A calculation of the evolution operator and of the quark structure of states in two-dimensional massless electrodynamics

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(Submitted 22 August 1979)
Zh. Eksp. Teor. Fiz. **78**, 1314–1331 (April 1980)

The finite-time evolution operator $S(T)$ is constructed for two versions of massless electrodynamics: the Schwinger model, and a model with several kinds of electrons. Diagonalization of $S(T)$ allows one to obtain all the physical states for both models and to describe their quark structure.

PACS numbers: 11.10.Mn, 12.40.Bb

1. INTRODUCTION

In the widely discussed current hypothesis of color confinement in nonabelian gauge theories the spectrum of physical states (hadrons) differs in a fundamental way from the spectrum of the noninteracting system (gluons and quarks). Among the questions which arise in connection with this hypothesis, answers to which are essential for an understanding of the physics of confinement, we would like to single out and analyze with a simple example the problem of the quark structure of physical states. For this it is necessary to construct the wave functions of physical states in terms of the operators of the noninteracting fields.

We solve this problem for a well-studied simplest model of confinement, namely two-dimensional quantum electrodynamics of massless electrons[1)] (QED_2[1-3]). For this model exact operator solutions have been found[2-5], allowing one to compute all the Green's functions. The characteristic traits of the spectrum can also be determined directly from the Schrödinger equation.[5] However, the complicated nature of this equation for QED_2 does not allow one to elucidate the quark structure of the states, therefore these methods are inadequate for our purposes.

One can avoid the necessity of a direct solution of the Schrödinger equation if one constructs for the given model the evolution operator $S(T)$ for finite time, and expands it in a series with respect to time exponentials. This operator has the expression

$$S(T)=e^{-iHT}=\sum_n |\Psi_n\rangle e^{-iE_nT}\langle\Psi_n|. \tag{1}$$

Here E_n is the energy of the n-th physical state, $\langle\Psi_n|$ and $|\Psi_n\rangle$ are the bra and the ket of this state, depending on the coordinates of the system, respectively at the initial time (0) and the final time (T). This method of solution was proposed some time ago by Feynman[6] for problems of quantum mechanics. The operator $S(T)$ determines the time evolution of arbitrarily formed wave packets and its diagonalization (1) allows one to obtain the physical states in terms of a complete set of arbitrary states, among those states generated by the free quark operators.

In the present paper we formulate necessary conditions for the determination of the operator $S(T)$ and find its expression in QED_2. The evolution operator and the physical states of a gauge theory are best constructed in a gauge which does not require the use of an indefinite metric, where the Hamiltonian is positive-definite, and the wave function does not depend on the variables which are eliminated by the subsidiary condition (cf. e.g.,[7]). All these conditions are satisfied by the Coulomb gauge, in which we construct the operator $S(T)$. For the clarification of some problems related to understanding the gauge invariance of the model we shall use the gauge where $A_0=0$ (Sec. 2).

In a field theory the operator $S(T)$ can be constructed as a functional integral,[7] where the integration is carried out between boundary values of the fields defined at the initial and final instants of time:

$$S(T)=\int DA(x)D\psi D\bar{\psi}\exp\left\{i\int_0^T \mathscr{L}(x)d^2x\right\}. \tag{2}$$

We must define this integral not only over a finite time interval but also in a finite space volume $V=2L$. Otherwise the physical wave functions will turn out to be orthogonal to the wave functions of the free theory, a fact which is related to the finite density of bare quarks in the physical vacuum. The boundary conditions at the edges of volume will be assumed to be periodic, which simplifies the discussion of the problem in the momentum representation.

On the other hand, in two-dimensional theories periodic boundary conditions forbid states with total charge $Q\neq 0$, whereas an investigation of such states is of interest from the point of view of charge confinement. For periodic boundary conditions states with $Q\neq 0$ are unphysical, since they do not satisfy the supplementary condition which selects physical states [Eq. (9), Sec. 2]. However, this difficulty has no relation to confinement, and the latter may also be investigated for periodic boundary conditions. Confinement means the impossibility of having a local charge (outside the confinement radius) and this property does not, of course, depend on the boundary conditions at the edges of the volume. It can be verified also for systems with $Q=0$, but with a local charge density in space. Therefore, in order to study properties of states with physical charge $Q\neq 0$ it suffices to introduce into the system in a convenient manner an outside charge, so that the total charge vanishes. We place the system of charges Q between the outside charges $-Q/2$ situated at the points $\pm L$ of the

 0038-5646/80/040663-08$02.40

space direction (a "capacitor"). Such charges will compensate the field of the physical charges outside the capacitor, without affecting the system inside it. This procedure is equivalent to considering the system in a finite volume with a cutoff of the nondecaying fields which appear in this case.

The operator $S(T)$ is computed in Secs. 3 and 4. The analysis of the spectrum of QED_2, the properties of the states, and the relation between these properties and the physics of confinement transcends, essentially, the framework of the present paper. The calculations are carried out for two versions of QED_2: the one-electron model (the Schwinger model[1]) and a model with several kinds of electrons interacting with the same field A_μ but with different charges g_i.[4]

To conclude this section we list several formulas and definitions which will be extensively used in this sequel. The Lagrange function of the Schwinger model is given by

$$\mathcal{L}(x) = -\tfrac{1}{4}(F_{\mu\nu})^2 + i\bar{\psi}\gamma^\mu\partial_\mu\psi + gA_\mu j^\mu. \tag{3}$$

The generalization to the case of several types of electrons is obvious. The current j^μ in Eq. (3) has to be defined by means of the Schwinger point splitting[8]:

$$j^\mu = \lim_{\varepsilon\to 0} \bar{\psi}(x+\varepsilon)\gamma^\mu\psi(x-\varepsilon) - \frac{g}{2\pi}A^\mu. \tag{4}$$

The expression (4) guarantees the gauge invariance of the current.

The Lagrangian (3) conserves helicity, therefore it is convenient to split ψ into parts with helicities ± 1:

$$\psi_i = \psi_R u_i^{(R)} + \psi_L u_i^{(L)},$$
$$u^{(R,L)} = \tfrac{1}{2}(1\pm\gamma_5)u, \quad \gamma_5 = \gamma^0\gamma^1. \tag{5}$$

Then the Fourier expansion of ψ_R and ψ_L

$$\psi_{R,L} = \frac{1}{2L}\sum_p [e^{\pm ipx}a_{R,L}(p) + e^{\mp ipx}b^+_{R,L}(p)] \tag{6}$$

defines the creation-annihilation operators of bare particles: the right-handed and left-handed quarks (R and L). The evolution operator will be expressed in terms of these operators.

2. DEFINITION OF THE EVOLUTION IN THE COULOMB GAUGE

The wave function $\Psi(T)$ and the evolution operator (1) in a gauge theory depend on the variables of the charged quark field and on the potentials of the electromagnetic (EM) field at the time T. The EM potentials contain "redundant" variables—the longitudinal fields. It is simplest to get rid of these by a transition to the Coulomb gauge, by defining the dependence of the wave function or of $S(T)$ on these variables in an appropriate gauge. For this purpose it is convenient to choose the gauge where $A_0 = 0$, in which the longitudinal variables (in the two-dimensional case there are no others, just the potential A_1) are dynamical variables. This gauge satisfies the conditions stated in the Introduction, and in it the Hamiltonian has a simple form. It follows from Eq. (3) that

$$H(x) = H_{em}{}^0 + H(A_1),$$
$$H_{em}{}^0 = \frac{1}{2}A_1^2, \quad H(A_1) = \frac{1}{2i}\bar{\psi}\gamma^1\overleftrightarrow{\partial_1}\psi + jA_1. \tag{7}$$

Here H^0_{em} is the Hamiltonian of the free electromagnetic field and $H(A_1)$ is the Hamiltonian of the charged particles in the field A_1.

In this gauge the evolution operator can be expressed in terms of the functional integral:

$$S(T) = e^{-iHT} = \int DA_1(x,t)\exp\left(\frac{i}{2}\int A_1^2 d^2x\right) T\left\{\exp\left(-i\int_0^T H(A_1)\,dt\,dx\right)\right\}. \tag{8}$$

This formula is valid in the second-quantization representation for the fermion operators. Here $H(A_1)$ is a second-quantized Hamiltonian of a charged particle in the field, defined by Eq. (7). The formula (8) can be brought to the form (2) by means of the standard transition to Grassmann variables,[7] i.e., it can be reduced to a functional integral over the fermion variables. In both integrals (2) and (8) the integration over the field $A_1(x, t)$ extends over the interval $0 < t < T$. For $t = 0$ and $t = T$ the field takes on definite values $A_1(x, 0)$ and $A_1(x, T)$. These are the values which play the role of the EM field variables on which the evolution operator depends.

The physical part $S_{ph}(T)$ of the evolution operator is subject to the supplementary condition[7]

$$\operatorname{div} E + \rho = 0, \tag{9}$$

which in the gauge $A_0 = 0$ takes the form

$$\left[\frac{\partial}{\partial x}\frac{1}{i}\frac{\delta}{\delta A_1(x,T)} - \rho(x,T)\right] S_{ph}(T) = 0,$$
$$E = -\dot{A}_1 = -\frac{1}{i}\frac{\delta}{\delta A_1}. \tag{10}$$

$S(T)$ is subject to a similar condition for $t = 0$. Both conditions are satisfied by the expression

$$S_{ph}(T) = \exp\left(+i\int \rho(x,T)f(x,T)\,dx\right) S_c(T)\exp\left(-i\int \rho(x,0)f(x,0)\,dx\right),$$
$$\partial f(x,t)/\partial x = -A_1(x,t), \tag{11}$$

in which $S_c(T)$ is independent of A_1. The operator $S_c(T)$ is the evolution operator in the Coulomb gauge. Indeed, an arbitrary gauge transformation of the evolution operator consists in the substitution

$$S(T) \to U(T)S(T)U^+(0), \quad A_\mu \to \tilde{A}_\mu = A_\mu + \partial f/\partial x_\mu, \tag{12}$$

where $U(T)$ is exactly the operator in front of $S_c(T)$ in Eq. (11), and $f(x)$ is the gauge function. Thus the operator $S_c(T)$ which does not depend on the EM field variables is the evolution operator in the gauge $A_1 = 0$, i.e., the Coulomb gauge. It is defined only for the physical states satisfying the condition (9).

For the reasons discussed in the Introduction we shall be interested in the possibility of transition to the Coulomb gauge when the system is enclosed in a finite spatial interval $(-L, +L)$. Let us see how one can generalize to this case the procedure of gauge change and the selection of physical states. We impose periodic boundary conditions on the fermion operators:

$$\psi(L) = \psi(-L), \tag{13}$$

which, as indicated, allow one easily to go over to the momentum representation (6) with discrete momentum

values $p_n = n\pi/L$, and which do not affect the local properties of the system.[6]

Since gauge transformations are not supposed to violate the condition (13) one must assume that the gauge functions $f(x)$ are periodic, and consequently so are the potentials A_μ. Therefore we can expand them in a Fourier series analogous to Eq. (6):

$$A_\mu(x,t)=\frac{1}{2L}\sum_n \exp(ik_n x)A_\mu(k_n,t). \tag{14}$$

In momentum space the admissible gauge transformations take on the form

$$A_1(p_n,t)\to A_1(p_n,t)-ip_n f(p_n,t),$$
$$A_0(p_n,t)\to A_0(p_n,t)+\dot{f}(p_n,t). \tag{15}$$

$A_1(0,t)$ does not change under such a transformation and being gauge invariant, becomes a dynamical variable which cannot be removed in any gauge. The dependence of $S(T)$ on this variable is arbitrary, and its dynamical influence on states is small as $L\to\infty$, since in the Lagrange function (3) and in the Hamiltonian (7) the contribution of this component of the potential enters with a factor $V^{-1}=(2L)^{-1}$. Therefore we shall ignore in the sequel the dependence of $S(T)$ on $A_1(0,t)$, although the contributions of this component can be easily taken into account at each step of the calculation. They do not lead to essential physical consequences.

Another peculiarity of the periodic boundary conditions was noted in the Introduction. Such conditions are incompatible with states of the system with $Q\neq 0$. Indeed, integrating the supplementary condition (9) over the interval $(-L, +L)$ we obtain

$$E(L)-E(-L)+Q=0. \tag{16}$$

Thus for all charged states for which condition (9) or (16) is satisfied we must have $E(L)\neq E(-L)$. The only way to consider such states, maintaining periodic boundary conditions, is to modify Eq. (9) and to introduce into the system charges from the outside so as to compensate exactly the charge Q. If this is done in such a manner as not to modify the physics of the local phenomena far from the external charges, it is obvious that the properties of the physical charges of the system in the two-dimensional model under consideration can be investigated in such unphysical [from the point of view of condition (9)] states.

The simplest method of introduction of an external charge into the system, which affects least the physics of internal local phenomena, consists in a modification of the supplementary condition to

$$\operatorname{div} E+\rho(x)-{}^1\!/_2 Q(\delta(x-L)+\delta(x+L))=0. \tag{17}$$

The additional term in Eq. (17) corresponds to a capacitor, as discussed in the Introduction. The supplementary condition (17) no longer contradicts the periodic boundary conditions. Going over to the momentum representation it is easy to obtain its solutions for the interval $(-L, L)$, so that now $S_{ph}(T)$ is represented by

$$S_{ph}(T)=\exp\left\{\frac{i}{2L}\sum_{n\neq 0}\left[\frac{\rho(p_n,T)-(-1)^n Q}{ip_n}\right]A_1(-p_n,T)\right\}$$
$$\times S_c(T)\exp\left\{-\frac{i}{2L}\sum_{n\neq 0}\left[\frac{\rho(p_n,0)-(-1)^n Q}{ip_n}\right]A_1(-p_n,0)\right\}; \tag{18}$$

$S_c(T)$ is again independent of A_1.

The exponential factor in Eq. (18) realizes the transition to the gauge

$$\bar{A}_1(p_n,t)=0,\qquad A_0(p_n,t)=-\frac{\dot{A}_1(p_n,t)}{ip_n},\quad n\neq 0,$$
$$A_0(0,t)=\sum_{n\neq 0}\frac{(-1)^n}{ip}A_1(p_n,t)=-\sum_{n\neq 0}(-1)^n A_0(p_n,t). \tag{19}$$

This is the gauge we shall call the Coulomb gauge. We note that in this gauge $A_0(0,t)$ is not an independent variable, but can be expressed in terms of the field components with $p_n\neq 0$.

In order to determine the expression for $S_c(T)$ in terms of a functional integral it is necessary, as expressed by Eqs. (11) and (18), to separate from the expression (8) a factor which corresponds to the gauge transformations (15) or (19). This is easily done owing to the fact that the operator exponential of the second-quantized Hamiltonian for a charged particle in an external field in Eq. (8) satisfies the relations

$$T\left\{\exp\left(-i\int_0^T H(A_\mu(x))\,dt\,dx\right)\right\}$$
$$=\prod_{i=1}^{N}\left[1-i\int_{t_i}^{t_{i+1}}H(A_\mu(x,t_i))\,dx\,dt_i\right]$$
$$=\exp\left(i\int\rho(x,T)f(x,T)\,dx\right)T\left\{\exp\left(-i\int_0^T H\left(A_\mu+\frac{\partial f}{\partial x_\mu}\right)dt\,dx\right\}\right.$$
$$\times\exp\left(-i\int\rho(x,0)f(x,0)\,dx\right),\qquad t_0=0,\quad t_N=T. \tag{20}$$

For the gauge transformation (19), starting with the gauge $A_0=0$, we obtain from Eq. (7) the result that the Hamiltonian in Eq. (20) has the form

$$H(A_0)=-\frac{1}{2i}\bar{\psi}\gamma^1\overleftrightarrow{\partial}_1\psi-\rho(x)A_0(x)+\frac{Q}{2}[\delta(x-L)+\delta(x+L)]A_0(x). \tag{21}$$

The term describing the interaction of the capacitor with the field comes from the $A_0(0,t)$ component of the field A_0, Eq. (19):

$$-\frac{1}{2L}A_0(0,t)\rho(0)=\frac{Q}{2L}\sum(-1)^n A_0(p_n,t)$$
$$=\frac{Q}{2}\int_{-L}^{L}dx A_0(x,t)[\delta(x-L)+\delta(x+L)]. \tag{22}$$

Replacing now the integration variables in the integral (8) according to Eq. (19)

$$A_0(p_n,t_i)=-\frac{A_1(p_n,t_i)-A_1(p_n,t_{i-1})}{ip_n\delta t_i}, \tag{23}$$

we obtain the following formula for the integration measure:

$$\prod_{i=1}^{N-1}DA_1(p_n,t_i)=(ip_n)^N\prod_{i=1}^{N}DA_0(p_n,t_i)\,\delta\left(ip_n\int_0^T A_0(p_n,t)\,dt\right.$$
$$\left.+A_1(p_n,T)-A_1(p_n,0)\right). \tag{24}$$

The expression (24) contains a delta function since we have introduced an integration over $A_0(p_n, T)$ in the right-hand side. The Jacobian of the transformation has been taken into account in the calculation of the common normalization factor in Appendix II.

We now substitute into Eq. (8) the expression (20) with

the Hamiltonian (21). We carry out in (8) a change of variables according to Eq. (24). This yields

$$S_{pl}(T)=\exp\left\{\frac{i}{2L}\sum_{n\neq 0}\left[\frac{\rho(p_n,T)-(-1)^nQ}{ip_n}\right]A_1(-p_n,T)\right\}$$
$$\times\int DA_0(p_n,t)\exp\left[\frac{i}{2}\int_0^T\left(\frac{\partial A_0}{\partial x}\right)^2dt\,dx\right]T\left\{\exp\left(-i\int_0^T H(A_0)\,dt\,dx\right)\right\}$$
$$\times\delta\left(ip_n\int_0^T A_0(p_n,t)\,dt+A_1(p_n,T)-A_1(p_n,0)\right)$$
$$\times\exp\left\{-\frac{i}{2L}\sum_{n\neq 0}\left[\frac{\rho(p_n,0)-(-1)^nQ}{ip_n}\right]A_1(-p_n,0)\right\}. \quad (25)$$

Here we have taken into account the fact that $\dot{A}_1^2=(\partial A_0/\partial x)^2=-\frac{1}{2}(F_{\mu\nu})^2$ is a gauge invariant. We emphasize the fact that there is no integration over $A_0(0,t)$ in Eq. (25). According to Eq. (19) that component is not independent.

The final formula for $S_c(T)$ is obtained if one separates in the functional integral with respect to A_0 in (25) the part which does not depend on the potentials $A_1(x,0)$ and $A_1(x,T)$. It is obvious that this part is obtained by averaging this integral with respect to $A_1(x,T)$. Then the delta function disappears from the integrand and we obtain the following formula:

$$S_c(T)=\int DA_0(x,t)\exp\left[\frac{i}{2}\int_0^T\left(\frac{\partial A_0}{\partial x}\right)^2dt\,dx\right]$$
$$\times T\left\{\exp\left(-i\int_0^T H(A_0)\,dt\,dx\right)\right\}. \quad (26)$$

Here the integration with respect to $A_0(x,t)$ is carried out over the whole interval $0\leq t\leq T$, including $t=T$ [according to Eq. (23) the integrand does not depend on $A_0(x,0)$].

To conclude this section we find the expression for the total Hamiltonian of the system in the Coulomb gauge in a finite volume also for the case $Q\neq 0$. Carrying out in (7) a gauge transformation (19) and making use of Eq. (9) in order to go over into the coordinate space, we obtain:

$$H=H_0+\frac{1}{4L}\sum_{n\neq 0}\frac{|\rho(p_n)-(-1)^nQ|^2}{p_n^2}=H_0-\frac{1}{4}\int_{-L}^{L}dx\,dx'\rho(x)\,|x-x'|\,\rho(x')$$
$$+\frac{1}{4}Q^2L-\frac{1}{4L}\left[\int_{-L}^{L}x\rho(x)\,dx\right]^2. \quad (27)$$

This Hamiltonian does not contain any arbitrary parameters (cf. Ref. 10). This has to do with the fixing of the periodic boundary conditions and the neglect of the dependence of the wave function on $A_1(0,t)$.

3. CALCULATION OF THE INTEGRAL OVER THE FERMI FIELDS

The "untangling" of the operator exponential in Eq. (26) with the help of the usual coherent-state technique[7] leads to the following expression of $S_c(T)$ in terms of a functional integral with respect to the fermion fields:

$$S_c(T)=\int DA_0D\psi D\bar{\psi}\exp\left\{i\int_0^T dt\int dx\left[\frac{1}{2}\left(\frac{\partial A_0}{\partial x}\right)^2+i\bar{\psi}(\hat{\partial}-igA_0\gamma^0)\psi\right]\right\} \quad (28)$$

The expression appearing in the exponential function is the QED_2 Lagrangian in the Coulomb gauge. For the model with several kinds of charged quarks[4] the corresponding Lagrangian is

$$\mathcal{L}=\frac{1}{2}\left(\frac{\partial A_0}{\partial x}\right)^2+i\sum_{i=1}^{N}\bar{\psi}_i\hat{\partial}\psi_i+A_0\sum_{i=1}^{N}g_i\bar{\psi}_i\gamma^0\psi_i. \quad (29)$$

Here ψ_i are the operators of the i-th kind of quarks, g_i are their charges. The integration in Eq. (28) is over all the ψ_i. The calculations are completely analogous in both cases. Carrying them out for one kind of quark we can then write down the answer for the model in Ref. 4.

The Gaussian integral over the fermion fields is calculated by means of the standard method of separating the dependence on the boundary conditions for $t=0$ and $t=T$. This is achieved by means of a change of integration variables

$$\psi=\psi_0+\psi',\quad \bar{\psi}=\bar{\psi}_0+\bar{\psi}'. \quad (30)$$

The finiteness of the time interval T introduces however an essential distinction from the calculations which are often encountered.

The steepest-descent field ψ_0 has to be chosen so that, in addition to being a solution of the Dirac equation in the external field, it should satisfy the boundary conditions imposed on ψ. These boundary conditions have to be chosen the same as the ones usually imposed on fermion fields: for $t=0$ the positive-frequency part of the fields ψ and $\bar{\psi}$ is given, whereas for $t=T$, their negative-frequency part is specified. This choice leads to the result that after the transition to the second-quantized form[7] we obtain a disentangled operator expression, where the creation operators of the particles and antiparticles, a^+, b^+ refer to the time $t=T$, and the annihilation operators a, b, refer to the time $t=0$. This form of $S_c(T)$ corresponds to the representation (1) of the evolution operator.

In Appendix I it is shown that the problem of solving the Dirac equation with the chosen boundary conditions reduces to the construction of a definite Green's function for this equation. For the case of the Coulomb gauge in which we are interested (in terms of ψ_R and ψ_L) the solution of the equations

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}-igA_0\right)\psi_{R0}=0,\quad \left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}-igA_0\right)\psi_{L0}=0, \quad (31)$$

satisfying the conditions

$$\psi_{R,L}^{(+)}(x,0)=a_{R,L}(x)=\int_0^\infty a_{R,L}(p)\,e^{\pm ipx}\frac{dp}{2\pi},$$
$$\psi_{R,L}^{(-)}(x,T)=b_{R,L}^+(x)=\int_0^\infty b_{R,L}^+(p)\,e^{\mp ipx}\frac{dp}{2\pi}, \quad (32)$$

can be written in the form

$$\psi_{R0}(x,t)=\int G_R^T(xt,x'0)\,a_R(x')\,dx'-\int G_R^{(T)}(xt,x'T)\,b_R^+(x')\,dx' \quad (33)$$

with a similar expression for ψ_{L0}. The Green's function $G_{RL}^{(T)}(xt,x't')$ satisfying the conditions formulated in Appendix I, is represented by

$$G_{R,L}^{(T)}(xt,x't')=G_{R,L}^{(0)}(xt,x't')\exp\left\{ig\int_0^T dt_1$$
$$\times\int dx_1A_0(x_1,t_1)\left[G_{R,L}^{(0)}(xt,x_1t_1)-G_{R,L}^{(0)}(x't',x_1t_1)\right]\right\}, \quad (34)$$

where $G^{(0)}_{R,L}$ denotes the usual Feynman Green's functions for free massless particles. They have the expressions

$$G^{(0)}_{R,L}=\frac{1}{2\pi i}\frac{1}{t-t'\mp(x-x')-i\delta\varepsilon(t-t')}. \tag{35}$$

After solving a similar problem for $\bar\psi$ and effecting the substitution (30) the equation (28) goes over into the following expression:

$$S_e(T)=\int DA_0(x,t)\exp\left[\frac{i}{2}\int_0^T\left(\frac{\partial A_0}{\partial x}\right)^2 d^2x\right]\exp[\Phi(a^+,b^+,a,b)]D(A), \tag{36}$$

where

$$\Phi(a^\pm,b^\pm)=\int dx\,dx'\sum_{i=R,L}[a_i^+(x)G_i^{(T)}(xT,x'0)a_i(x)$$
$$+b_i(x)G_i^{(T)}(x0,x'T)b_i^+(x')-a_i^+(x)G_i^{(T)}(x,T,x'T)b_i^+(x')$$
$$-b_i(x)G_i^{(T)}(x0,x'0)a_i(x')], \tag{37}$$

and $D(A)$ denotes the integral over ψ' and $\bar\psi'$, which is equal to the determinant of the Dirac equation in the external field A:

$$D(A)=\int D\bar\psi'\,D\psi\exp\left\{i\int_0^T dt\int dx\bar\psi'(\hat\partial-ig\hat A)\psi'\right\}. \tag{38}$$

The integral in (38) is over ψ' and $\bar\psi'$ with vanishing values for $t=0$ and $t=T$. As explained above, the a^*, b^* in Eq. (36) can again be considered as second-quantized operators; the a^*, b^* in (36) commute with the a and b, since they refer to different times.

It is well known that $D(A)$ can be expressed in terms of the Green's function of the Dirac equation in the field A, $\hat G^{(T)}(xt,xt)$, according to the formula

$$D(A)=\exp\left\{-ig\int_0^1 d\lambda\,\mathrm{Sp}\left[\int_0^T dt\int dx\hat G^{(T)}(g\lambda;xt,xt)\hat A(x,t)\right]\right\}. \tag{39}$$

However, the Green's function $\hat G^{(T)}(xt,x't')$ is undefined for equal arguments, as can be seen from the explicit expression (34), (35). The answer depends, generally speaking, on the method of taking the limit $x\to x'$, $t\to t'$. This indeterminacy, at least for $T\to\infty$, is directly related to the ultraviolet divergence of the only nonvanishing simplest loop diagram in QED_2, the one determining the photon mass.[11] The indeterminacy in the calculation of this diagram is removed by the requirement that it be gauge invariant. The same principle allows one to determine $D(A)$ for finite T also.

We require that the expression

$$S_{ex}(T)=D(A)\exp[\Phi(a,b,a^+,b^+)], \tag{40}$$

which obviously represents the evolution operator of the system of charged fermions in the external field A, should be invariant with respect to the gauge transformation (12):

$$S_{ex}(\bar A,a^\pm,b^\pm)=U(T)U^+(0)S_{ex}(A,a^\pm,b^\pm)U(0)U^\tau(T),$$
$$U(t)=\exp\left(-i\int\rho(x,t)f(x,t)dx\right). \tag{41}$$

The density $\rho(x,T)$ does not commute with $a^+(x)$ and $b^+(x)$, while $\rho(x,0)$ does not commute with $a(x)$ and $b(x)$. The result of applying the gauge transformation to the second factor in Eq. (40), to first order in the gauge function $f(x)$ is determined by the commutator of the operator $\int\rho(x)f(x)dx$ with $S_{ex}(T)$. We compute this commutator making use of the expression for the charge density [$a^*(x)$, $b^*(x)$ are defined in Eq. (32)]:

$$\rho(x)=g\sum_{i=R,L}[a_i^+(x)a_i(x)-b_i^+(x)b_i(x)+a_i^+(x)b_i^+(x)+b_i(x)a_i(x)], \tag{42}$$

and the anticommutation relations

$$\{a_R^+(x),a_R(x')\}=\{b_R^+(x),b_R(x')\}=\frac{1}{2\pi i}\frac{1}{x-x'-i\delta},$$
$$\{a_L^+(x),a_L(x')\}=\{b_L^+(x),b_L(x')\}=\frac{1}{2\pi i}\frac{1}{x'-x-i\delta}. \tag{43}$$

Separating in the commutator the terms which do not depend on the fermion operators we obtain

$$ig\int\frac{dx'}{2\pi i}\left[\frac{G_R^{(T)}(x'T,xT+\varepsilon)f(x',T)}{x'-x-i\delta}\qquad\frac{G_R^{(T)}(x\varepsilon,x'0)f(x',0)}{x-x'-i\delta}\right]$$
$$+(\text{terms with } R\to L),\quad \varepsilon\to 0. \tag{44}$$

On account of Eq. (41) these terms must be compensated by a gauge transformation of $D(A)$ (the terms with the operators a^*, b^* yield the transformation law for the Green's functions). The Green's functions $G^{(T)}$ in the arbitrary gauge entering Eq. (44) can be defined according to Eq. (34) if one replaces A_0 there by A_0-A_1 for $G_R^{(T)}$ and by A_0+A_1 for $G_L^{(T)}$. Integrating in Eq. (44) with respect to x and x', we obtain

$$D(\bar A)-D(A)=-\frac{g^2}{2\pi}\int\frac{dp}{2\pi}\int_0^T dt[A_0(p,t)-\varepsilon(p)A_1(p,t)]$$
$$\times[f(-p,T)\exp(-i|p|(T-t))-f(-p,0)\exp(-i|p|t)]. \tag{45}$$

The condition (45) leads to a unique expression for $D(A)$:

$$\ln D(A)=-\frac{1}{2}m^2\int\frac{dp}{2\pi}\int_0^T dt\,dt'\left\{p^2A_0(p,t)A_0(-p,t')\right.$$
$$+A_1(p,t)A_1(-p,t')\frac{\partial}{\partial t}\frac{\partial}{\partial t'}+ip\left(A_1(p,t)A_0(-p,t')\frac{\partial}{\partial t}\right.$$
$$\left.\left.-A_0(p,t)A_1(-p,t')\frac{\partial}{\partial t'}\right)\right\}\frac{\exp(-i|p||t-t'|)}{2|p|},$$
$$m^2=g^2/\pi. \tag{46}$$

This expression is relativistically invariant and for $T\to\infty$ it goes over into the usual expression for the simplest loop diagram of QED_2[11]

$$\ln D(A)=\frac{m^2}{2}\int\left(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)A_\mu(p)A_\nu(-p)\frac{d^2p}{(2\pi)^2 i}. \tag{47}$$

One could have obtained the result (46) in another way, requiring that the evolution operator $S_{ex}(T)$ should satisfy the time-dependent Schrödinger equation with a Hamiltonian describing the interaction of charged particles with an external EM field. The corresponding calculation leads to the following interesting expression of $D(A)$ in terms of the external-field Green's function:

$$D(A)=\exp\left\{-ig\int dx\int_0^T dt\lim_{x\to x'}\lim_{\varepsilon\to 0}\mathrm{Sp}[\hat G^{(t)}(xt,x't+\varepsilon)\hat A(x,t)]\right\}. \tag{48}$$

In distinction from Eq. (39), the limiting procedure for equal values of the arguments in $G^{(t)}(xt,x't')$ in (48) is completely determined.

The fact that the system has been considered in a fin-

ite volume did not introduce any modifications of principle in the calculations of this section.

4. THE FINAL FORM OF THE EVOLUTION OPERATOR IN THE COULOMB GAUGE

In order to obtain a concrete matrix element of the evolution operator it is necessary to expand the operator exponential in Eq. (36) in a series with respect to the operators $a^*(x)$, $b^*(x)$ and, separating the appropriate term, take the corresponding integral with respect to A_0. Since the Green's function $G^{(T)}_{R,L}(xt, x't')$ depends exponentially on A_0, this integral will be of the Gaussian type:

$$I=\int DA_0(p_n,t)\exp\left[is_{eff}(A_0)+\frac{ig}{2L}\sum_n\int_0^T dtR(-p_n,t)A_0(p_n,t)\right]; \quad (49)$$

here s_{eff} is the effective action for the EM field A_0, taking account of the determinant $D(A)$, (46):

$$s_{\text{eff}}(A_0)=\frac{1}{2L}\sum_{n\neq 0}\frac{p_n^2}{2}\int_0^T A_0(p_n,t)\left[\delta(t-t')+\frac{im^2}{2|p_n|}\exp(-i|p_n||t-t'|)\right]A_0(-p_n,t')\,dt\,dt'. \quad (50)$$

We carry out the calculations in a finite volume, hence the integrals in Eqs. (49), (50) with respect to p have been replaced by sums.

The element $S_c(x_k,y_k)$ of the evolution operator, being the coefficient of the operators

$$\left(\prod_{k=1}^{n_R}\int dx_k a_R^+(x_k)\right)\left(\prod_{k=1}^{m_R}\int dx_k' b_R^+(x_k')\right)\left(\prod_{k=1}^{n_L}\int dy_k a_L^+(y_k)\right)\left(\prod_{k=1}^{m_L}\int dy_k' b_L^+(y_k')\right)|0\rangle,$$

$$\langle 0|\left(\prod_{k=1}^{\tilde n_R}\int d\bar x_k a_R(\bar x_k)\right)\left(\prod_{k=1}^{\tilde m_R}\int d\bar x_k' b_R(\bar x_k')\right)\left(\prod_{k=1}^{\tilde n_L}\int d\bar y_k a_L(\bar y_k)\right)\left(\prod_{k=1}^{m_L}\int d\bar y_k' b_L(\bar y_k')\right)\times S_c(x_k;x_k';y_k;y_k';\bar x_k;\bar x_k';\bar y_k;\bar y_k'), \quad (51)$$

turns out to be equal to the product

$$S_c(x_k,y_k)=S_0(x_k,y_k)I(x_k,y_k), \quad (52)$$

where $S_0(x_k,y_k)$ is the indicated element of the evolution operator in a free theory. It can be obtained by series expansion of the expression

$$S_0(T)=\exp\left\{\int\frac{dx\,dx'}{2\pi i}[(T-x+x'-i\delta)^{-1}[a_R^+(x)a_R(x')+b_R^+(x)b_R(x')+a_L^+(x')a_L(x)+b_L^+(x')b_L(x)]+(x-x'-i\delta)^{-1}[a_R^+(x)b_R^+(x')+a_R(x)b_R(x')+a_L^+(x')b_L^+(x)+a_L(x')b_L(x)]]\right\} \quad (53)$$

[the equation (40), with the free Green's functions $G^{(0)}_{R,L}$, (35), substituted for $G^{(T)}_{R,L}$ and with the determinant $D(A)$ omitted]. $I(x_k,y_k)$ is an integral of the form (49). The role of the source $R(p_n,t)$ in it is played by the coefficients with which the EM field A_0 enters into the Green's functions $G^{(T)}_{R,L}(xt,x't')$ which appear in a series expansion of Eq. (36). In order to determine them we rewrite Eq. (34) in the momentum representation:

$$G^{(T)}_{R,L}(xt,x't')=G^0_{R,L}(xt,x't')\exp\left\{\frac{ig}{2L}\sum_n\int_0^T dt_1A_0(\pm p_n,t_1)[\theta(p_n)(\theta(t-t_1)\times\exp(\pm ip_nx)-\theta(t'-t_1)\exp(\pm ip_nx'))-\theta(-p_n)(\theta(t-t_1)\exp(\pm ip_nx)-\theta(t_1-t')\exp(\pm ip_nx'))]\right\}. \quad (54)$$

Thus, for the matrix element (51) of the evolution operator the source $R(p_n,t)$ turns out to be equal to

$$R(p_n,t)=R_i(p_n)\exp(-i|p_n|t)+R_f(p_n)\exp(-i|p_n|(T-t)),$$
$$R_i(p_n)=\sum_k\{\theta(p_n)[\exp(-ip_n\tilde x_k)-\exp(-ip_n\tilde x_k')]+\theta(-p_n)[\exp(-ip_n\tilde y_k)-\exp(-ip_n\tilde y_k')]\},$$
$$R_f(p_n)=\sum_k\{\theta(-p_n)[\exp(-ip_nx_k)-\exp(-ip_nx_k')]+\theta(p_n)[\exp(-ip_ny_k)-\exp(-ip_ny_k')]\}. \quad (55)$$

In particular, for zero momentum, we obtain from here

$$gR_i(0)=gR_f(0)=Q, \quad (56)$$

where Q is the total charge of the state under consideration.

The Gaussian integral (49) can be done in general form. It should be taken into account that, as already noted in Sec. 2, the variable $A_0(0,t)$ is not independent and must be expressed in terms of the other variables according to Eq. (19). We then obtain, making use of Eq. (56):

$$I(x_k,y_k)=\exp\left\{\frac{i}{4L}\int_0^T dt\sum_{n\neq 0}\bar A_0(p,t)[gR(-p_n,t)-(-1)^nQ]\right\}, \quad (57)$$

where the steepest-descent field $\bar A_0(p_n,t)$ satisfies the equation

$$\bar A_0(p_n,t)+\frac{im^2}{2|p_n|}\int_0^T\exp(-i|p_n||t-t'|)\bar A_0(p_n,t')\,dt'=\frac{(-1)^nQ-gR(p_n,t)}{p_n^2}. \quad (58)$$

It is easy to pass from the integral equation (58) to a differential equation. Applying the operator $\partial^2/\partial t^2+p_n^2$ to both sides we arrive at the expression

$$\left[\frac{\partial^2}{\partial t^2}+\omega_n^2\right]\bar A_0(p_n,t)=Q(-1)^n,\qquad \omega_n^2=p_n^2+m^2. \quad (59)$$

Boundary conditions to Eq. (59) are determined from Eq. (58) and have the form

$$\dot{\bar A}_0(p_n,T)+i|p_n|\bar A_0(p_n,T)=\frac{2i}{|p_n|}[Q(-1)^n-gR_f(p_n)],$$
$$\dot{\bar A}_0(p_n,0)-i|p_n|\bar A_0(p_n,0)=-\frac{2i}{|p_n|}[Q(-1)^n-gR_i(p_n)]. \quad (60)$$

Solving the equation (59) we find the steepest-descent field:

$$\bar A_0(p_n,t)=\frac{Q(-1)^n}{\omega_n^2}+\frac{m^2\omega_n^{-2}Q(-1)^n-gR_i(p_n)-gR_f(p_n)}{|p_n|(\omega_n+|p_n|)}\times\frac{\exp(-i\omega_nt)+\exp(-i\omega_n(T-t))}{1-\Omega_n}+g\frac{R_f(p_n)-R_i(p_n)}{|p_n|(\omega_n+|p_n|)}\times\frac{\exp(-i\omega_n(T-t))-\exp(-i\omega_nt)}{1+\Omega_n} \quad (61)$$

where we have used the notation

$$\Omega_n=\left(\frac{\omega_n-|p_n|}{\omega_n+|p_n|}\right)\exp(-i\omega_nT).$$

We note that in distinction from the initial Coulomb potential $\bar A_0(p_n,t)$ contains the singularity $1/|p_n|$ rather than $1/p_n^2$ which leads only to a logarithmic, rather than linear, growth of the potential at infinity.

Substituting Eq. (61) into Eq. (57) and integrating with respect to time, we obtain, finally

$$I(x_k,y_k)=[\operatorname{Det}s_{eff}^{-1}]^{1/2},$$
$$S_Q\exp\left\{-\frac{\pi}{2L}\sum_{n\neq 0}[2F_1(p_n)R_f(-p_n)R_i(p_n)+F_2(p_n)(R_i(p_n)R_i(-p_n)+R_f(p_n)R_f(-p_n))]\right\}, \quad (62)$$

where the functions $F_1(p_n)$ and $F_2(p_n)$ have the form

$$F_1(p_n)=\frac{\exp(-i|p_n|T)}{|p_n|}-\frac{4\omega_n}{(\omega_n+|p_n|)^2}\frac{\exp(-i\omega_n T)}{1-\Omega_n^2},$$
$$F_2(p_n)=\frac{1}{|p_n|}\frac{\omega_n-|p_n|}{\omega_n+|p_n|}\frac{1-\exp(-2i\omega_n T)}{1-\Omega_n^2}, \quad (63)$$

and S_Q is that part of the evolution operator for $Q\neq 0$ which is related to the presence of the capacitor:

$$S_Q=\exp\Big[-\frac{i}{4L}\sum_{n\neq 0}\frac{Q^2}{\omega_n^2}T\Big]\exp\Big\{\frac{1}{2L}\sum_{n\neq 0}\frac{1-\exp(-i\omega_n T)}{1-\Omega_n}$$
$$\times\frac{m^2\omega_n^{-2}Q^2+2gQ(-1)^n(R_i(p_n)+R_j(p_n))}{|p_n|\omega_n(\omega_n+|p_n|)}\Big\}. \quad (64)$$

The equations (52), (53), (62), (64) make it possible to obtain an arbitrary matrix element of the evolution operator (51) in the Schwinger model. The determinant Det s_{eff} is calculated in Appendix II:

$$[\text{Det}\, s_{\text{eff}}]^{-1/2}=Z^{-1}e^{-iE_0T}. \quad (65)$$

The quantity E_0 represents the energy of the physical vacuum, with the energy of the bare vacuum $|0\rangle$ being taken as zero. According to (A.II.13) it equals

$$E_0=\frac{1}{2}V\int(\omega-|p|)\frac{dp}{2\pi}. \quad (66)$$

E_0 diverges logarithmically in the ultraviolet region. The equation (66) agrees with the answer which can be obtained directly considering the only nonvanishing vacuum diagram in QED_2. The normalization constant

$$Z=\Big(\frac{2}{mV}\Big)^{1/2}\exp\Big(\frac{4-\pi}{2\pi}mV\Big) \quad (67)$$

is the partition function for the physical QED_2 vacuum. At the same time $Z^{-1/2}$ is the transition matrix element from the bare vacuum into the physical vacuum. The fact that it becomes exponentially small with the volume V signifies that in the physical vacuum there appears a finite density of bare particles, approximately equal to the exponent of the exponential in Eq. (67), divided by V.

We finally generalize the results we have obtained to the Segrè-Weisberger model.[4] As shown by Eq. (36), for this it is necessary to obtain expressions for the Green's functions $G^{(T)}(xt,x't')$ of each kind of quark, and for the determinant $D(A)$ in this model. It is obvious that the expressions for the Green's functions remain in force if one replaces in Eq. (33) the charge g by g_i, where g_i is the charge of the i-th kind of quark. Now all types of quarks contribute to the determinant $D(A)$. For this determinant we obtain again the formula (46), where m has to be replaced by

$$m^2=\frac{1}{\pi}\sum_{i=1}^{N}g_i^2, \quad (68)$$

where m is the mass of the vector boson which appears in this model.

We now obtain an expression for the matrix element of the evolution operator of the form (51). We attach to each spatial coordinate entering into Eq. (51) an additional superscript j, i.e., we denote them by $x_k^{(j)}$, $y_k^{(j)}$, superscript which labels the N kinds of quarks. Repeating verbatim the calculations which were carried out above we obtain

$$S_c(x_k^{(j)},y_k^{(j)})=I(x_k^{(j)},y_k^{(j)})\prod_{j=1}^{N}S_0(x_k^{(j)},y_k^{(j)}),$$
$$I(x_k^{(j)},y_k^{(j)})=[\text{Det}\, s_{eff}]^{-1/2}$$
$$\times S_Q\exp\Big\{-\frac{\pi}{4L}\sum_{n\neq 0}\sum_{j,l=1}^{N}\eta_j\eta_l[2F_1(p_n)R_f^{(j)}(-p_n)R_i^{(l)}(p_n)$$
$$+F_2(p_n)(R_i^{(j)}(p_n)R_i^{(l)}(-p_n)+R_f^{(j)}(p_n)R_f^{(l)}(-p_n))]\Big\}. \quad (69)$$

Here we have introduced the dimensionless constants

$$\eta_j^2=g_j^2/\pi m^2=g_j^2\Big/\sum_{j=1}^{N}g_j^2 \quad (70)$$

in place of the coupling constants g_j. The quantities $R_i^{(j)}(p_n)$ and $R_f^{(j)}(p_n)$ for each kind of quark are determined by Eq. (55); Det s_{eff} does not depend on η_j and is given by Eq. (65).

The diagonalization of the evolution operators (57) and (69) allows us to determine all the physical properties of both models in the quark representation and to describe their properties. Without getting involved here in this tedious probelm, we indicate only that the appearance in $S(T)$ of a time-dependence of the type $e^{-i\omega T}$ and $e^{-i\rho T}$ [cf. Eq. (63)] means that the spectrum of these models consists of massive [with mass (46) or (68)] and massless excitations (cf. Ref. 4). One can show, however, that in the Schwinger model the massless excitations disappear in fact from the spectrum.[2-3]

In conclusion we would like to thank Ya. I. Azimov, V. N. Gribov, and L. L. Frankfurt for numerous useful discussions.

APPENDIX I

The solution of the Dirac equation

$$[\hat{\partial}-igA(x)]\psi=0, \quad (I.1)$$

depending on an arbitrary function

$$\psi(x)=\Phi(x)-\int_0^T\hat{G}^{(T)}(x,x')(\hat{\partial}'-igA(x'))\Phi(x')d^2x', \quad (I.2)$$

can be expressed, on account of the equations for the external-field Green's function:

$$[\hat{\partial}-igA(x)]\hat{G}^{(T)}(x,x')=\delta^{(2)}(x-x'),$$
$$G^{(T)}(x,x')[\hat{\partial}'+igA(x')]=-\delta^{(2)}(x-x') \quad (I.3)$$

in terms of the boundary values $\Phi(x,0)$ and $\Phi(x,T)$:

$$\psi(x,t)=\int G^{(T)}(xt,x'0)\gamma^0\Phi(x',0)dx'-\int G^{(T)}(xt,x'T)\gamma^0\Phi(x',T). \quad (I.4)$$

In order that the solution $\psi(x,t)$ should be determined by prescribed positive-frequency part of $\Phi(x,0)$ and negative frequency part of $\Phi(x,T)$ one must select a Green's function satisfying sufficiently complicated boundary conditions.

We introduce the notations $\hat{G}_{++}^{(T)}$, $\hat{G}_{+-}^{(T)}$, $\hat{G}_{-+}^{(T)}$, $\hat{G}_{--}^{(T)}$, where the first subscript denotes the positive-frequency (+) or negative-frequency (−) part of the function $\hat{G}^{(T)}(x,t,x',t')$ related to its dependence on the variable t. The second subscript has a similar interpretation for the variable t'. Then the boundary conditions for the Green's function we are looking for can be written in the form:

a) for $t=0$

$$\hat{G}^{(T)}_{+-}=\hat{G}^{(T)}_{++}=0 \text{ for } 0<t'\leqslant T,$$
$$\hat{G}^{(T)}_{+-}(x\varepsilon, x'0)=\gamma^0\delta(x-x') \quad (\varepsilon\to+0); \tag{I.5a}$$

b) for $t=T$

$$\hat{G}^{(T)}_{-+}=\hat{G}^{(T)}_{--}=0 \text{ for } 0\leqslant t'<T,$$
$$\hat{G}^{(T)}_{-+}(xT, x'T+\varepsilon)=-\gamma^0\delta(x-x'). \tag{I.5b}$$

These conditions mean that outside the time interval $(0, T)$ the system is in the vacuum state with switched-off field. Therefore from the surfaces 0 and T signals can propagate into the interval $(0, T)$ only if on these surfaces there are sources at x and x', defined by the well-known representation

$$\hat{G}(x, x')=\langle 0|T\{\psi(x)\bar{\psi}(x')\}|0\rangle \tag{I.6}$$

in terms of the creation-annihilation operators of particles and antiparticles in terms of which $\psi(x)$ and $\psi(x')$ are expressed at the times 0 and T.

A Green's function with such boundary conditions can be uniquely determined. Its properties guarantee indeed the selection of a solution determined by the boundary conditions above for $\Phi_+(x, 0)$ and $\Phi_-(x, 0)$.

APPENDIX II

The operator s_{eff} in Sec. 4 was represented as the ratio of two differential operators. Therefore its determinant can be written in the form

$$\operatorname{Det} s_{eff}=\prod_p \frac{\operatorname{Det}\Delta_\omega}{\operatorname{Det}\Delta_p}\equiv\prod_p\frac{D_\omega}{D_p}, \tag{II.1}$$

where we have introduced the notations

$$\Delta_\omega=\partial^2/\partial t^2+\omega^2, \quad \Delta_p=\partial^2/\partial t^2+p^2. \tag{II.2}$$

These operators are defined on the class of functions satisfying the homogeneous boundary conditions (60):

$$A(p, T)+i|p|A(p, T)=0, \quad A(p, 0)-i|p|A(p, 0)=0. \tag{II.3}$$

By definition

$$D_\omega^{-1/2}=\int DA(p, t)\exp\left\{i\int_0^T A(p, t)\Delta_\omega A(-p, t)dt\right\}. \tag{II.4}$$

The integration $DA(p, t)$ in (II.4) is taken only over functions satisfying (II.3). In order to carry out this integration we represent the arbitrary $A(p, t)$ in the form

$$A(p, t)=\varphi(p, t)+a(p, t), \tag{II.5}$$

where

$$\Delta_\omega\varphi(p, t)=0, \quad \varphi(p, 0)=A(p, 0), \quad \varphi(p, T)=A(p, T). \tag{II.6}$$

Solving Eq. (II.6) we obtain

$$\varphi(p, t)=A(p, 0)\frac{\sin\omega(T-t)}{\sin\omega T}+A(p, T)\frac{\sin\omega t}{\sin\omega T}. \tag{II.7}$$

In turn $a(p, t)=0$ for $t=0$ and $t=T$. We substitute (II.5–7) into (II.4) and separate the integration with respect to $A(p, 0)$ and $A(p, T)$. Then

$$D_\omega^{-1/2}=\int DA(p, 0)DA(p, T)\exp\{i[\varphi(T)\dot{a}(T)-\varphi(0)\dot{a}(0)]\}$$
$$\times\int Da(p, t)\exp\left[i\int_0^T a(-p, t)\Delta_\omega a(p, t)dt\right]. \tag{II.8}$$

According to (II.3), (II.7–8) we have

$$\varphi(T)\dot{a}(T)-\varphi(0)\dot{a}(0)=-[A^2(p, T)+A^2(p, 0)](i|p|+\omega\operatorname{ctg}\omega T)$$
$$+2A(p, T)A(p, 0)\omega/\sin\omega T. \tag{II.9}$$

Taking the integral with respect to $A(p, 0)$ and $A(p, T)$ and taking into account the fact that the determinant of the functional integral in (II.8) equals[8]

$$D_\omega^{(0)}=\frac{\sin\omega T}{\omega}, \tag{II.10}$$

we obtain for D the expression

$$D_\omega=(p^2-2i\omega|p|\operatorname{ctg}\omega T-\omega^2)\frac{\sin\omega T}{\omega}. \tag{II.11}$$

Consequently Det s_{eff} equals

$$\operatorname{Det} s_{\text{eff}}=\prod_p\frac{D_\omega}{D_p}=\prod_p\frac{(\omega+|p|)^2}{4|p|\omega}\exp[i(\omega-|p|)T]$$
$$\times\left[1-\left(\frac{\omega-|p|}{\omega+|p|}\right)^2\exp(-2i\omega T)\right]. \tag{II.12}$$

The product is over all p except $p=0$, since as was indicated in Section 2, there is no integration with respect to $A_0(0, t)$ in the integrals (25), (26). In a finite volume the product in Eq. (II.12) is over integral values of n: $p_n=n\pi/L$. The calculation of (II.23) for $T\gg 1/m$ up to terms of order $1/L$ leads to the result

$$[\operatorname{Det} s_{eff}]^{-1/2}=(mL)^{1/2}\exp\left(-\frac{4-\pi}{\pi}mL\right)\exp\left[-iTL\int\frac{dp}{2\pi}(\omega-|p|)\right]. \tag{II.13}$$

[1)]Owing to the obvious analogy we shall call them quarks in the sequel.

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Translated by Meinhard E. Mayer