\times\theta(\Delta_0(\mathbf{p}+\mathbf{q}_c)+\omega_\lambda^{\pm}(\mathbf{q}_c,\mathbf{p})-\omega),$$
$$A^k(\mathbf{p},\omega)=\pi^2\Delta_0(\mathbf{p}+\mathbf{q}_c)\{(\omega-\omega_\lambda^{\pm}(\mathbf{q}_c,\mathbf{p}))^2-\Delta_0^2(\mathbf{p},\mathbf{q})\}^{-1/2}$$
$$\times\theta(\omega-\omega_\lambda^{\pm}(\mathbf{q}_c,\mathbf{p})-\Delta_0(\mathbf{p}+\mathbf{q}_c)),$$

where $\theta(x)=1$ at $x\geqslant0$ and $\theta(x)=0$ at $x<0$.

If the $l_{\mathbf{q}}$ line is an ellipse, the minus sign corresponds to a positive definite form of $C_{\alpha\beta}x_\alpha x_\beta$, and the plus sign to a negative definite from of $C_{\alpha\beta}x_\alpha x_\beta$.

2. It is possible that near the tangency point the coefficients $C_{\alpha\beta}x_\alpha x_\beta$ of the quadratic form vanish, i.e., if we confine ourselves to the second-order approximation, then the tangency takes place over a section of the surface (flattening point). In this case the change of the topology of the $l_{\mathbf{q}}^{\lambda}$ line leads to stronger singularities in $\delta\psi_{\mathbf{q}_c}'(\mathbf{p},\omega)$ and $\delta Z_{\mathbf{q}_c}'(\mathbf{p},\omega)$ compared with (8).

Assume for the sake of argument that the equation for the $l_{\mathbf{q}}^{\lambda}$ line is of the form

$$c_5x^4+2c_7x^2y^2+c_9y^4+\delta_{\mathbf{p}}(\eta)=0, \tag{9}$$

where the coefficients $c_i$ are expressed in obvious fashion in terms of the corresponding coefficients of the functions $f_1(x,y)$ and $f_2(x,y)$ [expression (6)]. The calculations lead to the result

$$g_{\mathbf{q}_c}(\mathbf{p},\eta)\sim[\delta_{\mathbf{p}}(\eta)]^{-1/2}\theta(\delta_{\mathbf{p}}(\eta)), \tag{10a}$$

$$\delta\psi_{\mathbf{q}_c}'(\mathbf{p},\omega)=\sum_{\lambda}a_\lambda^1\nu_0(\varepsilon_F)\begin{cases}A_1^k(\mathbf{p},\omega)+iB_1^k(\mathbf{p},\omega)\\ B_1^k(\mathbf{p},\omega)+iA_1^k(\mathbf{p},\omega)\end{cases}, \tag{10b}$$

where

$$A_1^k(\mathbf{p},\omega)=\pi(^1/_2\Delta_0(\mathbf{p}+\mathbf{q}_c))^{3/2}[\omega_\lambda^{\pm}(\mathbf{q}_c,\mathbf{p})+\Delta_0(\mathbf{p}+\mathbf{q}_c)-\omega]^{-1},$$
$$B_1^k(\mathbf{p},\omega)=\pi^2(^1/_2\Delta_0(\mathbf{p}+\mathbf{q}_c))^{1/2}\delta(\omega-\Delta_0(\mathbf{p}+\mathbf{q}_c)-\omega_\lambda^{\pm}(\mathbf{q}_c,\mathbf{p})),$$
$$a_\lambda^1=a_\lambda^0|S_{\mathbf{q}_c}^{\lambda}(\omega_c^*)|^{1/2}|c_7|^{-1/2}.$$

The upper line in the right-hand side of (10b) corresponds to the case $c_5, c_9<0$, and the lower to the case $c_5, c_9>0$, while the function $\omega_\lambda^{\pm}(\mathbf{q}_c,\mathbf{p})$ is of the same form as in (7).

In the derivation of (8) and (10) as well as later on we make the substitution

$$\mathrm{Re}\left\{\frac{\Delta_0(\mathbf{p},\omega)}{[\omega^2-\Delta_0^2(\mathbf{p},\omega)]^{1/2}}\right\}\to\frac{\Delta_0(\mathbf{p})\theta(\omega-\Delta_0(\mathbf{p}))}{[\omega^2-\Delta_0^2(\mathbf{p},\omega)]^{1/2}}.$$

At $\omega\gg\Delta_0(\mathbf{p})$ the main contribution to the integral of (4) is made by the values $\omega'\approx\Delta_0(\mathbf{p})$, where $\Delta_0(\mathbf{p})=\Delta_0[\mathbf{p},\Delta(\mathbf{p})]$ is obtained by solving the equations $\omega^2-\Delta_0^2(\mathbf{p},\omega)=0$.

Thus, it follows from (8) and (10) that the functions $\delta\psi_{\mathbf{q}_c}'(\mathbf{p},\omega)$ and $\delta Z_{\mathbf{q}_c}'(\mathbf{p},\omega)$ have singularities at $\omega=\omega_\lambda^{\pm}(\mathbf{q}_c,\mathbf{p})+\Delta_0(\mathbf{p}+\mathbf{q}_c)$. The analytic forms of these singularities are very sensitive to the types of the topological changes of the $l_{\mathbf{q}}^{\lambda}$ line near the tangency point. For example, elliptic and hyperbolic tangency points correspond to square-root singularities, while a flattening point corresponds to a singularity of the type $1/x$.

A schematic representation of some of the obtained types of singularity is shown in Fig. 1.

## 3. $\Delta'(\mathbf{p},\omega)$ SINGULARITIES CONNECTED WITH THE CHANGE OF THE TOPOLOGY OF THE SURFACE $\omega_\lambda(\mathbf{q})=\omega$

To obtain expressions for $\mathrm{Im}\,\delta\psi_{\mathbf{q}_k}(\mathbf{p},\omega)$ and $\mathrm{Im}\,\delta Z_{\mathbf{p}_k}(\mathbf{p},\omega)$ [Eqs. (5)] we choose the origin in $q$-space to be the point $\mathbf{q}=\mathbf{q}_k$. Near this point, it suffices to retain in the

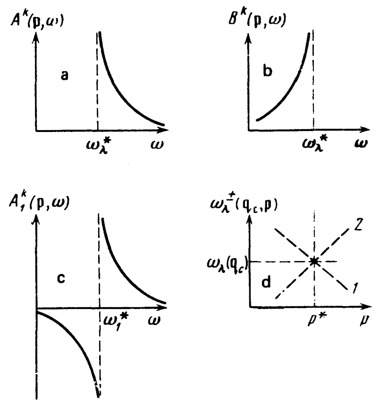


FIG. 1. a–c) Plots of the functions $A^k(\mathbf{p},\omega)$, $B^k(\mathbf{p},\omega)$ and $A_1^k(\mathbf{p},\omega)$ against the frequency $\omega$ as $\omega\to\omega_\lambda^*$, where $\omega_\lambda^*\equiv\omega_\lambda^{\pm}(\mathbf{q}_c,\mathbf{p})+\Delta_0(\mathbf{p}+\mathbf{q}_c)$; d) dependence of $\omega_\lambda^{\pm}(\mathbf{q}_c,\mathbf{p})$ on $\mathbf{p}$: curve 1—for $\omega_\lambda^{+}(\mathbf{q}_c,\mathbf{p})$, curve 2—for $\omega_\lambda^{-}(\mathbf{q}_c,\mathbf{p})$.

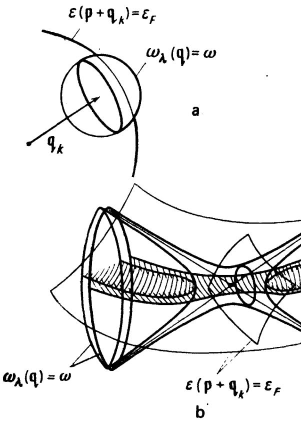


FIG. 2. Variation of the surfaces $\varepsilon(\mathbf{p}+\mathbf{q})=\varepsilon_F$ and $\omega_\lambda(\mathbf{q})=\omega$ near $\mathbf{q}=\mathbf{q}_k$: a) the function $\omega_\lambda(\mathbf{q})=\omega$ has a maximum or a minimum at $\mathbf{q}=\mathbf{q}_k$; b) the function $\omega_\lambda(\mathbf{q})=\omega$ has a conical point at $\mathbf{q}=\mathbf{q}_k$.

function $\omega_\lambda(\mathbf{q}_\lambda+\mathbf{q})$ the quadratic terms. The expressions for $g_{\mathbf{q}_k}(\mathbf{p},\eta)$ can be represented in the form

$$g_{\mathbf{q}_k}(\mathbf{p},\eta)=\sum_\lambda|g^\lambda_{\mathbf{p},\,\mathbf{p}+\mathbf{q}_k}|^2\int\limits_{\delta\Omega_{\mathbf{q}_k}}d\mathbf{q}\,\delta\left(\omega_\lambda(\mathbf{q}_k)-\eta+\sum_i\varepsilon_i\frac{\mathbf{q}_i^2}{2M_i}\right)\times\delta(\varepsilon(\mathbf{p}+\mathbf{q}_k)-\varepsilon_F+\mathbf{v}_{\mathbf{p}+\mathbf{q}_k}\mathbf{q}),\qquad(11)$$

where $(M_i)^{-1}=\partial^2\omega_\lambda(\mathbf{q})/\partial\mathbf{q}^2$ at $\mathbf{q}=\mathbf{q}_k$ are the principal values of the effective-mass tensor, and $\varepsilon_i=\pm1$ is a parameter that takes into account the different topological changes in the surface $\omega_\lambda(\mathbf{q})=\omega$, namely formation or vanishing of individual sections of the surface, or else formation or vanishing of the "bridge" between the sections of this surface. The investigation of the singularities of the function $g_{\mathbf{q}_k}(\mathbf{p},\eta)$ is similar to the investigation of the singularities of the absorption coefficient of ultrasound in metals in a phase transition of order $2\frac{1}{2}$.[6]

We consider now the singularities in $\delta\psi'_{\mathbf{q}_k}(\mathbf{p},\omega)=\omega\delta Z'_{\mathbf{q}_k}(\mathbf{p},\omega)$.

1. Let the function $\omega_\lambda(\mathbf{q})$ have at $\mathbf{q}=\mathbf{q}_k$ a maximum ($\varepsilon_i<0$) or a minimum ($\varepsilon_i>0$); then the $l^\lambda_\mathbf{q}$ line in the expression for $g_{\mathbf{q}_k}(\mathbf{p},\eta)$ is an ellipse (Fig. 2a). Integrating in this case we arrive at the result

$$\begin{aligned}\delta\psi'_{\mathbf{q}_k}(\mathbf{p},\omega)&=\pm\sum_\lambda b\Delta_0(\mathbf{p}_0+\mathbf{q}_k)\times(B^{\mathbf{q}_k}(\mathbf{p},\omega)+iA^{\mathbf{q}_k}(\mathbf{p},\omega)),\\ B^{\mathbf{q}_k}(\mathbf{p},\omega)&=\{\Delta_0^2(\mathbf{p}+\mathbf{q}_k)-(\omega-\omega_\lambda^\pm(\mathbf{q}_k,\mathbf{p}))^2\}^{-1/2}\times\theta(\Delta_0(\mathbf{p}+\mathbf{q}_k)+\omega_\lambda^\pm(\mathbf{q}_k,\mathbf{p})-\omega),\\ A^{\mathbf{q}_k}(\mathbf{p},\omega)&=\{(\omega-\omega_\lambda^\pm(\mathbf{q}_k,\mathbf{p}))^2-\Delta_0^2(\mathbf{p}+\mathbf{q}_k)\}^{-1/2}\times\theta(\omega-\omega_\lambda^\pm(\mathbf{q}_k,\mathbf{p})-\Delta_0(\mathbf{p}+\mathbf{q}_k)),\\ \omega_\lambda^\pm(\mathbf{q}_k,\mathbf{p})&=\omega_\lambda(\mathbf{q}_k)\pm(\varepsilon(\mathbf{p}+\mathbf{q}_k)-\varepsilon_F)^2/2|\mathbf{v}^*|,\\ b&=(M_1M_2M_3)^{1/2}/4\pi^2|\mathbf{v}^*|,\end{aligned}\qquad(12)$$

where $\mathbf{v}^*$ is a vector with components

$$\mathbf{v}^*=\{v^x_{\mathbf{p}+\mathbf{q}_k}M_1^{1/2},\,v^y_{\mathbf{p}+\mathbf{q}_k}M_2^{1/2},\,v^z_{\mathbf{p}+\mathbf{q}_k}M_3^{1/2}\}.$$

The plus and minus signs correspond to the minimum and maximum of the function $\omega_\lambda(\mathbf{q})$, respectively.

2. Let the function $\omega_\lambda(\mathbf{q})$ have near $\mathbf{q}=\mathbf{q}_k$ a conical point (for example, $\varepsilon_1\varepsilon_2>0,\varepsilon_3<0$); then the $l^\lambda_\mathbf{q}$ line in the expression for $g_{\mathbf{q}_k}(\mathbf{p},\eta)$ can be either an ellipse ($\theta^\circ<\theta_{\rm cr}^\circ$) or a hyperbola ($\theta^\circ>\theta_{\rm cr}^\circ$)—see Fig. 2b [$\theta^\circ$ is the angle between the vector $\mathbf{v}_{\mathbf{p}+\mathbf{q}_k}$ and the $q_3$ axis, $\tan\theta^\circ_{\rm cr}=(M_3/M_2)^{1/2}$]. In these cases the expressions for $\delta\psi'_{\mathbf{q}_k}(\mathbf{p},\omega)=\omega\delta Z'_{\mathbf{q}_k}(\mathbf{p},\omega)$ take the form

$$\delta\psi'_{\mathbf{q}_k}(\mathbf{p},\omega)=\begin{cases}\sum\limits_\lambda b_1^{(+)}(B_+^{\mathbf{q}_k}(\mathbf{p},\omega)+iA_+^{\mathbf{q}_k}(\mathbf{p},\omega))\\ \\ \sum\limits_\lambda b_1^{(-)}(A_-^{\mathbf{q}_k}(\mathbf{p},\omega)+iB_-^{\mathbf{q}_k}(\mathbf{p},\omega))\end{cases},\qquad(13)$$

where

$$b_1^{(\pm)}=\frac{\Delta_0(\mathbf{p}+\mathbf{q}_k)}{8\pi^2}\frac{(M_1M_2)^{1/2}}{|\mathbf{v}_{\mathbf{p}+\mathbf{q}_k}|\cos\theta^\circ}\frac{1}{R_\pm^{1/2}},$$
$$R_\pm=\pm\left(1-\frac{M_2}{M_3}\mathrm{tg}^2\,\theta^\circ\right).$$

The upper and lower lines in the right-hand side of (13) pertain to the case when the $l_\mathbf{q}$ line is an ellipse and a hyperbola, respectively.

The expressions for $A_\pm^{\mathbf{q}_k}(\mathbf{p},\omega)$ and $B_\pm^{\mathbf{q}_k}(\mathbf{p},\omega)$ coincide with $A^{\mathbf{q}_k}(\mathbf{p},\omega)$ and $B^{\mathbf{q}_k}(\mathbf{p},\omega)$ if we replace $\omega_\lambda^\pm(\mathbf{q}_k,\mathbf{p})$ by

$$\omega_\lambda^\pm(\mathbf{q}_k,\mathbf{p})=\omega_\lambda(\mathbf{q}_k)+\xi_\pm(\mathbf{p}),\qquad\xi_\pm(\mathbf{p})=\mp\frac{(\varepsilon(\mathbf{p}+\mathbf{q}_k)-\varepsilon_F)^2}{2M_3\cos^2\theta^\circ|\mathbf{v}_{\mathbf{p}+\mathbf{q}_k}|^2R_\pm}.$$

Consequently, the functions $\delta\psi_{\mathbf{q}_k}(\mathbf{p},\omega)$ and $\delta Z'_{\mathbf{q}_k}(\mathbf{p},\omega)$ have at $\omega\to\omega_\lambda^\pm(\mathbf{q}_k,\mathbf{p})+\Delta_0(\mathbf{p}+\mathbf{q}_k)$ square-root singularities due to the fact the $l^\lambda_\mathbf{q}$ line experiences in the general case, near the directions $\mathbf{q}=\mathbf{q}_k$ and following the considered topological changes in $\omega_\lambda(\mathbf{q})=\omega$, only two types of topological changes: 1) the appearance or disappearance of an $l^\lambda_\mathbf{q}$ line (the $l^\lambda_\mathbf{q}$ line is an ellipse), and 2) the $l^\lambda_\mathbf{q}$ line has a self-intersection point (the $l^\lambda_\mathbf{q}$ line is a hyperbola). In this case the singularities differ from those due to tangency points in the form of the dependences of the functions $\omega_\lambda^\pm(\mathbf{q}_c,\mathbf{p})$ and $\omega_\lambda^\pm(\mathbf{q}_k,\mathbf{p})$ on the direction of the vector $\mathbf{p}$ (see Figs. 1 and 3).

Thus, from the calculations of Secs. 2 and 3 it follows that the singularities of the order parameter $\Delta'(\mathbf{p},\omega)$ of the superconductor in $\mathbf{p},\omega$ space are due to the change of the topology of the line of intersection of the Fermi surface with the equal-energy surfaces of the phonons.

## 4. SINGULARITIES IN $\Delta'(\omega)$

Using the expressions obtained in Secs. 2 and 3, we can obtain the singularities of $\Delta'(\omega)$. In fact, it follows

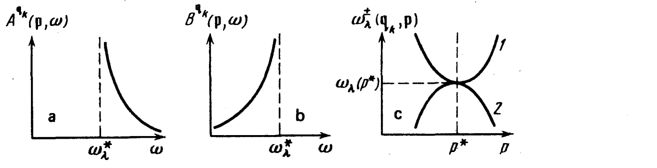


FIG. 3. a,b) Plots of the functions $A^{\mathbf{q}_k}(\mathbf{p},\omega)$ and $B^{\mathbf{q}_k}(\mathbf{p},\omega)$ against the frequency $\omega$ as $\omega\to\omega_\lambda^*$, where $\omega_\lambda^*=\omega_\lambda^\pm(\mathbf{q}_k,\mathbf{p})+\Delta_0(\mathbf{p}+\mathbf{q}_k)$; c) dependence of $\omega_\lambda^+(\mathbf{q}_k,\mathbf{p})$ on $\mathbf{p}$—curve 1—and dependence of $\omega_\lambda^-(\mathbf{q}_k,\mathbf{p})$ on $\mathbf{p}$—curve 2.

from (3) that

$$\mathrm{Im}\,\psi(\omega)=\mathrm{Im}\langle\psi(\mathbf{p},\omega)\rangle_{\mathbf{p}}=\mathrm{Im}\,\psi_0(\omega)+\sum_i \mathrm{Im}\,\delta\psi_{\mathbf{q}_i}(\omega),$$

$$\mathrm{Im}\,Z(\omega)=\mathrm{Im}\langle Z(\mathbf{p},\omega)\rangle_{\mathbf{p}}=\mathrm{Im}\,Z_0(\omega)+\sum_i \mathrm{Im}\,\delta Z_{\mathbf{q}_i}(\omega),$$

with the functions $\mathrm{Im}\psi_0(\omega)$ and $\mathrm{Im}Z_0(\omega)$ known.[2] We express the singular part of the function $\mathrm{Im}\delta\psi_{\mathbf{q}_i}(\omega)$ at $T=0$ in the form

$$\mathrm{Im}\,\psi'_{\mathbf{q}_i}(\omega)=\pi\int_0^\omega d\omega'\,\mathrm{Re}\left\{\frac{\Delta_0(\mathbf{p}^*+\mathbf{q}_i,\omega')}{[\omega'^2-\Delta_0^2(\mathbf{p}^*+\mathbf{q}_i,\omega')]^{1/2}}\right\}\frac{\partial g_{\mathbf{q}_i}(\omega-\omega')}{\partial\omega},$$

$$g_{\mathbf{q}_i}(\eta)\equiv\langle g_{\mathbf{q}_i}(\mathbf{p},\eta)\rangle_{\mathbf{p}}=\int_{\delta\Omega_{\mathbf{q}_i}} d\mathbf{q}\,\delta(\omega_\lambda(\mathbf{q}_i+\mathbf{q})-\eta)A(\mathbf{q}_i+\mathbf{q}),\quad(15)$$

$$A(\mathbf{q})=\frac{1}{(2\pi)^3}\int_\Omega d\mathbf{p}\,|g_{\mathbf{p},\mathbf{p}+\mathbf{q}}|^2\delta(\varepsilon(\mathbf{p})-\varepsilon_F)\delta(\varepsilon(\mathbf{p}+\mathbf{q})-\varepsilon_F).$$

1. We determine the singular part of the function $\delta\psi'_{\mathbf{q}_c}(\omega)=\omega\delta Z'_{\mathbf{q}_c}(\omega)$. We note that we have assumed that at a definite value $\mathbf{p}=\mathbf{p}^*$ the surfaces $\varepsilon(\mathbf{p}^*+\mathbf{q})=\varepsilon_F$ and $\omega_\lambda(\mathbf{q})=\omega^*$ are tangent at the point $\mathbf{q}=\mathbf{q}_c$. From the form of the function $g_{\mathbf{q}_c}(\eta)$ it follows that:

1) if the vector $\mathbf{p}=\mathbf{p}^*$ does not satisfy the conditions of the Kohn singularity for the surfaces $\varepsilon(\mathbf{p}^*)=\varepsilon_F$ and $\varepsilon(\mathbf{p}^*+\mathbf{q}_c)=\varepsilon_F$, then the function $A(\mathbf{q})$ can be regarded as constant in the vicinity of the point $\mathbf{q}_i=\mathbf{q}_c$;

2) if the vector $\mathbf{p}=\mathbf{p}^*$ satisfies the conditions of the Kohn singularity for the surfaces $\varepsilon(\mathbf{p}^*)=\varepsilon_F$ and $\varepsilon(\mathbf{p}^*+\mathbf{q}_c)=\varepsilon_F$, then the function $A(\mathbf{q}_c+\mathbf{q})$ has a singular part that depends on the character of the topological changes of the line of intersection of the surfaces $\varepsilon(\mathbf{p}^*)=\varepsilon_F$ and $\varepsilon(\mathbf{p}^*+\mathbf{q}_c+\mathbf{q})=\varepsilon_F$ (referred to henceforth as the $l_{\mathbf{p}}$ line for brevity).

Consequently, in the second case the function $\partial g_{\mathbf{q}_c}(\eta)/\partial\eta$ has a singular part $\partial\tilde{g}_{\mathbf{q}_c}(\eta)/\partial\eta$ due to the change of the topologies of the $l_{\mathbf{q}}^\lambda$ and $l_{\mathbf{p}}$ lines. We present a number of concrete calculations of the function $\partial\tilde{g}_{\mathbf{q}_c}(\eta)/\partial\eta$.

Assume for the sake of argument that the $l_{\mathbf{p}}$ line is described near the tangency point by an expression analogous to (7), in which

$$\delta_{\mathbf{p}}(\eta)\to\delta_{\mathbf{p}}^k(\eta)=(\varepsilon(\mathbf{p}^*+\mathbf{q})-\varepsilon_F)(\mathbf{n}\mathbf{n}_0)/|\mathbf{v}_{\mathbf{p}^*}|,$$

$$C_{\alpha\beta}\to C_{\alpha\beta}^k=\frac{1}{2}\left\{\frac{\mathbf{n}\mathbf{n}_0}{|v_{\mathbf{p}^*+\mathbf{q}}|}\frac{\partial^2\varepsilon(\mathbf{p}+\mathbf{q})}{\partial p_\alpha\partial p_\beta}-\frac{1}{|\mathbf{v}_{\mathbf{p}^*}|}\frac{\partial^2\varepsilon(\mathbf{p})}{\partial p_\alpha\partial p_\beta}\right\}_{\mathbf{p}=\mathbf{p}^*}$$

where $\mathbf{n}=\mathbf{v}_{\mathbf{p}^*+\mathbf{q}}/|v_{\mathbf{p}^*+\mathbf{q}}|$. We shall assume that $C_{\alpha\beta}^k x_\alpha x_\beta$ is positive-definite. Then the function $\partial\tilde{g}_{\mathbf{q}_c}(\eta)/\partial\eta$ takes the following form:

a) If the $l_{\mathbf{q}}$ line is described by Eq. (7), then

$$\frac{\partial\tilde{g}_{\mathbf{q}_c}(\eta)}{\partial\eta}=\sum_\lambda A_\lambda\begin{cases}-\pi^2\theta(\omega_c^*-\eta)\\+\pi\ln|\omega_c^*-\eta|\end{cases},\quad(16)$$

where

$$A_\lambda=\tilde{a}_\lambda a_\lambda^{(2)},$$

$$a_\lambda^{(2)}=|C_{xx}^kC_{yy}^k-(C_{xy}^k)^2|^{-1/2}/(2\pi)^3|\mathbf{S}_{\mathbf{q}_c}^\lambda(\omega_c^*)||\mathbf{v}_{\mathbf{p}^*}|.$$

Here $\tilde{a}_\lambda$ coincides with $a_\lambda$ of expression (7), in which we must assume that $\mathbf{n}=\mathbf{n}_0$ and $\mathbf{p}=\mathbf{p}^*$. The upper line of (16) pertains to the case when the form $\tilde{C}_{\alpha\beta}x_\alpha x_\beta$ is positive definite (the $l_{\mathbf{q}}^\lambda$ line is an ellipse); the lower line pertains to the case when the $l_{\mathbf{q}}^\lambda$ line is a hyperbola.

b) If the $l_{\mathbf{q}}^\lambda$ line is described by Eq. (10), then

$$\frac{\partial\tilde{g}_{\mathbf{q}_c}(\eta)}{\partial\eta}=-\pi^2\sum_\lambda C_\lambda[\pm(\eta-\omega_c^*)]^{-1/2}\theta(\pm(\eta-\omega_c^*)),$$

$$C_\lambda=\tilde{a}_\lambda^{(1)}a_\lambda^{(2)}.\quad(17)$$

Here $\tilde{a}_\lambda^{(1)}$ coincides with $a_\lambda^1$ of (10), in which $\mathbf{n}=\mathbf{n}_0$ and $\mathbf{p}=\mathbf{p}^*$. The plus and minus signs correspond respectively to the cases $c_5, c_9<0$ and $c_5, c_9>0$ [$c_5$ and $c_9$ are the coefficients in Eq. (9)]. We can determine analogously the form of the function $\partial\tilde{g}_{\mathbf{q}_c}(\eta)/\partial\eta$ and of other different forms of the $l_{\mathbf{q}}^\lambda$ and $l_{\mathbf{p}}$ lines. It follows from (16) and (17) that the form of the function $\partial\tilde{g}_{\mathbf{q}_c}(\eta)/\partial\eta$ depends on the character of the topological changes of the $l_{\mathbf{p}}$ and $l_{\mathbf{q}}^\lambda$ lines.

Knowing the function $\partial\tilde{g}_{\mathbf{q}_c}(\eta)/\partial\eta$, we can determine the singular parts of the functions $\delta\psi_{\mathbf{q}_c}'(\omega)$ and $\delta Z_{\mathbf{q}_c}'(\omega)$. We introduce the variable

$$x_{\mathbf{q}_c}=\omega-\omega_c^*-\Delta_0(\mathbf{p}+\mathbf{q}_c).$$

a) If $\partial\tilde{g}_{\mathbf{q}_c}(\eta)/\partial\eta$ is determined by formulas (16), then

$$\delta\psi'_{\mathbf{q}_c}(\omega)=\pi\sum_\lambda A_\lambda\begin{cases}A^k(\omega)+iB^k(\omega), & l_{\mathbf{q}}^\lambda\text{-line is an ellipse}\\B^k(\omega)+iA^k(\omega), & l_{\mathbf{q}}^\lambda\text{-line is a hyperbola}\end{cases}$$

$$A^k(\omega)=-(2\Delta_0(\mathbf{p}^*+\mathbf{q}_c))^{1/2}\pi^2(-x_{\mathbf{q}_c})^{1/2}\theta(-x_{\mathbf{q}_c}),$$
$$B^k(\omega)=(2\Delta_0(\mathbf{p}^*+\mathbf{q}_c))^{1/2}\pi^2(x_{\mathbf{q}_c})^{1/2}\theta(x_{\mathbf{q}_c}).\quad(18)$$

b) If $\partial\tilde{g}_{\mathbf{q}_c}(\eta)/\partial\eta$ is determined by formula (17), then

$$\delta\psi'_{\mathbf{q}_c}(\omega)=\pi\sum_\lambda C_\lambda\begin{cases}A_1^k(\omega)+iB_1^k(\omega), & \text{sign } + \text{ in (17)}\\B_1^k(\omega)+iA_1^k(\omega), & \text{sign } - \text{ in (17)}\end{cases},$$

$$B_1^k(\omega)=-(\Delta_0(\mathbf{p}^*+\mathbf{q}_c)/2)^{1/2}\pi^3\theta(x_{\mathbf{q}_c}),$$
$$A_1^k(\omega)=(\Delta_0(\mathbf{p}^*+\mathbf{q}_c)/2)^{1/2}\pi^2\ln|x_{\mathbf{q}_c}|.\quad(19)$$

It is seen from (18) and (19) that the analytic forms of $\delta\psi_{\mathbf{q}_c}'(\omega)$ and $\delta Z_{\mathbf{q}_c}'(\omega)$ are very sensitive to the changes in the topologies of both the $l_{\mathbf{p}}$ line and $l_{\mathbf{q}}^\lambda$ line.

2. We proceed now to consider the expression for the singular part of the function $\delta\psi'_{\mathbf{q}_k}(\omega)$ at $T=0$.

The analytic form of the function $\partial\tilde{g}_{\mathbf{q}_k}(\eta)/\partial\eta$ depends on the character of the topological changes of the $l_{\mathbf{p}}$ and $l_{\mathbf{q}}^\lambda$ lines. By way of example, we consider certain possible cases.

a) Let the surfaces $\varepsilon(\mathbf{p})=\varepsilon_F$ and $\varepsilon(\mathbf{p}+\mathbf{q}_k)=\varepsilon_F$ intersect in a certain interval of the values of the vector $\mathbf{p}$. Then

$$\frac{\partial\tilde{g}_{\mathbf{q}_k}(\eta)}{\partial\eta}=\sum_\lambda N^\lambda\begin{cases}\pm(\omega_\lambda(\mathbf{q}_k)-\eta)^{-1/2}\theta(\omega_\lambda(\mathbf{q}_k)-\eta)\\\pm(\eta-\omega_\lambda(\mathbf{q}_k))^{-1/2}\theta(\eta-\omega_\lambda(\mathbf{q}_k))\end{cases},\quad(20)$$

$$N^\lambda=\frac{I_0(2M_1M_2M_3)^{1/2}}{4\pi^2},\quad I_0=\int dl_{\mathbf{p}}\frac{|g_{\mathbf{p},\mathbf{p}+\mathbf{q}_k}^\lambda|^2}{|\mathbf{v}_{\mathbf{p}}||\mathbf{v}_{\mathbf{p}+\mathbf{q}_k}|\sin\alpha_{\mathbf{q}_k}},$$

where $\alpha_{\mathbf{q}_k}$ is the angle between the vectors $\mathbf{v}_{\mathbf{p}}$ and $\mathbf{v}_{\mathbf{p}+\mathbf{q}_k}$.

The upper line in the right-hand side of (20) pertains to the case when an increase of $\omega$ is accompanied by a transition from a closed surface $\omega_\lambda(\mathbf{q})=\omega$ to an open one (with a minus sign), or to the vanishing of one of the parts of the surface $\omega_\lambda(\mathbf{q})=\omega$ (with a minus sign), while the lower part pertains to the cases of the opposite transitions. Consequently, in this case the singularities in $\partial\tilde{g}_{\mathbf{q}_k}(\eta)/\partial\eta$ coincide with the singularities in the derivative of the irregular increment to the density of the number of phonon states with respect to energy with changing topology of the surface $\omega_\lambda(\mathbf{q})=\omega$.

b) The surfaces $\varepsilon(\mathbf{p})=\varepsilon_F$ and $\varepsilon(\mathbf{p}+\mathbf{q}_k)=\varepsilon_F$ have only a tangency point at $\mathbf{p}=\mathbf{p}^*$. We consider several tangency cases. If the $l_\mathbf{p}$ line near the point $\mathbf{p}=\mathbf{p}^*$ is an ellipse, then the function $\partial\tilde{g}_{\mathbf{q}_k}(\eta)/\partial\eta$ is determined by formulas (20), in which it is necessary to replace $N^\lambda$ by $N_1^\lambda$. To obtain an expression for $N_1^\lambda$, it is necessary to replace the integral with respect to $dl_\mathbf{p}$ in the expression for $N^\lambda$ by the following:

$$I_0=\frac{\pi}{2}\frac{|\tilde{g}^\lambda_{\mathbf{p}^*,\mathbf{p}^*+\mathbf{q}_k}|^2}{|v_{\mathbf{p}^*}||v_{\mathbf{p}^*+\mathbf{q}_k}|}|\tilde{C}^k_{xx}\tilde{C}^k_{yy}-(\tilde{C}^k_{xy})^2|^{-1/2}, \tag{21}$$

where the coefficients $\tilde{C}^k_{\alpha\beta}$ coincide with the coefficients $C^k_{\alpha\beta}$ of expression (16), in which we must put $\mathbf{q}=\mathbf{q}_k$ and $\mathbf{n}=\mathbf{n}_0$.

If the $l_\mathbf{p}$ line near the point $\mathbf{p}=\mathbf{p}^*$ is a hyperbola, then the calculations lead to the following result. When the $l_\mathbf{q}^\lambda$ line is an ellipse, then

$$\frac{\partial\tilde{g}_{\mathbf{q}_k}(\eta)}{\partial\eta}=\sum_\lambda N_1^\lambda\begin{cases}\pm\dfrac{\theta(\omega_\lambda(\mathbf{q}_k)-\eta)}{(\omega_\lambda(\mathbf{q}_k)-\eta)^{1/2}}\ln\dfrac{(2\sigma)^4}{\omega_\lambda(\mathbf{q}_k)-\eta}\\ \pm\dfrac{\theta(\eta-\omega_\lambda(\mathbf{q}_k))}{(\eta-\omega_\lambda(\mathbf{q}_k))^{1/2}}\ln\dfrac{(2\sigma)^4}{\eta-\omega_\lambda(\mathbf{q}_k)}\end{cases}. \tag{22}$$

The plus and minus signs have the same meaning as in (20); $\sigma$ is a parameter that defines a certain region where the hyperbolic form for the $l_\mathbf{p}$ line is a good approximation.

In the case when the $l_\mathbf{q}^\lambda$ line is a hyperbola we have

$$\frac{\partial\tilde{g}_{\mathbf{q}_k}(\eta)}{\partial\eta}=\sum_\lambda N_1^\lambda\begin{cases}\pi\dfrac{\theta(\pm[\omega_\lambda(\mathbf{q}_k)-\eta])}{[\pm(\omega_\lambda(\mathbf{q}_k-\eta))]^{1/2}}\\ \dfrac{\theta(\pm[\eta-\omega_\lambda(\mathbf{q}_k)])}{[\pm(\eta-\omega_\lambda(\mathbf{q}_k))]^{1/2}}\ln\dfrac{4\sigma^4}{\pm[\eta-\omega_\lambda(\mathbf{q}_k)]}\end{cases}. \tag{23}$$

The plus sign in (23) corresponds to the case when the closed surface $\omega_\lambda(\mathbf{q})=\omega$ becomes open with increasing $\omega$, and the minus sign corresponds to the opposite topological transition.

We note that the change of the topology of the $l_\mathbf{p}$ line enhances the contribution made to the function $\partial\tilde{g}_{\mathbf{q}_k}(\eta)/\partial\eta$ by the topological singularities of the density of the number of phonon states. In these cases $\omega_\lambda(\mathbf{q})=\omega$ can undergo topological changes that lead weaker singularities in the density of the number of phonon states. If the surfaces $\varepsilon(\mathbf{p})=\varepsilon_F$ and $\varepsilon(\mathbf{p}+\mathbf{q}_k)=\varepsilon_F$ intersect in some interval of the values of $\mathbf{p}$ and have also a tangency point at $\mathbf{p}=\mathbf{p}^*$, then the function $\partial\tilde{g}_{\mathbf{q}_k}(\eta)/\partial\eta$ consists of two terms, the first of which is given by (21) and the second by (22) or (23), depending on the type of topological changes of the $l_\mathbf{q}^\lambda$ and $l_\mathbf{p}$ lines. Knowing the function $\partial\tilde{g}_{\mathbf{q}_k}(\eta)/\partial\eta$, we can determine $\delta\psi'_{\mathbf{q}_k}(\omega)=\omega\delta Z'_{\mathbf{q}_k}(\omega)$. We substitute for this purpose (20) in (15). Putting $x_{\mathbf{q}_k}=\omega-\omega_\lambda(\mathbf{q}_k)-\Delta_0(\mathbf{p}+\mathbf{q}_k)$, we obtain

$$\begin{gathered}\delta\psi'_{\mathbf{q}_k}(\omega)=\sum_\lambda N^\lambda\begin{cases}\pm(A^{\mathbf{q}_k}(\omega)+iB^{\mathbf{q}_k}(\omega))\\ \pm(B^{\mathbf{q}_k}(\omega)+iA^{\mathbf{q}_k}(\omega))\end{cases}\\ A^{\mathbf{q}_k}(\omega)=\pi(\Delta_0(\mathbf{p}+\mathbf{q}_k)/2)^{1/2}\theta(x_{\mathbf{q}_k}),\\ B^{\mathbf{q}_k}(\omega)=-(\Delta_0(\mathbf{p}+\mathbf{q}_k)/2)^{1/2}\ln|x_{\mathbf{q}_k}|.\end{gathered} \tag{24}$$

It is seen from (24) that, depending on the type of the topological transition in the surface $\omega_\lambda(\mathbf{q})=\omega$, the real and imaginary parts of the functions $\delta\psi'_{\mathbf{q}_k}(\omega)$ and $\delta Z'_{\mathbf{q}_k}(\omega)$ have either a logarithmic singularity at $\omega=\omega_\lambda(\mathbf{q}_k)+\Delta_0(\mathbf{p}^*+\mathbf{q}_k)$, or a discontinuity.

It follows from (22) and (15) that

$$\begin{gathered}\delta\psi'_{\mathbf{q}_k}(\omega)=\pi\sum_\lambda N_1^\lambda\begin{cases}\pm(A_1^{\mathbf{q}_k}(\omega)+iB_1^{\mathbf{q}_k}(\omega))\\ \pm(B_1^{\mathbf{q}_k}(\omega)+iA_1^{\mathbf{q}_k}(\omega))\end{cases}\\ A_1^{\mathbf{q}_k}(\omega)=\left(\frac{\Delta_0(\mathbf{p}^*+\mathbf{q}_k)}{2}\right)^{1/2}\pi\ln\frac{(2\sigma)^4}{x_{\mathbf{q}_k}}\theta(x_{\mathbf{q}_k}),\\ B_1^{\mathbf{q}_k}(\omega)=\left(\frac{\Delta_0(\mathbf{p}^*+\mathbf{q}_k)}{2}\right)^{1/2}\frac{1}{2}\ln|x_{\mathbf{q}_k}|\ln\frac{|x_{\mathbf{q}_k}|}{(2\sigma)^8}.\end{gathered} \tag{25}$$

Using the results (24) and (25), we can easily obtain the form of or $\delta Z_{\mathbf{q}_k}'(\omega)$ for the case of the topological changes of the $l_\mathbf{q}$ and $l_\mathbf{p}$ lines corresponding to (23).

It is seen from the results of the present section that the singularities of $\Delta'(\omega)$ are due to the change in the topology of the $l_\mathbf{q}^\lambda$ and $l_\mathbf{p}$ lines of intersection of three surfaces: $\varepsilon(\mathbf{p})=\varepsilon_F$, $\varepsilon(\mathbf{p}+\mathbf{q})=\varepsilon_F$, and $\omega_\lambda(\mathbf{q})=\omega$ in $\mathbf{p},\mathbf{q}$ space. The results obtained by Scalapino and Anderson[2] can be attributed to a particular case that is realized when, with change in the parameter $\omega$, a change takes place in the topology of the surface $\omega_\lambda(\mathbf{q})=\omega$, while the surfaces $\varepsilon(\mathbf{p})=\varepsilon_F$ and $\varepsilon(\mathbf{p}+\mathbf{q})=\varepsilon_F$ intersect or have an elliptic tangency point.

## 5. DISCUSSION OF RESULTS

1. It is known[1] that the main contribution to the tunnel current in a system consisting of two metals interlined by an insulator is made by electrons that move practically perpendicular to the separation boundary (barrier). Therefore the singularities in $\Delta'(\mathbf{p},\omega)$, considered in Secs. 2 and 3, can apparently be experimentally observed in investigations of the tunnel effects in single crystals. The tunnel effect in single crystals has been investigated relatively little. Principal attention was paid to the anisotropy of $\Delta(\mathbf{p},0)$, which is determined at $\omega=eV=\Delta(\mathbf{p},0)$.[7] The existing experimental technique makes it possible to investigate tunnel effects in a single crystal–insulator–polycrystal (film) system. Observation of the singularities of $\Delta'(\mathbf{p},\omega)$ would uncover new possibilities of investigating singularities of both the phonon and electron spectra of superconductors.

2. In the case of S–N tunnel junctions between polycrystals, the first derivative of the tunnel current ($j_T$) with respect to the voltage ($V$) is proportional to the tunnel density $N_T(\omega)/N(0)$, which takes at $\omega\gg\Delta(\omega)$ the form[2]

$$\frac{\partial j_T}{\partial V}\sim\frac{N_T(\omega)}{N(0)}=1+\frac{1}{2\omega^2}\operatorname{Re}\Delta^2(\omega). \tag{26}$$

Taking (14) into account, we can rewrite (26) in the form

$$\frac{N_T(\omega)}{N(0)}=1+\frac{\operatorname{Re}\Delta_0^2(\omega)}{2\omega^2}+\frac{1}{\omega^2}\operatorname{Re}\left\{\frac{\Delta_0(\omega)}{Z_0(\omega)}[\delta\psi_{\mathbf{q}_c}(\omega)+\delta\psi_{\mathbf{q}_k}(\omega)]\right\}. \tag{27}$$

Consequently, singularities of two types can be observed in the dependence of $\partial j_T/\partial V$ on $V$. The first type, due to $\delta\psi_{\mathbf{q}_c}(\omega)$, manifests itself primarily in $\partial^3 j_T/\partial V^3$, and in some cases, depending on the character of the topological changes in the $l_\mathbf{p}$ and $l_\mathbf{q}$ lines, also in $\partial^2 j_T/\partial V^2$. The second type of singularity, due to $\delta\psi_{\mathbf{q}_k}(\omega)$, will be observed in $\partial^2 j_T/\partial V^2$.

The geometric interpretation of the singularities of $\Delta'(\omega)$ is the following.

First type of singularities of $\Delta'(\omega)$. There exists a set of vectors $\mathbf{q}=\mathbf{q}_c$ at which the surfaces $\varepsilon(\mathbf{p})=\varepsilon_F$ and $\varepsilon(\mathbf{p}+\mathbf{q})=\varepsilon_F$ are tangent. Knowing the vectors $\mathbf{q}_c$, we can construct the surface $\varepsilon(\mathbf{q}_c)=\varepsilon_F$, which constitutes a surface of Kohn singularities. The singularities in $\Delta'(\omega)$ can arise at a definite value of the parameter $\omega=\omega_c^*+\Delta_0(\mathbf{p}^*+\mathbf{q}_c)$, if the surfaces $\varepsilon(\mathbf{q}_c)=\varepsilon_F$ and $\omega_\lambda(\mathbf{q})=\omega^*$ are tangent at the point $\mathbf{q}=\mathbf{q}_c$.

Second type of singularities of $\Delta'(\omega)$. It is brought about by the fact that a change of $\omega$ is accompanied by a change of the topology of the surface $\omega_\lambda(\mathbf{q})=\omega$, in which case the surfaces $\varepsilon(\mathbf{p})=\varepsilon_F$ and $\varepsilon(\mathbf{p}+\mathbf{q}_k)=\varepsilon_F$ should either intersect or by tangent. These singularities are observed at $\omega=\Delta(\mathbf{p}^*+\mathbf{q}_k)+\omega_\lambda(\mathbf{q}_k)$.

We note that in some cases the analytic forms of these types of singularity can coincide in some cases [see expressions (19) and (24)]. Therefore for an unambiguous interpretation of the observed singularities with respect to the tunnel characteristics of superconductors it is necessary to use neutron investigations of the phonon spectrum. In the general case, the numbers of the singularities in $\Delta'(\omega)$ and in the phonon state density are not equal.

We wish to point out that the tunnel effect in superconductors is presently used only to determine the function of the electron-phonon interaction $g_0(\omega)$, which is obtained from the dependence of $\partial j_T/\partial V$ on $V$. The quantity $\partial^2 j_T/\partial V^2$ is resorted to for a qualitative assessment of the singularities in $\Delta'(\omega)$. It seems that modern computation techniques make it possible to use the dependence of $\partial^2 j_T/\partial V^2$ on $V$ to determine $\mathrm{Re}\Delta'(\omega)$ and $\mathrm{Im}\Delta'(\omega)$, and consequently to determine the nature of the singularities in $\Delta'(\omega)$. Such experiments would be the next step in the study of electron-phonon interactions.

3. We note in conclusion that the recently developed microjunction spectroscopy makes it possible to determine the "transport" electron-phonon interaction function[8,9] $G(\omega)$, which we represent in the form

$$\begin{gathered}
G(\omega)=\sum_\lambda \int_\Omega d\mathbf{q}\,\delta(\omega_\lambda(\mathbf{q})-\omega)A^*(\mathbf{q}),\\
A^*(\mathbf{q})=\int_\Omega d\mathbf{p}\,\delta(\varepsilon(\mathbf{p})-\varepsilon_F)\delta(\varepsilon(\mathbf{p}+\mathbf{q})-\varepsilon_F)|\tilde{g}^\lambda_{\mathbf{p},\mathbf{p}+\mathbf{q}}|^2K(\mathbf{v}_\mathbf{p},\mathbf{v}_{\mathbf{p}+\mathbf{q}}),\\
K(\mathbf{v}_\mathbf{p},\mathbf{v}_{\mathbf{p}+\mathbf{q}})=\frac{|\mathbf{v}_\mathbf{p}^z||\mathbf{v}_{\mathbf{p}+\mathbf{q}}^z|}{|\mathbf{v}_\mathbf{p}^z\mathbf{v}_\mathbf{p}-\mathbf{v}^z_{\mathbf{p}+\mathbf{q}}\mathbf{v}_{\mathbf{p}+\mathbf{q}}|}\theta(-\mathbf{v}_\mathbf{p}^z\mathbf{v}_{\mathbf{p}+\mathbf{q}}^z).
\end{gathered} \tag{28}$$

We investigate now the singularities of $G(\omega)$. We represent for this purpose this function in the form

$$\begin{gathered}
G(\omega)=G_0(\omega)+\sum_i G_{\mathbf{q}_i}(\omega),\\
G_0(\omega)=\sum_\lambda \int_{\Omega_0} d\mathbf{q}\,\delta(\omega_\lambda(\mathbf{q})-\omega)A^*(\mathbf{q}),\\
G_{\mathbf{q}_i}(\omega)=\sum_\lambda \int_{\delta\Omega_{\mathbf{q}_i}} d\mathbf{q}\,\delta(\omega_\lambda(\mathbf{q}_i+\mathbf{q})-\omega)\,A^*(\mathbf{q}_i+\mathbf{q}),\\
\Omega=\Omega_0+\sum_i \delta\Omega_{\mathbf{q}_i}.
\end{gathered} \tag{29}$$

Here $G_0(\omega)$ is the smooth part of the function $G(\omega)$, while $G_{\mathbf{q}_i}(\omega)$ are the singular parts due to the change of the topology of the $l^\lambda_\mathbf{q}$ and $l_\mathbf{p}$ lines near the points $\mathbf{q}_i=\mathbf{q}_c$ and $\mathbf{q}_i=\mathbf{q}_k$.

If the function $K(\mathbf{v}_\mathbf{p},\mathbf{v}_{\mathbf{p}+\mathbf{q}})$ at the points $\mathbf{q}=\mathbf{q}_c$ and $\mathbf{q}=\mathbf{q}_k$ do not vanish and have no singularities, then the analytic form of the singularities of $\delta G'_\mathbf{q}(\omega)$ coincides with $\partial g_\mathbf{q}$ $\partial\tilde{g}_{\mathbf{q}_i}(\omega)/\partial\omega$. According to Kulik *et al.*,[9] the function $G(\omega)$ can be obtained from the second derivative of the current $\mathbf{j}_j$ with respect to the voltage applied to the microjunction. It is seen from (16), (17), and (20)–(23) that the singularities appear in the quantity $G'(\omega)$, which determines the third derivative of the current with respect to voltage.

It is of interest to investigate jointly the current-voltage characteristics of tunnel junctions of superconductors ($\partial^2\mathbf{j}_T/\partial V^2$) and of microjunctions ($\partial^3\mathbf{j}_j/\partial V^3$), which can yield some information on the function $K(\mathbf{v}_\mathbf{p},\mathbf{v}_{\mathbf{p}+\mathbf{q}})$. Thus, for example, if the total number of singularities in $\partial^2\mathbf{j}_T/\partial V^2$ is larger than in $\partial^3\mathbf{j}_j/\partial V^3$, then $K(\mathbf{v}_\mathbf{p},\mathbf{v}_{\mathbf{p}+\mathbf{q}})$ vanishes at some points of $p$ space. If at a definite value of $V$ a stronger singularity appears in $\partial^3\mathbf{j}_j/\partial V^3$ than in $\partial^2\mathbf{j}_T/\partial V^2$, then this can serve as an indication that $K(\mathbf{v}_\mathbf{p},\mathbf{v}_{\mathbf{p}+\mathbf{q}})$ has a singularity at some point of p space.

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