Two-dimensional isotopic model of a fermion field with broken $SU(2)$ symmetry

V. N. Dutyshev

Gor'kiĭ Physicotechnical Research Institute, Gor'kiĭ State University

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In two-dimensional space-time a fermion isotopic model of field theory with zero rest mass is investigated in the case of broken isotopic symmetry. It is shown that the quantization can be carried out exactly for arbitrary ratios of the constants which determine the isotopic interaction. For the case of $U(1)$ symmetry the eigenvalues and eigenfunctions of the energy operator in the space of pseudoparticles are found.

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1. INTRODUCTION

A remarkable property of many nontrivial physical models in two-dimensional space-time is that they are completely integrable. There is a broad class of such models, including among others well-known ones described by the nonlinear Schrödinger equation and by the sine-Gordon equation. The classical solution of these equations is obtained by means of the method of the inverse scattering problem.[1-7] There has recently been much development of methods for quantizing such models.[8-12] Belavin[12] has considered a massless $SU(2)$ symmetric model of a Fermi field. He showed that the additional γ_5 invariance of this model leads to a factorization of the scattering S matrix and makes it possible to determine all the eigenstates and the spectrum of the Hamiltonian in the language of pseudoparticles. An analysis of Belavin's equations shows that because of the filling of the pseudo-particle states the interaction in the system results in giving the physical particles a mass determined by the isotropic interaction constant g and the cutoff parameter Λ.

The present paper proposes the solution of a more general problem, namely the case in which the $SU(2)$ symmetry is broken. The isotopic interaction is determined by three different constants $g_a (A=1, 2, 3)$. The interaction Lagrangian is

$$\mathscr{L}_I=-{}^1/{}_2 g_0(\bar\psi\gamma_\mu\psi)^2-{}^1/{}_2 g_a(\bar\psi\gamma_\mu\tau^a\psi)^2. \tag{1}$$

Here $\psi_i^\alpha(x)$ is an isodoublet of Fermi fields (i is the spinor index taking the values ± 1, and α is the isotopic index), γ_μ are the Dirac matrices $[\gamma_0=\sigma^x, \gamma_1=-i\sigma^y, \gamma_5=\sigma^z)$, and τ^a are the isotopic matrices for the group $SU(2)]$.

In this paper it will be shown that for arbitrary constants g_a there is an infinite set of integrals of the motion; their existence indicates the presence of a hidden symmetry of the system. The existence of this symmetry enables us to find the exact quantum solution. The case of $U(1)$ symmetry, i.e., with $g_1=g_2 \neq g_3$, is treated analogously. For this case all the eigenstates and the spectrum of the Hamiltonian in the language of pseudoparticles are found.

2. THE EQUATIONS FOR THE EIGENFUNCTIONS OF THE HAMILTONIAN

The original Lagrangian is

$$\mathscr{L}=i\bar\psi\gamma_\mu\partial_\mu\psi-m_0\bar\psi\psi+\mathscr{L}_I. \tag{2}$$

In constructing the eigenfunctions of the Hamiltonian operator we let the rest mass go to zero; that is, we actually consider the massless case. We subject the operators ψ to the commutation relations

$$\{\psi_i{}^\alpha(x),\ \psi_j{}^{+\beta}(y)\}=\delta_{\alpha\beta}\delta_{ij}\delta(x-y), \tag{3}$$

and then the Hamiltonian of the system in the second-quantization representation can be written

$$\hat H=\int dx\{-i\psi_{i_1}{}^{+\alpha}\overset{z}{\sigma}_{i_1i_2}\partial_1\psi_{i_2}{}^\alpha+m_0\psi_{i_1}{}^{+\alpha}\overset{x}{\sigma}_{i_1i_2}\psi_{i_2}{}^\alpha$$
$$+{}^1/{}_2\,\psi_{i_1}^{+\alpha_1}\psi_{j_1}^{+\beta_1}(g_0\delta_{\alpha_1\alpha_2}\delta_{\beta_1\beta_2}+g_a\overset{a}{\tau}_{\alpha_1\alpha_2}\overset{a}{\tau}_{\beta_1\beta_2})(\delta_{i_1i_2}\delta_{j_1j_2}-\overset{z}{\sigma}_{i_1i_2}\overset{z}{\sigma}_{j_1j_2})\psi_{i_2}{}^{\alpha_2}\psi_{j_2}{}^{\beta_2}\}. \tag{4}$$

We denote with $|0\rangle$ the pseudovacuum eigenstate of the Hamiltonian, which satisfies the conditions

$$\psi_i{}^\alpha(x)|0\rangle=0,\quad \hat H|0\rangle=0. \tag{5}$$

Because the total number of particles $N=\int\psi^+\psi dx$ is conserved, we can look for an arbitrary eigenstate of the Hamiltonian in the form

$$|N\rangle=\int dx_1\ldots dx_N\,\Phi_{\alpha_1\ldots\alpha_N}^{i_1\ldots i_N}(x_1\ldots x_N)\psi_{i_1}^{+\alpha_1}(x_1)\ldots\psi_{i_N}^{+\alpha_N}(x_N)|0\rangle. \tag{6}$$

The functions Φ satisfy the condition

$$\hat H_N\Phi^{i_1\ldots i_N}_{\alpha_1\ldots\alpha_N}(x_1\ldots x_N)=E_N\Phi^{i_1\ldots i_N}_{\alpha_1\ldots\alpha_N}(x_1\ldots x_N) \tag{7}$$

with the Hamiltonian $\hat H_N$, which acts in the N-particle sector of Fock space and is given by

$$\hat H_N=\sum_{n=1}^N\left(-i\sigma_n{}^z\frac{\partial}{\partial x_n}+m_0\sigma_n{}^x\right)+\sum_{n<m}(g_0+g_a\tau_n{}^a\tau_m{}^a)(I-\sigma_n{}^z\sigma_m{}^z)\delta(x_n-x_m). \tag{8}$$

The indices n and m on the matrices σ and τ indicate the number of a particle.

Since the operators $\psi_i^{+\alpha}(x)$ and $\psi_j^{+\beta}(x)$ anticommute, the functions Φ are antisymmetric:

$$\Phi^{i_{q_1}\cdots i_{q_N}}_{\alpha_{q_1}\cdots\alpha_{q_N}}(x_{q_1}\ldots,x_{q_N})=(-1)^{\eta_Q}\Phi^{i_1\ldots i_N}_{\alpha_1\ldots\alpha_N}(x_1\ldots x_N), \tag{9}$$

where η_Q is the parity of the permutation $Q=(q_1,\ldots, q_N)$ of the numbers from 1 to N.

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Let us restrict our system to a length L and assume that the functions Φ satisfy periodic boundary conditions in each argument

$$\Phi(\ldots,x_n+L,\ldots)=\Phi(\ldots,x_n,\ldots). \tag{10}$$

Accordingly, the functions Φ that define eigenstates (6) of the Hamiltonian operator (4) must satisfy the conditions (7), (9), and (10).

3. CONSTRUCTION OF EIGENFUNCTIONS OF THE HAMILTONIAN

In the one-particle sector ($N=1$) we have the free Dirac equation

$$(-i\sigma^z\partial/\partial x+m_0\sigma^x)\Phi_\alpha{}^i(x)=E_1\Phi_\alpha{}^i(x), \tag{11}$$

with its solution in the form

$$\Phi_\alpha(x)=A_\alpha u(\theta)e^{ikx},\quad E_1=m_0\,\mathrm{ch}\,\theta,\quad k=m_0\,\mathrm{sh}\,\theta, \tag{12}$$

where k is the wave vector, θ is the rapidity of the particle, and $u(\theta)$ is a Dirac spinor of the form

$$u(\theta)=(2\,\mathrm{ch}\,\theta)^{-1/2}\begin{pmatrix}e^{\theta/2}\\ e^{-\theta/2}\end{pmatrix}. \tag{13}$$

We note that when θ in Eq. (12) is replaced with $i\pi-\theta$, the solution changes from one with positive energy into one with negative energy.

To look for a wave function Φ in a sector with $N\geqslant 2$, we break up the region of the variables $x_1, x_2, \ldots, x_N$ into $N!$ regions. We denote by $P=(p_1,p_2,\ldots,p_N)$ the region which satisfies the conditions $x_{p_1}<x_{p_2}<\ldots<x_{p_N}$, and by Φ^P the wave function in this region. Inside each of the regions P Eq. (8) does not contain any δ functions and is a Dirac equation for N free particles.

For $N=2$ the wave functions $\Phi^{(12)}$ and $\Phi^{(21)}$ are defined by the same set of wave vectors k_1 and k_2, since the total energy $E_2=m_0\cosh\theta_1+m_0\cosh\theta_2$ and the total momentum $K_2=k_1+k_2=m_0\sinh\theta_1+m_\theta\sinh\theta_2$ are conserved. Following Bethe's hypothesis[13] that for $N>2$ the set of momenta is the same in all the regions, and remembering the antisymmetry condition (9), we write the wave function in an arbitrary region P in the form

$$\Phi^P_{\alpha_1\ldots\alpha_N}(x_1\ldots x_N)=\sum_Q(-1)^{\nu_Q}A^{PQ}_{\alpha_{p_1}\ldots\alpha_{p_N}}\prod_{n=1}^N u_n(\theta_{q_n})\exp(ik_{q_n}x_n), \tag{14}$$

where $k_1,\ldots,k_N$ is a fixed set of momenta, $\theta_1,\ldots,\theta_N$ is the set of rapidities of the particles, PQ is the product of the permutations, and $u_n(\theta)$ includes the spinor indices of the n-th particle. For example, for $N=2$ we have

$$\begin{aligned}\Phi^{(12)}_{\alpha_1\alpha_2}(x_1,x_2)&=A^{12}_{\alpha_1\alpha_2}u_1(\theta_1)u_2(\theta_2)\exp(ik_1x_1+ik_2x_2)\\&\quad-A^{21}_{\alpha_1\alpha_2}u_1(\theta_2)u_2(\theta_1)\exp(ik_2x_1+ik_1x_2),\\ \Phi^{(21)}_{\alpha_1\alpha_2}(x_1,x_2)&=A^{21}_{\alpha_2\alpha_1}u_1(\theta_1)u_2(\theta_2)\exp(ik_1x_1+ik_2x_2)\\&\quad-A^{12}_{\alpha_2\alpha_1}u_1(\theta_2)u_2(\theta_1)\exp(ik_2x_1+ik_1x_2).\end{aligned} \tag{15}$$

Clearly, the hypothesis about the form of N-particle wave functions shown in Eq. (14) requires verification for $N>2$. It will be shown subsequently that for the rest mass $m_0=0$ it is valid.

The functions Φ^P must be matched at the boundaries of the regions P. The matching conditions can be obtained from Eq. (7). For $N=2$ we introduce linear combinations of wave functions defined by the relations

$$\begin{aligned}&G_0=\Phi_{12}-\Phi_{21},\quad G_1=\Phi_{11}+\Phi_{22},\\&G_2=\Phi_{11}-\Phi_{22},\quad G_3=\Phi_{12}+\Phi_{21}.\end{aligned} \tag{16}$$

For the spinor components G_μ^{+-} Eq. (7) takes the form

$$-i\partial_1G_\mu{}^{+-}+i\partial_2G_\mu{}^{+-}+m_0(G_\mu{}^{++}+G_\mu{}^{--})+2f_\mu G_\mu{}^{+-}\delta(x_1-x_2)=E_2G_\mu{}^{+-}, \tag{17}$$

with $\mu=0,1,2,3$. The constants f_μ are given by the equations

$$\begin{aligned}&f_0=\frac{g_0-g_3-g_1-g_2}{2},\quad f_1=\frac{g_0+g_3+g_1-g_2}{2},\\&f_2=\frac{g_0+g_3-g_1+g_2}{2},\quad f_3=\frac{g_0-g_3+g_1+g_2}{2}.\end{aligned} \tag{18}$$

The matching conditions at the boundary $x_1=x_2$ that follow from Eq. (17) are

$$G_\mu{}^{+-}(x_1<x_2)\,|_{x_1=x_2}=\exp(2if_\mu)\,G_\mu{}^{+-}(x_2<x_1)\,|_{x_1=x_2}. \tag{19}$$

Replacing i with $-i$ gives the matching formula for the components G_μ^{-+}. The remaining components of the function $G_\mu^{i_1i_2}$ are continuous, since the equations for them do not contain any δ functions.

From the matching conditions (10) we can obtain the connection between the coefficients A^{12} and A^{21} of the wave function. We write

$$\begin{aligned}&B_0{}^P=A_{12}{}^P-A_{21}{}^P,\quad B_1{}^P=A_{11}{}^P+A_{22}{}^P,\\&B_2{}^P=A_{11}{}^P-A_{22}{}^P,\quad B_3{}^P=A_{12}{}^P+A_{21}{}^P;\end{aligned} \tag{20}$$

then, using Eqs. (15), (16), and (19), we get

$$B_\mu{}^{12}=\gamma_\mu(\theta_{12})B_\mu{}^{21}, \tag{21}$$

where $\theta_{12}=\frac{1}{2}(\theta_1-\theta_2)$, and the quantities $\gamma_\mu(\theta)$ are defined as follows:

$$\begin{aligned}&\gamma_a(\theta)=\frac{\mathrm{cth}\,\theta+i\lambda_a}{\mathrm{cth}\,\theta-i\lambda_a},\quad a=1,2,3;\\&\gamma_0(\theta)=-\frac{1+i\lambda_0\,\mathrm{cth}\,\theta}{1-i\lambda_0\,\mathrm{cth}\,\theta},\quad \lambda_\mu=\mathrm{tg}\,f_\mu.\end{aligned} \tag{22}$$

For the coefficients A^{12} and A^{21} we get from Eq. (21)

$$A^{12}_{\alpha_1\alpha_2}=\hat{K}_{12}(\theta_{12})A^{21}_{\alpha_2\alpha_1}. \tag{23}$$

The operator K is of the form

$$\hat{K}_{nm}(\theta)=v^\mu(\theta)\,\tau_n{}^\mu\tau_m{}^\mu, \tag{24}$$

where we have used the notations

$$\begin{aligned}&v^0={}^1/_4(\gamma_1+\gamma_2+\gamma_3-\gamma_0),\ v^1={}^1/_4(\gamma_1-\gamma_2+\gamma_3+\gamma_0),\\&v^2={}^1/_4(-\gamma_1+\gamma_2+\gamma_3+\gamma_0),\quad v^3={}^1/_4(\gamma_1+\gamma_2-\gamma_3+\gamma_0).\end{aligned} \tag{25}$$

Let us derive the corresponding formulas which connect A^{12} and A^{21} in the case when the rest mass vanished, $m_0=0$. In taking the limit $m_0\to 0$ we replace θ_n with $\sigma_n\theta_0$ for states with positive energy and with $i\pi-\sigma_n\theta_0$ for states with negative energy. We let the parameter θ_0 go to $+\infty$ together with $m_0\to 0$ in such a way that $k_n=m_0\sinh\theta_n$ remains constant. The variables σ_n will play the role of the helicities of the particles; they take the values ± 1. In the limit we get

$$\begin{aligned}&u^i(\theta)\to u_\sigma{}^i=\delta_{i\sigma},\quad E_2(\theta_1,\theta_2)\to E_2=\sigma_1k_1+\sigma_2k_2,\\&E_N\to E_N=\sum_{n=1}^N\sigma_nk_n,\quad \mathrm{cth}\,\theta_{nm}\to\sigma_{nm},\end{aligned} \tag{26}$$

where $\sigma_{nm}=\frac{1}{2}(\sigma_n-\sigma_m)$. The connection between the coefficients A^{12} and A^{21} will be of the form

$$A^{12}_{\alpha_1\alpha_2}=\hat{S}_{12}(\sigma_{12})A^{21}_{\alpha_2\alpha_1}. \tag{27}$$

We write the matrix S in a form analogous to that of the matrix K:

$$\hat{S}_{nm}(\sigma)=w^\mu(\sigma)\tau_n^\mu\tau_m^\mu=\begin{pmatrix} w^0+w^3\tau_n^3; & w^1\tau_n^1-iw^2\tau_n^2 \\ w^1\tau_n^1+iw^2\tau_n^2; & w^0-w^3\tau_n^3\end{pmatrix}=\begin{pmatrix} a & 0 & 0 & d \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ d & 0 & 0 & a\end{pmatrix} \tag{28}$$

with the matrix elements

$$\begin{aligned} a(\sigma)&=w^0+w^3=(\sigma^2+\lambda_1\lambda_2)/(\sigma-i\lambda_1)(\sigma-i\lambda_2),\\ b(\sigma)&=w^0-w^3=\sigma(1+\lambda_0\lambda_3)/(\sigma-i\lambda_3)(1-i\lambda_0\sigma),\\ c(\sigma)&=w^1+w^2=i(\lambda_3-\lambda_0\sigma^2)/(\sigma-i\lambda_3)(1-i\lambda_0\sigma),\\ d(\sigma)&=w^1-w^2=\sigma i(\lambda_1-\lambda_2)/(\sigma-i\lambda_1)(\sigma-i\lambda_2).\end{aligned} \tag{29}$$

In the case $N>2$ we also get from the matching conditions for N-particle wave functions

$$A^{\cdots q_nq_{n+1}\cdots}_{\cdots\alpha_n\alpha_{n+1}\cdots}=\hat{S}_{n,n+1}\left(\frac{\sigma_{q_n}-\sigma_{q_{n+1}}}{2}\right)A^{\cdots q_{n+1}q_n\cdots}_{\cdots\alpha_{n+1}\alpha_n\cdots}. \tag{30}$$

Applying these relations repeatedly, we can express any coefficient $A^P_{\alpha_1\ldots\alpha_N}$ in terms of $A^{12\cdots N}_{\alpha_1\cdots\alpha_N}$, which we hereafter denote by the symbol Ω. This relation is unambiguous if the factorization conditions on the matrix $\hat{S}$,

$$\hat{S}_{12}(\sigma)\hat{S}_{13}(\sigma+\sigma')\hat{S}_{23}(\sigma')=\hat{S}_{23}(\sigma')\hat{S}_{13}(\sigma+\sigma')\hat{S}_{12}(\sigma), \tag{31}$$

where $\sigma=\sigma_{12}$, $\sigma'=\sigma_{23}$, $\sigma+\sigma'=\sigma_{13}$, are satisfied.

The factorization conditions (31) can be derived easily for the case $N=3$ if we note that there are two ways to express the coefficient A^{321} in terms of A^{123}:

$$(123)\to(213)\to(231)\to(321)$$

and

$$(123)\to(132)\to(312)\to(321).$$

Setting the two results equal to each other, we get the condition (31)

4. PROOF OF THE FACTORIZATION CONDITIONS

The factorization condition (31) is identical with the factorization condition for the scattering matrix given by Zamolodchikov.[15] According to Refs. 14 and 15, to verify that it is satisfied it is sufficient to verify that the ratios cd/ab and $(a^2+b^2-c^2-d^2)/2ab$ do not depend on the variable σ, which takes the values 0, ± 1. This is indeed so, since

$$\begin{aligned} cd/ab&=(\lambda_1-\lambda_2)(\lambda_0-\lambda_3)/(1+\lambda_1\lambda_2)(1+\lambda_0\lambda_3)=\operatorname{tg}(g_2-g_1)\operatorname{tg}(g_1+g_2),\\ \frac{a^2+b^2-c^2-d^2}{2ab}&=1-\frac{(\lambda_1-\lambda_0)(\lambda_2-\lambda_3)+(\lambda_1-\lambda_3)(\lambda_2-\lambda_0)}{(1+\lambda_1\lambda_2)(1+\lambda_0\lambda_3)}\\ &=\frac{\cos(2g_3)}{\cos(g_1-g_2)\cos(g_1+g_2)}.\end{aligned} \tag{32}$$

This concludes the proof of the factorization conditions. In what follows, however, we require that these conditions hold for arbitrary values of σ and σ', so that we shall continue the matrix $\hat{S}(\sigma)$ from the integer points 0, ± 1 to arbitrary values of σ in such a way that the conditions (31) are not violated. The general solution of the equation (31) for arbitrary u and v is given in Refs. 14 and 15; it is

$$\begin{aligned} &a=\rho(u)\,\mathrm{sn}\,(u+2\eta),\quad b=\rho(u)\,\mathrm{sn}\,(u),\\ &c=\rho(u)\,\mathrm{sn}\,(2\eta),\quad d=k\rho(u)\,\mathrm{sn}\,(2\eta)\,\mathrm{sn}\,(u)\,\mathrm{sn}\,(u+2\eta).\end{aligned} \tag{33}$$

Here η is an arbitrary parameter, $\mathrm{sn}(u)$ is the Jacobian elliptic function of modulus k (k is arbitrary), and $\rho(u)$ is an arbitrary function.

We set $u=-if\sigma$; then by choosing the parameters η, k,f and the function $\rho(u)$ we can produce a solution of the given problem. The parameters are determined from the equations

$$k\,\mathrm{sn}^2(2\eta)=(\lambda_1-\lambda_2)(\lambda_0-\lambda_3)/(1+\lambda_1\lambda_2)(1+\lambda_0\lambda_3), \tag{34}$$

$$\mathrm{cn}(2\eta)\,\mathrm{dn}(2\eta)=1-\frac{(\lambda_1-\lambda_0)(\lambda_2-\lambda_3)+(\lambda_1-\lambda_3)(\lambda_2-\lambda_0)}{(1+\lambda_1\lambda_2)(1+\lambda_0\lambda_3)}, \tag{35}$$

$$\frac{\mathrm{sn}(if)}{\mathrm{sn}(2\eta-if)}=-\frac{1+\lambda_0\lambda_3}{1+\lambda_1\lambda_2}\frac{(1-i\lambda_1)(1-i\lambda_2)}{(1-i\lambda_3)(1-i\lambda_0)}, \tag{36}$$

where cn and dn are Jacobi's elliptic cosine and delta function of modulus k.

The function $\rho(u)$ is of the form

$$\rho(u)=a(\sigma)/\mathrm{sn}\,(2\eta+u). \tag{37}$$

Accordingly, the matrix $\hat{S}(\sigma)$ with the elements

$$\begin{aligned} &a(\sigma)=\frac{\sigma^2+\lambda_1\lambda_2}{(\sigma-i\lambda_1)(\sigma-i\lambda_2)},\quad b(\sigma)=-a(\sigma)\frac{\mathrm{sn}(if\sigma)}{\mathrm{sn}(2\eta-if\sigma)},\\ &c(\sigma)=a(\sigma)\frac{\mathrm{sn}(2\eta)}{\mathrm{sn}(2\eta-if\sigma)},\quad d(\sigma)=-ka(\sigma)\,\mathrm{sn}(2\eta)\,\mathrm{sn}(if\sigma)\end{aligned} \tag{38}$$

satisfies the factorization condition (31) for arbitrary σ, and is identical with our earlier matrix at the points 0, ± 1.

Equations (38) show that for arbitrary constants g_a the matrix $\hat{S}$ differs only by a factor from Baxter's local matrix for the eight-vertex model (the XYZ model in field theory). If $m_0\neq 0$, the ratios cd/ab and $(a^2+b^2-c^2-d^2)/2ab$ are not constant; they depend on the variable θ, and the conditions (31) are not satisfied. Only in the case when all the g_a are zero do we get trivial agreement with these conditions. The isotopic spin then plays no part; the corresponding model with $m_0\neq 0$ has been considered by Berezin and Sushko.[16]

We note two special cases:

1) If we take $g_1=g_2=g$ ($\lambda_1=\lambda_2=\lambda$), i.e., consider the case of $U(1)$ symmetry of the original Lagrangian (2), we have from Eqs. (34)–(36)

$$k=0,\quad \cos(2\eta)=1-2(\lambda-\lambda_0)(\lambda-\lambda_3)/(1+\lambda^2)(1+\lambda_3\lambda_0), \tag{39}$$

$$\frac{\sin(2\eta)}{\mathrm{th}\,f}=\frac{\lambda_3-\lambda_0}{1+\lambda_0\lambda_3}-2\frac{(\lambda_3-\lambda)(1+\lambda_0\lambda)}{(1+\lambda^2)(1+\lambda_0\lambda_3)}. \tag{40}$$

The elements of the matrix $\hat{S}$ are

$$\begin{aligned} &a(\sigma)=\frac{\sigma+i\lambda}{\sigma-i\lambda},\quad b(\sigma)=a(\sigma)\frac{\mathrm{sh}(f\sigma)}{\mathrm{sh}(f\sigma+2i\eta)},\\ &c(\sigma)=a(\sigma)\frac{i\sin(2\eta)}{\mathrm{sh}(f\sigma+2i\eta)},\quad d(\sigma)=0.\end{aligned} \tag{41}$$

In this case the matrix $\hat{S}$ differs by a factor from the corresponding matrices for Baxter's ice model and the model described by the sine-Gordon equation.[10,11]

2) In the case of $SU(2)$ symmetry, i.e., when $g_1=g_2=g_3=g$ ($\lambda_1=\lambda_2=\lambda_3=\lambda$), we have

$$k=0,\quad \cos(2\eta)=1,\quad \text{i.e.,}\quad \eta\to 0. \tag{42}$$

It follows from Eq. (40) that f also goes to zero, and that also

$$\lim_{\eta\to 0}\frac{2\eta}{f}=\frac{\lambda-\lambda_0}{1+\lambda_0\lambda}=\varkappa=\mathrm{tg}(2g). \tag{43}$$

For the elements of the matrix S we get

$$\begin{gathered} a(\sigma)=\frac{\sigma+i\lambda}{\sigma-i\lambda},\quad b(\sigma)=a(\sigma)\frac{\sigma}{\sigma+i\varkappa},\\ c(\sigma)=a(\sigma)\frac{i\varkappa}{\sigma+i\varkappa},\quad d(\sigma)=0. \end{gathered} \tag{44}$$

In this symmetric model the matrix $\hat{S}_{nm}$ can be expressed simply in terms of the permutation operator P_{nm} which interchanges the isotopic indices α_n and α_m:

$$\hat{S}_{nm}(\sigma)=b(\sigma)+c(\sigma)\hat{P}_{nm}, \tag{45}$$

which is the form in which this matrix was used by Belavin.[12] Here the matrix S differs by a factor from that in the model described by the nonlinear Schrödinger equation.[11]

5. APPLICATION OF THE BOUNDARY CONDITIONS

Up to now we have made no use of the periodic boundary conditions (10). Substituting the wave function (14) in Eq. (10) and using Eq. (30), we get a system of equations for the isospin vector Ω:

$$\hat{T}_n\Omega=e^{ik_nL}\Omega; \tag{46}$$

$$\hat{T}_n=\hat{S}_{n,n+1}\ldots\hat{S}_{nN}\hat{S}_{n1}\ldots\hat{S}_{n,n-1}. \tag{47}$$

The argument of each matrix $\hat{S}_{nm}$ in Eq. (47) is $\sigma_{nm}=\frac{1}{2}(\sigma_n-\sigma_m)$. All of the N equations (46) are compatible, since $[T_nT_m]=0$ by virtue of the factorization conditions (31). These equations determine the isotopic vector Ω and the eigenvalues of the set of momenta $k_1, k_2, \ldots, k_N$.

We shall carry out the solution of Eqs. (46) by a method like that used in a number of papers.[8-12,17,18] We introduce an auxiliary operator

$$\hat{L}(v)=-\prod_{n=1}^{N}\hat{S}_{0n}\left(\frac{v-\sigma_n}{2}\right); \tag{48}$$

$$\hat{S}_{0n}\left(\frac{v-\sigma_n}{2}\right)=w^{\mu}\left(\frac{v-\sigma_n}{2}\right)\tau_0^{\mu}\tau_n^{\mu}. \tag{49}$$

The operator $\hat{L}(v)$ acts in a space of 2^{N+1} dimensions, with an additional particle numbered 0.

We use a symbol for the trace of the operator over the indices of the additional particle:

$$\hat{T}(v)=\mathrm{Sp}_0\hat{L}(v), \tag{50}$$

$\hat{T}(v)$ is analogous to Baxter's transfer matrix.[14] As he showed,[14] it follows from the factorization conditions (31) that

$$[\hat{T}(v)\hat{T}(u)]=0 \tag{51}$$

for arbitrary u, v. Besides this, it is easy to verify that

$$\hat{T}(\sigma_n)=\hat{T}_n. \tag{52}$$

We represent the operator $\hat{L}(v)$ in the form

$$\hat{L}(v)=\begin{pmatrix}A(v) & B(v)\\ C(v) & D(v)\end{pmatrix}=-\prod_{n=1}^{N}\begin{pmatrix}w_n^0+w_n^3\tau_n^3; & w_n^1\tau_n^1-iw_n^2\tau_n^2\\ w_n^1\tau_n^1+iw_n^2\tau_n^2; & w_n^0-w_n^3\tau_n^3\end{pmatrix} \tag{53}$$

where

$$w_n^{\mu}=w^{\mu}\left(\frac{v-\sigma_n}{2}\right).$$

An important property of the operators $\hat{L}(v)$ and $\hat{L}(u)$ is their commutation relations, which can be put in the form

$$\hat{R}\left(\frac{v-u}{2}\right)(\hat{L}(v)\otimes\hat{L}(u))=(\hat{L}(u)\otimes\hat{L}(v))\hat{R}\left(\frac{v-u}{2}\right) \tag{54}$$

(cf. Refs. 10 and 11). The product $\hat{L}(v)\otimes\hat{L}(u)$ is a four-row (*sic*) matrix made up of blocks:

$$\hat{L}(v)\otimes\hat{L}(u)=\begin{pmatrix}\hat{A}(v)\ \hat{L}(u); & \hat{B}(v)\hat{L}(u)\\ \hat{C}(v)\hat{L}(u); & \hat{D}(v)\hat{L}(u)\end{pmatrix}. \tag{55}$$

$\hat{R}(u)$ is a four-row matrix with numerical elements:

$$\hat{R}(u)=\begin{pmatrix}a(u) & 0 & 0 & d(u)\\ 0 & c(u) & b(u) & 0\\ 0 & b(u) & c(u) & 0\\ d(u) & 0 & 0 & a(u)\end{pmatrix}. \tag{56}$$

Let us solve the equations (46) in the case of the symmetry U(1). The solution for arbitrary constants g_a is more cumbersome and is produced by a method analogous to that used for the solution for the XYZ model, which is explained in a paper by Faddeev.[11] In our case ($d=0$) Eq. (54) leads to the following commutation relations for the operators $\hat{A}$, $\hat{B}$, and $\hat{D}$:

$$\begin{gathered} \hat{A}(v)\hat{B}(u)=\alpha(u-v)\hat{B}(u)\hat{A}(v)-\beta(u-v)\hat{B}(v)\hat{A}(u),\\ \hat{D}(v)\hat{B}(u)=\alpha(v-u)\hat{B}(u)\hat{D}(v)-\beta(v-u)\hat{B}(v)\hat{D}(u),\\ [\hat{B}(v),\hat{B}(u)]=0, \end{gathered} \tag{57}$$

where

$$\begin{gathered} \alpha(u)=a\left(\frac{u}{2}\right)\Big/b\left(\frac{u}{2}\right)=\mathrm{sh}\left(f\frac{u}{2}+2i\eta\right)\Big/\mathrm{sh}\left(f\frac{u}{2}\right),\\ \beta(u)=c\left(\frac{u}{2}\right)\Big/b\left(\frac{u}{2}\right)=i\sin(2\eta)/\mathrm{sh}\left(f\frac{u}{2}\right). \end{gathered} \tag{58}$$

Let us denote by Ω_0 a state in which all of the spins are directed upward, i.e.,

$$\tau_n^z\Omega_0=\Omega_0. \tag{59}$$

It is easy to verify that Ω_0 is an eigenvector of the operators $\hat{A}$, $\hat{D}$, and $\hat{C}$ and that

$$\begin{gathered} \hat{A}(v)\Omega_0=-\prod_{n=1}^{N}a\left(\frac{v-\sigma_n}{2}\right)\Omega_0,\quad \hat{D}(v)\Omega_0=-\prod_{n=1}^{N}b\left(\frac{v-\sigma_n}{2}\right)\Omega_0,\\ \hat{C}(v)\Omega_0=0 \end{gathered} \tag{60}$$

and, that consequently

$$\hat{T}(v)\Omega_0=-\left\{\prod_{n=1}^{N}a\left(\frac{v-\sigma_n}{2}\right)+\prod_{n=1}^{N}b\left(\frac{v-\sigma_n}{2}\right)\right\}\Omega_0. \tag{61}$$

We shall look for other eigenvectors of the operator $\hat{T}(v)$ in the form

$$\Omega(v_1,v_2,\ldots,v_M)=\prod_{n=1}^{M}\hat{B}(v_n)\Omega_0. \tag{62}$$

Applying the commutation relations (57), we find that

$$\hat{T}(v)\Omega=(\hat{A}(v)+\hat{D}(v))\Omega=\Lambda(v,v_1,v_2,\ldots,v_M)\Omega, \tag{63}$$

if the set of numbers $q_1, q_2, \ldots, q_N$ ($q_n=v_n+2i\eta/f$) satisfies the system of equations

$$\begin{gathered} \prod_{n=1}^{N}\left[\mathrm{sh}\left(f\frac{q_k-\sigma_n}{2}+i\eta\right)\Big/\mathrm{sh}\left(f\frac{q_k-\sigma_n}{2}-i\eta\right)\right]\\ =-\prod_{n=1}^{M}\left[\mathrm{sh}\left(f\frac{q_k-q_n}{2}+2i\eta\right)\Big/\mathrm{sh}\left(f\frac{q_k-q_n}{2}-2i\eta\right)\right], \end{gathered} \tag{64}$$

$k=1, 2, \ldots, N$. The eigenvalues of the operator $T(v)$ are given by

$$\Lambda(v, v_1, v_2, \ldots, v_M) = -\left\{\prod_{k=1}^{N} a\left(\frac{v-\sigma_k}{2}\right)\prod_{k=1}^{M}\alpha(v_k-v) + \prod_{k=1}^{N} b\left(\frac{v-\sigma_k}{2}\right)\prod_{k=1}^{M}\alpha(v-v_k)\right\}. \quad (65)$$

Substituting here $v=\sigma_n$ and recalling that $a(0)=-1$ and $b(0)$, we get for the eigenvalues of the operator T_n the expression

$$e^{ik_nL} = -\prod_{m=1}^{N}\frac{\sigma_n-\sigma_m+2i\lambda}{\sigma_n-\sigma_m-2i\lambda}\prod_{m=1}^{M}\left[\operatorname{sh}\left(f\frac{q_m-\sigma_n}{2}+i\eta\right)\Big/\operatorname{sh}\left(f\frac{q_m-\sigma_n}{2}-i\eta\right)\right]. \quad (66)$$

The system of equations (64) and (66) enables us to determine the possible values of the set of momenta $k_1, k_2, \ldots, k_N$. Substitution of the expression (62) in the wave function (14) gives the eigenfunctions of the Hamiltonian operator in the case $m_0=0$. For the energy and total momentum of the system we get

$$E_N=\sum_{n=1}^{N}\sigma_n k_n, \quad K_N=\sum_{n=1}^{N}k_n. \quad (67)$$

The energy of these states can take both positive and negative values and has no lower bound. To obtain the physical spectrum it is necessary to define the physical vacuum of the system, i.e., to fillup all the states with negative energies, introducing a cutoff momentum to remove divergences (the cutoff momentum Λ is a maximum allowable energy of an individual particle). By considering the various excitations against the background of this vacuum state, we can obtain the spectrum of these physical states of excitation.

6. CONCLUSION

A model like that examined above has been investigated by Ansel'm.[19] It was shown that there is no "zero-of charge" problem in the model and that it possesses asymptotic freedom in the main logarithmic approximation of perturbation theory. In this same approximation Vaks and Larkin[20] concluded that the particles spontaneously acquire a mass. An analysis of Eqs. (64) and (66) shows that the exact solution gives the particles a mass, and yields the same value of the mass.

This model, as we have stated in the text, is equivalent to the sine-Gordon model for an arbitrary value of the coupling constant β^2. The $SU(2)$-invariant case considered earlier by Belavin corresponds to $\beta^2=8\pi$. In the sine-Gordon model with $\beta^2<4\pi$ "fermion-antifermion" bound states appear. The exact solution allows us to trace this effect out in such cases.

It is also necessary to note the important fact, repeatedly emphasized by A. M. Polyakov, that two-dimensional models of the type we have considered here are analogous to four-dimensional gauge theories. As has been shown here, a particularly interesting property of our present model is the presence of an infinite series of conservation laws. This is extremely important in connection with the possibility of a similar phenomenon in four-dimensional gauge theories in a less trivial aspect (absence of production of particles).

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