Distribution function of molecular fields in a disordered Ising model

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The distribution function of molecular fields in a disordered Ising model is studied in the Bethe-Peierls approximation. It is shown that the equation for this function has branch points, at which states different from the paramagnetic are generated. The density of the distribution is investigated in the vicinity of these points. The results are discussed in the case of exchange interactions between magnetic atoms in alloys of the type PdFe and CuMn.

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§1. INTRODUCTION

In disordered magnetic systems, the statistical properties are primarily determined by the character of the distributions of the random parameters of the system, such as the exchange integrals, the coordinates of the magnetic atoms, the anisotropy constants, etc. The thermodynamic quantities are then functionals of the distribution densities of these parameters. Finding the densities, which are not given, becomes unavoidable at some stage of the study of the thermodynamics of such systems.

In the method of fluctuating molecular fields, the chief random parameter with an unknown distribution is the molecular field $\mathbf{H}_j$ acting on a localized magnetic moment (j is the number of the moment). The distribution density $P_H(\mathbf{H}_j)$ of this random field, in the majority of theoretical papers, has been investigated in the high- and low-temperature ranges.[1-3] Meanwhile, study of the behavior of the molecular-field distribution near a transition point [see, for example, Ref. 4, where $P_H(\mathbf{H}_j)$ in a Bethe lattice is treated near the percolation threshold] may shed light on some interesting problems of the thermodynamics of disordered magnetic materials.

In the present paper, the distribution density of the molecular fields near a transition point (the ordering temperature is determined in § 2) is studied for the Ising model in the Bethe-Peierls approximation. It is shown that to the Ginzburg-Landau equation there corresponds an equation of branching[5] of the nonlinear integral equation for $P_H(\mathbf{H}_j)$, whose solution makes it possible to find the cumulants of the distribution over a quite wide band of the phase plane.

In § 2 the model is described, the equations for $P_H(\mathbf{H}_j)$ and for the cumulants are given, and the bifurcation problem is formulated. The bifurcation points, at which states different from the paramagnetic are generated, are calculated in § 3. In § 4 the branching equation is solved. In § 5 the final expressions are given for the cumulants of the molecular-field distribution in the vicinity of the bifurcation points, for an arbitrary exchange interaction. In §6 the results obtained are discussed in the case of the interactions that occur in alloys of the type PdFe and CuMn.

We note that the molecular-field distribution function can be obtained experimentally from analysis of Mössbauer spectra,[6] from experiments on inelastic scattering of neutrons,[7] from depolarization of μ-mesons,[8] and from numerical experiments on computers.[9]

§2. EQUATIONS FOR THE MOLECULAR-FIELD DISTRIBUTION DENSITY AND FOR THE CUMULANTS; THE BIFURCATION PROBLEM

We considered a model of a disordered magnet in which randomly located magnetic atoms, with a spatial distribution density $V^{-1}P_R(|\mathbf{R}_i-\mathbf{R}_j|)$, interact with an arbitrary exchange interaction $J(|\mathbf{R}_i-\mathbf{R}_j|)$, where $\mathbf{R}_i$ is the coordinate of the ith magnetic atom, and where V is the volume of the system.

In the Bethe-Peierls approximation,[10] the system is described by the Hamiltonian

$$\mathscr{H}=-\sum_{j=1}^{N-1} J(|\mathbf{R}_0-\mathbf{R}_j|)\sigma_0\sigma_j-\sum_{j=1}^{N-1} H_j\sigma_j,$$

where σ_j are Ising operators, H_j is the molecular field, and N is the number of spins in the system. The exchange interaction $J(R_{ij})$, where $R_{ij}=|\mathbf{R}_i-\mathbf{R}_j|$, may be either constant or variable in sign. For example,

$$J(R_{ij})=J_0\exp(-R_{ij}/r_0), \tag{1}$$

$$J(R_{ij})=J_0F(2k_FR_{ij}),\quad F(x)=(\sin x-x\cos x)/x^4, \tag{2}$$

(k_F is the Fermi momentum).

The exponentially decreasing interaction (1) and the oscillating Ruderman-Kittel-Kasuya-Yosida (RKKY) interaction (2) occur between magnetic impurities in alloys of transition and normal metals, of the type PdFe and CuMn.

The self-consistency procedure $\langle\sigma_0\rangle_T=\langle\sigma_i\rangle_T$, i 1,2,..., $N-1$, where the angular brackets denote thermodynamic averaging (T is the temperature), connects the molecular fields on different spins and the random distances between them through the equations

$$\begin{gathered} q_i=\frac{H_i}{T},\quad q_i-\sum_{j\neq i} B(q_j,R_{0j};T)=0,\quad i=1,2,\ldots,N-1, \\ B(q_j,R_{0j};T)=\frac{1}{2}\ln\frac{1+\operatorname{th}(q_j)\operatorname{th}[T^{-1}J(R_{0j})]}{1-\operatorname{th}(q_j)\operatorname{th}[T^{-1}J(R_{0j})]}. \end{gathered} \tag{3}$$

In the molecular-field approximation,

$$B(q_j,R_{0j};T)=T^{-1}J(R_{0j})\operatorname{th}(q_j). \tag{4}$$

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Obviously the distribution density $P_q(q_i)$ of the dimensionless random molecular field q_i should be expressible in terms of $P_R(R_{0j})$ by some equation based on (3). For $N \to \infty$ and $V^{-1}N = n = \text{const}$ (n is the concentration of spins), this equation has the following form[1-4]

$$P_q(q) = \frac{1}{2\pi}\int d\rho \exp\left\{-i\rho q + \nu \int P_q(x)\langle \exp[i\rho B(x,R;T)] - 1\rangle_R\, dx\right\}. \quad (5)$$

The angular brackets denote configurational averaging:

$$\nu\langle \ldots \rangle_R = n\int \ldots P_R(R_{ij})\, d^3R_{ij},$$

where ν is the mean number of magnetic atoms in the "interaction volume": $\nu = \frac{4}{3}\pi r_0^3 n$ and $\nu = \frac{4}{3}\pi n/(2k_F)^3$ for the interactions (1) and (2), respectively. For the lattice model of disordered magnets, according to which the magnetic atoms are located on sites of a regular lattice, while the exchange constants J_{ij} fluctuate, the configurational average must be replaced by an average over the distribution density $P_J(J_{ij})$:

$$\nu\langle \ldots \rangle_R \to \langle \ldots \rangle_J.$$

Equation (5) is derived on the assumption of statistical independence of the molecular fields. This assumption is evidently justified for long-range interactions within whose sphere of action there are a large number of randomly and independently distributed spins, which create the molecular field at the given spin. It is discussed in detail in Refs. 3 and 4 and is a basic approximation of the model.

It is convenient to work not with $P_q(q)$ but with the logarithm $\Lambda(y)$ of its characteristic function:

$$P_q(q) = \frac{1}{2\pi}\int \exp[\Lambda(y) - iqy]\, dy.$$

From (5) we have the integral equation for $\Lambda(y)$:

$$\Lambda(\rho) = \nu\int K(\rho, y; \theta) \exp \Lambda(y)\, dy \quad (6)$$

with kernel

$$K(\rho, y; \theta) = \frac{1}{2\pi}\int \langle \exp[i\rho B(x, R; \theta)] - 1\rangle_R\, e^{-ixy}\, dx,$$

where $\theta = T/J_0$ is the dimensionless temperature.

Starting from the properties of the function $B(x, R; \theta)$, one can find a representation (the $\varkappa$ representation) for which the linear integral operator with kernel $K(\rho, y; \theta)$ corresponds to a triangular matrix. On carrying out a cumulant expansion in (6), we get

$$\varkappa_k = \nu\langle\langle B^k(q, R; \theta)\rangle_R\rangle_q, \quad k = 1, 2, \ldots, \quad (7)$$

or

$$\varkappa_k = \nu\int \Gamma_k(y) \exp\left[\sum_{m=1}^{\infty} \frac{\varkappa_m}{m!}(iy)^m\right] dy,$$
$$\Gamma_k(y) = \frac{1}{2\pi}\int \langle B^k(x, R; \theta)\rangle_R\, e^{-ixy}\, dx, \quad (8)$$

where $\varkappa_k$ are the cumulants of the distribution of the dimensionless molecular field. Furthermore, on passing from cumulants to moments $\eta_m = \langle q^m\rangle_q$ in the right side of (8), we arrive at the following algebraic equation:

$$\varkappa = \nu\gamma(\theta)\eta(\varkappa), \quad (9)$$

where $\varkappa$ and η are vectors with components $\varkappa_k$ and η_k, $k = 1, 2, \ldots$, and where γ is a triangular matrix with elements

$$\gamma_{km} = \frac{1}{m!}\left\langle \frac{d^m}{dx^m} B^k(x, R; \theta)\big|_{x=0}\right\rangle_R, \quad k, m = 1, 2, \ldots, \quad (10)$$

$$\gamma_{kk} = \langle \text{th}^k [T^{-1}J(R)]\rangle_R.$$

In the molecular-field approximation, one can obtain from equation (8) the Sherrington-Kirkpatrick (SK) equation[11,12] for the lattice model of a spin glass with a normal distribution of exchange constants J_{ij}. In fact, it was shown by Klein[3] that the SK model corresponds to a normal distribution of molecular fields, $\varkappa_k = 0$ for $k \geq 3$. In the molecular-field approximation (4), we have from (7)

$$\varkappa_k = \nu T^{-k}\langle J^k(R)\rangle_R \langle \text{th}^k(q)\rangle_q, \quad k = 1, 2, \ldots,$$

and the first two equations (8) coincide with the SK equations.

The algebraic equation (9) is more graphic than the integral equation (6). From it one can trace the character of the solutions obtained in § 5. But in a study of the questions of existence of the necessary solutions, it is more convenient to operate with integral operators than with large matrices.

It is easy to see that

$$\int K(\rho, y; \theta)\, dy = 0,$$

therefore the paramagnetic solution $\Lambda(y) \equiv 0$ is satisfied over the whole phase plane (ν, θ). Suppose that we move along the phase plane from a paramagnetic point that is not a characteristic point for the kernel $K(\rho, y; \theta)$ (Fig. 1). The function $\Lambda(y) \equiv 0$ is an analytical solution of equation (6) in the vicinity of this point. This trivial solution can be continued analytically as far as the first real eigenvalue $\nu_c^{-1}(\theta)$ of the kernel $K(\rho, y; \theta)$. The point of generation of a nonparamagnetic solution can be only $\nu_c^{-1}(\theta)$. If among the branches that appear there is a nontrivial one, then $\nu_c^{-1}(\theta)$ will be a bifurcation point of equation (6). Thus the ordering temperature must be sought among the points of bifurcation of the equation for the logarithm of the eigenfunction of the distribution density of the molecular fields. It is in general difficult to predict what will happen to the states that originate at the bifurcation points on further motion along the

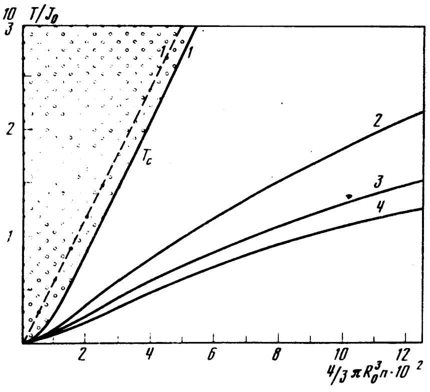

FIG. 1. Phase diagram for the model with the exponential interaction (1). The circles denote the paramagnetic region, the solid curves the eigenvalues $\nu_k(T/J_0)$ ($k = 1, 2, 3, 4$) of the operator $\mathbf{K}(T/J_0)$, Curve 1 the line of ferromagnetic ordering; the dotted straight line corresponds to the molecular-field approximation. On the axis of abscissas, $R_0 = 1/2k_F$ is written r_0 in the text of this article.

phase plane. Possibilities are a second branching before attainment of the following eigenvalue, a new branching from this eigenvalue, a continuous transition of different branches one to another, etc.

Let (ν, θ) be an arbitrary point of the phase plane. At this point the paramagnetic solution $\Lambda(y) \equiv 0$ is known. We are interested in all continuous solutions of equation (6) at a sufficiently near point

$$(\nu+\operatorname{sign}(\lambda)|\lambda|\nu,\ \theta+\operatorname{sign}(\tau)|\tau|\theta)$$

[here sign (x) is the sign of x] that tend to the paramagnetic solution when $|\lambda|$ and $|\tau|$ tend to zero. We also require boundedness and hermiticity of the eigenfunctions.

The smallness of $|\lambda|$ and $|\tau|$ permits expansion of the kernel of the integral equation (6) in these parameters. As a result we arrive at the equation

$$\Lambda-\nu\mathbf{K}(\theta)\Lambda=\sum_{m+n\geqslant 2}\operatorname{sign}^n(\varepsilon)|\varepsilon|^n\mathbf{K}_{mn}(\theta)\Lambda^m, \tag{11}$$

where $\mathbf{K}(\theta)$ and $\mathbf{K}_{mn}(\theta)$ are linear integral operators with kernels $K(\rho, y; \theta)$ and $K_{mn}(\rho, y; \theta)=\mathbf{\Omega}_{mn}(\rho, y; \theta)$; here

$$\mathbf{\Omega}_{mn}=\frac{1}{m!n!}\theta^n\frac{d^n}{d\theta^n}$$

in the case of fixed concentration $\varepsilon\equiv\tau$, $\lambda=0$ and

$$\mathbf{\Omega}_{mn}=\frac{1}{m!}(\delta_{0n}+\delta_{1n})$$

in the case of fixed temperature $\varepsilon\equiv\lambda$, $\tau=0$. We shall not discuss the trivial case of analytic continuation from this paramagnetic point but shall consider only branching that go out from singular points. For this purpose, it is necessary first to find these points.

§3. EIGENVALUES AND EIGENFUNCTIONS OF THE INTEGRAL OPERATOR $K(T/J_0)$

In the $\varkappa$ representation, a linear integral equation

$$\mathbf{K}(\theta)L=\nu^{-1}L$$

can be obtained from (9) by linearization of the function $\eta(\varkappa)$:

$$\gamma(\theta)\varkappa=\nu^{-1}\varkappa.$$

In this representation, the operator $\mathbf{K}(\theta)$ is triangular, and this enables us to find its eigenvalues directly. They form a real spectrum

$$\nu_k^{-1}(\theta)=\langle\operatorname{th}^k[T^{-1}J(R)]\rangle_R, \quad k=1, 2, \ldots \tag{12}$$

We consider the "associated" equation

$$\mathbf{K}^T(\theta)M=\nu^{-1}M,$$

where $\mathbf{K}^T(\theta)$ is the transposed operator; $\operatorname{Im}K(\rho, y; \theta)=0$. We introduce the Fourier transforms of the eigenfunction,

$$M(y)=\frac{1}{2\pi}\int m(x)e^{-ixy}\,dx$$

and of the kernel $K(\rho, y; \theta)$ according to formula (6). Then the "associated" equation transforms to the following:

$$\langle m[B(x, R; \theta)]-m(0)\rangle_R=\nu^{-1}m(x).$$

It is easy to verify that the eigenfunctions of this equation,

$$m_k(x)=\chi_0^{(k)}\operatorname{th}^k(x), \quad k=1, 2, \ldots,$$

where $\chi_0^{(k)}$ are arbitrary constants, are realized at the eigenvalues (12).

In order to find the eigenfunctions of the operator $\mathbf{K}(\theta)$, we shall use their orthogonality to M_k. In the $\varkappa$ representation,

$$(M_k, L_i)_1=\chi^{(k)}\varkappa^{(i)},$$

where the index 1 means that the scalar product is taken with weighting function 1,

$$\frac{\chi_m^{(k)}}{\chi_0^{(k)}}=\frac{(-1)^m}{m!}\cdot\frac{d^m}{dx^m}\operatorname{th}^k(x)|_{x=0}, \quad m=1, 2, \ldots,$$

and $\varkappa_m^{(i)}$ are the cumulants of the eigenfunction L_i, which are to be determined from the orthogonality conditions. We write the first several eigenvectors $\varkappa^{(i)}/\varkappa_0^{(i)}$ ($\varkappa_0^{(i)}$ are normalizing constants):

$$\begin{matrix}1\\0\\0\\0\\0\\0\\0\\0\\\vdots\end{matrix},\quad\begin{matrix}0\\1\\0\\0\\0\\0\\0\\0\\\vdots\end{matrix},\quad\begin{matrix}1/3\\0\\1\\0\\0\\0\\0\\0\\\vdots\end{matrix},\quad\begin{matrix}0\\2/3\\0\\1\\0\\0\\0\\0\\\vdots\end{matrix},\quad\begin{matrix}1/5\\0\\1\\0\\1\\0\\0\\0\\\vdots\end{matrix},\quad\begin{matrix}0\\23/45\\0\\4/3\\0\\1\\0\\0\\\vdots\end{matrix}$$

Thus the eigenfunctions L_i of the integral operator $\mathbf{K}(\theta)$ are linearly independent nonorthogonal polynomials. To fix the normalizing constants $\varkappa_0^{(i)}$, we define the scalar product with weighting function $\varphi(x)=2\pi^{-1/2}x^2\exp(-x^2)$. This function is so chosen that all the scalar products encountered in the problem converge. On requiring the normalization $(L_i, L_i)_\varphi=1$, we get, in particular, $\varkappa_0^{(1)}=\sqrt{6}/3$, $\varkappa_0^{(2)}=4\sqrt{15}/15$, $\varkappa_0^{(3)}=4\sqrt{66}/11$, etc.

§4. THE BRANCHING EQUATION

Let $\nu_k^{-1}(\theta)$ be a nondegenerate eigenvalue of the operator $\mathbf{K}(\theta)$, corresponding to eigenfunction L_k. By the methods of the theory of the branching of solutions of nonlinear equations,[5] it can be shown that all solutions of equation (11) branching from this eigenvalue can be represented in a single form, as a convergent series

$$\Lambda=\xi L_k+\sum_{m+n\geqslant 2}\operatorname{sign}^n(\varepsilon)|\varepsilon|^n\Lambda_{mn}\xi^m, \tag{13}$$

the number and form of the branches are determined by the auxiliary parameter $\xi=(L_k, \Lambda)_\varphi$. Here $\Lambda_{mn}(\rho)$ are functions subject to determination. They can be found by the method of undetermined coefficients. We obtain the equation for the parameter ξ (the branching equation) by multiplying (13) scalarly by L_k:

$$\sum_{m+n\geqslant 2}\operatorname{sign}^n(\varepsilon)|\varepsilon|^n\omega_{mn}^{(k)}\xi^m=0, \quad \omega_{mn}^{(k)}=(L_k, \Lambda_{mn})_\varphi. \tag{14}$$

We write the most important coefficients of the branching equation:

$$\omega_{0n}^{(k)}=0, \quad n=2, 3, \ldots, \quad \omega_{11}^{(k)}=d_k(M_k, L_k)_1,$$

$$\omega_{20}^{(1)}=0, \quad \omega_{20}^{(2)}=-\frac{8\sqrt{15}}{15}(M_2, L_2)_1, \quad \omega_{30}^{(1)}=-\frac{2}{9}(1+3w_{12})(M_1, L_1)_1, \tag{15}$$

$$w_{ik}=(\nu_i^{-1}\nu_k-1)^{-1}, \quad d_k=\nu_k\Omega_{11}\nu_k^{-1}=\begin{cases}\dfrac{d\ln\nu_k^{-1}(0)}{d\ln\theta} & \text{when } \lambda=0\\ 1 & \text{when } \tau=0\end{cases}.$$

We are interested in the real solutions of equation (14) that vanish with $|\varepsilon|$. Reality of ξ is necessary for satisfaction of the hermiticity of the characteristic function. We shall solve the branching equation by the method of Newton diagrams.[13] We shall seek the value of ξ in the form of a series

$$\xi(|\varepsilon|)=\xi_{\beta_1}|\varepsilon|^{\beta_1}+\xi_{\beta_2}|\varepsilon|^{\beta_2}+\ldots, \tag{16}$$

where $\beta_1, \beta_2, \ldots$ are an increasing sequence of rational numbers. The Newton diagrams for the first term of the series (16), plotted by use of the coefficients (15) of the branching equation, are shown in Fig. 2. The projection of the Newton diagram on the axis of abscissas is equal to the number of possible values (branches) ξ_{β_1}; the value of the slope with the negative direction of this axis is equal to β_1. The method also permits automatically writing the equation for ξ_{β_1}. From Fig. 2 we have

$$\begin{aligned}&\beta_1={}^1/_2:\ \omega_{30}^{(1)}\xi_{1/2}^3+\operatorname{sign}(\varepsilon)\,\omega_{11}^{(1)}\xi_{1/2}=0,\\ &\beta_1=1:\ \omega_{20}^{(2)}\xi_1^2+\operatorname{sign}(\varepsilon)\,\omega_{11}^{(2)}\xi_1=0,\end{aligned} \tag{17}$$

in the cases of branching from the eigenvalues $\nu_1^{-1}(\theta)$ and $\nu_2^{-1}(\theta)$. To find β_2 and ξ_{β_2}, it is necessary to substitute the already known β_1 and ξ_{β_1} in (16) and again supply the indicated procedure, and so on. It can be shown that ξ_{β_k}, $k=2,3,\ldots$, are subject to linear equations; that is, a second branching does not occur, and the number of branches is determined solely by equations (17). Here $\beta_k=k/2$ and $\beta_k=k$ in the cases of branching from $\nu_1^{-1}(\theta)$ and $\nu_2^{-1}(\theta)$ respectively.

Thus in the case of branching from the eigenvalue $\nu_1^{-1}(\theta)$, the branching equation [and consequently the integral equation (6)] has one trivial branch $\xi=0$ and two nontrivial:

$$\begin{aligned}&\xi=\xi_{1/2}|\varepsilon|^{1/2}+\sum_{k=2}^{\infty}\xi_{k/2}|\varepsilon|^{k/2},\\ &\xi_{1/2}=\pm\left(\frac{9}{2}\,\frac{d_1\operatorname{sign}(\varepsilon)}{1+3w_{12}}\right)^{1/2},\\ &d_1=\begin{cases}-\nu_1\left\langle\dfrac{J(R)}{T}\operatorname{ch}^{-2}\dfrac{J(R)}{T}\right\rangle_R & \text{when }\ \lambda=0\\ 1 & \text{when }\ \tau=0\end{cases}.\end{aligned} \tag{18}$$

For sign constancy of the potential, d_1 is negative at $\lambda=0$. When $J(R)>0$, $w_{12}\geqslant 0$; therefore $\xi_{1/2}$ is real only when $\tau<0$ or $\lambda>0$. For sufficiently small $|\varepsilon|$, the value of $\frac{1}{3}\sqrt{6}\,\xi_{1/2}|\varepsilon|^{1/2}$ coincides with the center of the distribution. This means that at the value $\nu_1^{-1}(\theta)$ a ferromagnetic state is created on decrease of temperature or increase of concentration. It is clear that the appearance of two distribution functions with centers that differ in sign is a consequence of the symmetry of the Ising model.

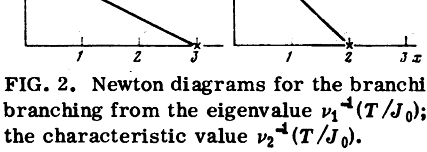

FIG. 2. Newton diagrams for the branching equation: a, branching from the eigenvalue $\nu_1^{-1}(T/J_0)$; b, branching from the characteristic value $\nu_2^{-1}(T/J_0)$.

We now consider the case of branching from $\nu_2^{-1}(\theta)$. This case is of interest for the problem with a potential of variable sign. For such a interaction, it can be shown that $\nu_1^{-1}(\theta)$ does not exist or $\nu_1(\theta)>\nu_2(\theta)$, and on moving from the paramagnetic state we first hit $\nu_2(\theta)$. The branching equation has one trivial and one nontrivial solution:

$$\xi=\xi_1|\varepsilon|+\sum_{k=2}^{\infty}\xi_k|\varepsilon|^k,\qquad \xi_1=\frac{\sqrt{15}}{8}d_2\operatorname{sign}(\varepsilon), \tag{19}$$

$$d_2=\begin{cases}-\nu_2\left\langle\dfrac{4J(R)}{T}\operatorname{sh}^{-1}\left[\dfrac{2J(R)}{T}\right]\operatorname{th}^2\dfrac{J(R)}{T}\right\rangle_R & \text{when }\ \lambda=0,\\ 1 & \text{when }\ \tau=0.\end{cases}$$

For sufficiently small $|\varepsilon|$, the value of $\frac{4}{15}\sqrt{15}\,\xi_1|\varepsilon|$ coincides with the dispersion of the distribution. Therefore the sign rule $\operatorname{sign}(d_2)=\operatorname{sign}(\varepsilon)$ must be satisfied. When $\lambda=0$, the value of d_2 is negative; therefore the sign rule is satisfied when $\tau<0$ or $\lambda>0$.

§5. CALCULATION OF HIGH-ORDER CUMULANTS

In the calculation of cumulants of high order by formula (13), we encounter quite cumbersome expressions. The properties of equation (11) brought out in the preceding paragraphs indicate a simpler method of calculating them.

According to (18) and (19), solutions of (11) are to be sought in the form of series

$$\Lambda(\rho)=\sum_{k=1}^{\infty}|\varepsilon|^{k/2}\Lambda_{k/2}(\rho)\text{and}\Lambda(\rho)=\sum_{k=1}^{\infty}|\varepsilon|^{k}\Lambda_{k}(\rho)$$

in the cases of branchings from $\nu_1^{-1}(\theta)$ and $\nu_2^{-1}(\theta)$ respectively. On substituting these expansions in (11), we find recurrent systems of linear integral equations for the functions $\Lambda_{k/2}(\rho)$ and $\Lambda_k(\rho)$:

$$\begin{aligned}&\Lambda_{k/2}-\nu_1\mathbf{K}(\theta)\Lambda_{k/2}=F_{k/2},\quad \Lambda_k-\nu_2\mathbf{K}(\theta)\Lambda_k=F_k,\quad k=1,2,\ldots,\\ &F_{1/2}=\operatorname{sign}(\varepsilon)\nu_1\mathbf{K}_{01}1,\quad F_1=\operatorname{sign}(\varepsilon)\nu_2\mathbf{K}_{01}1,\\ &F_1=\operatorname{sign}(\varepsilon)\nu_1\mathbf{K}_{01}1+\nu_1\mathbf{K}_{20}\Lambda_{1/2}^2,\quad F_2=\nu_2\mathbf{K}_{02}1+\operatorname{sign}(\varepsilon)\nu_2\mathbf{K}_{11}\Lambda_1+\nu_2\mathbf{K}_{20}\Lambda_1^2,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned} \tag{20}$$

The first equations of these systems are homogeneous. They were solved in § 3. For the inhomogeneous equations one can write

$$\begin{aligned}&\Lambda_{k/2}=\xi_{k/2}L_1+F_{k/2}+\mathbf{R}^{(1)}F_{k/2},\quad \Lambda_k=\xi_kL_2+F_k+\mathbf{R}^{(2)}F_k,\\ &\xi_{k/2}=(L_1,\Lambda_{k/2})_\varphi,\quad \xi_k=(L_2,\Lambda_k)_\varphi,\end{aligned} \tag{21}$$

where $\mathbf{R}^{(i)}$ is the resolvent of the kernel

$$N^{(i)}(\rho,y)=\nu_i(\theta)K(\rho,y;\theta)-\varphi(\rho)\varphi(y)M_i(\rho)L_i^*(y). \tag{22}$$

Furthermore, by using the linear independence of the eigenfunctions $L_k(\rho)$ we can express all powers of L_k in terms of linear combinations of them. For example, $L_1^3=\frac{1}{3}\sqrt{11}\,L_3-\frac{4}{3}L_1$. Then the last terms in (21) are easily calculated by the formula

$$\mathbf{R}^{(i)}L_k=w_{ik}L_k+w_{ki}(L_i,L_k)_\varphi L_i,\quad i\neq k, \tag{A.1}$$

the derivation of which is given in the Appendix. The conditions for solvability of the recurrent systems (20) uniquely determine the constants ξ, the system of equations for which are equivalent to the branching equation (14). Omitting the intermediate calculations, we shall give the final expressions, in the $\varkappa$ representation, for the solutions of equation (11).

Branching from $\nu_1^{-1}(\theta) \equiv \nu^{-1}$. The bifurcation curve $T_c(\nu)$ is determined by the condition

$$\nu^{-1} = \langle \mathrm{th}\,[T_c^{-1} J(R)]\rangle_R. \tag{23}$$

When $\tau < 0$ and $\lambda > 0$, there are two nontrivial solutions of the equation

$$\varkappa_k(\varepsilon) = \varkappa_k'(T_c)\,|\varepsilon|^{k/2} + \varkappa_k''(T_c)\,|\varepsilon|^{1+k/2} + \ldots \tag{24}$$

$(k = 1, 2, \ldots)$, where for the principal term of the expansion there is the recurrence formula

$$\varkappa_1' = \frac{\sqrt{6}}{3}\xi_{1/2}, \qquad \varkappa_k' = w_{1k}\sum_{m=1}^{k-1}\binom{k-1}{m}\varkappa_{k-m}'\eta_m(\varkappa_1', \ldots, \varkappa_m') \tag{25}$$

$(k = 2, 3, \ldots)$. The second order in $|\varepsilon|$ of the center of the distribution and of the dispersion is determined by the formulas

$$\varkappa_1'' = \frac{\sqrt{6}}{3}\xi_{3/2} + \frac{5\sqrt{6}}{54}\xi_{1/2}^3 w_{13}(1+3w_{12}),$$
$$\varkappa_2'' = {}^4/_3\,\xi_{1/2}\xi_{3/2}w_{12} + {}^2/_3\,\mathrm{sign}(\varepsilon)\,d_2\xi_{1/2}w_{12}(1+w_{12}) + {}^2/_{27}\,\xi_{1/2}^4[w_{13}(1+3w_{12})(16w_{13} - 11w_{12}) + 4(1+6w_{12}+3w_{12}^2)(w_{13}-w_{12})], \tag{26}$$

where

$$\xi_{3/2} = {}^9/_8\, b_1(1+3w_{12})^{-1}\xi_{1/2}^{-1} - {}^3/_2\,\mathrm{sign}(\varepsilon)\,d_2 w_{12}(1+w_{12})(1+3w_{12})^{-1}\xi_{1/2} + {}^1/_{90}[5(17w_{13}-2)(1+3w_{12}) + 60w_{12}(1+6w_{12}+3w_{12}^2)(1+3w_{12})^{-1} + 12(1+10w_{12}+15w_{12}^2)(1+3w_{12})^{-1}]\xi_{1/2}^3,$$

$$b_1 = \begin{cases} \nu T_c^2 \dfrac{d^2}{dT_c^2}\langle \mathrm{th}[T_c^{-1}J(R)]\rangle_R & \text{when } \lambda = 0 \\ 0 & \text{when } \tau = 0 \end{cases}.$$

We recall that the quantities ν_k^{-1}, w_{ik}, d_k, and $\xi_{1/2}$ were defined above by formulas (12), (15), and (18).

We consider the first order in $|\varepsilon|$ in the range of high concentrations ν, which corresponds to high temperatures $T_c(\nu)$ [numerical estimates for specific interaction potentials $J(R)$ and spatial spin distribution $P_R(R)$ will be made in the next paragraph]. When $\nu^{-1}\nu_k(T_c) \gg 1$, we have

$$w_{1k} = \nu\nu_k^{-1}(T_c) \ll 1, \quad k = 2, 3, \ldots\ . \tag{27}$$

Keeping only terms of the first degree in w_{ik}, we get $\varkappa_k' = w_{1k}\varkappa_1'^k$, whence we have directly a generalized Poisson distribution

$$\Lambda(\rho) = \nu\langle \exp\{i\rho\varkappa_1\,\mathrm{th}\,[T_c^{-1}J(R)]\} - 1\rangle_R,$$
$$\varkappa_1 = \varkappa_1'(T_c)\,|\varepsilon|^{1/2}. \tag{28}$$

In the molecular-field approximation (4),

$$\Lambda(\rho) = \nu\left\langle \exp\left\{ i\rho\frac{\varkappa_1}{\nu}\frac{J(R)}{\langle J(R)\rangle_R}\right\} - 1\right\rangle_R.$$

The distribution (28) has the finite sum

$$q = \varkappa_1\sum_{j=1}^{z}\mathrm{th}[T_c^{-1}J(R_{0j})],$$

where z is an independent random quantity, taking nonnegative integer values and having a Poisson distribution with parameter ν. Hence it follows that $z + 1$ has the meaning of effective number of "nearest neighbors." The center of the distribution and the dispersion of z are equal to ν.

Branching from $\nu_2^{-1}(\theta) \equiv \nu^{-1}$. The bifurcation curve $T_g(\nu)$ is determined by the condition

$$\nu^{-1} = \langle \mathrm{th}^2\,[T_g^{-1}J(R)]\rangle_R. \tag{29}$$

When $\tau < 0$ and $\lambda > 0$, there is one nontrivial solution of equation (11):

$$\varkappa_{2k-1}(\varepsilon) = 0, \quad \varkappa_{2k}(\varepsilon) = \varkappa_{2k}'(T_g)\,|\varepsilon|^k + \varkappa_{2k}''(T_g)\,|\varepsilon|^{1+k} + \ldots$$

$(k = 1, 2, \ldots)$, where for the principal term of the expansion there is the recurrence formula

$$\varkappa_1' = 0, \qquad \varkappa_2' = \frac{1}{2}d_2\,\mathrm{sign}(\varepsilon),$$

$$\varkappa_k' = w_{2k}\sum_{m=1}^{k-1}\binom{k-1}{m}\varkappa_{k-m}'\eta_m(\varkappa_1', \ldots, \varkappa_m') \tag{30}$$

$(k = 3, 4, \ldots)$. The second order in $|\varepsilon|$ of the first two nonvanishing cumulants is determined by the formulas

$$\varkappa_2'' = {}^1/_4\,b_2 + {}^1/_{24}\,d_2^2(5+39w_{24}),$$
$$\varkappa_4'' = {}^3/_4\,\mathrm{sign}(\varepsilon)\,b_2 d_2 w_{24} + {}^3/_4\,\mathrm{sign}(\varepsilon)\,d_2^2 d_4 w_{24}(1+w_{24}) + {}^1/_8\,\mathrm{sign}(\varepsilon)\,d_2^3[20w_{26}(1+3w_{24}) - 3w_{24}(5+7w_{24})], \tag{31}$$
$$b_2 = \begin{cases} \nu T_g^2 \dfrac{d^2}{dT_g^2}\langle \mathrm{th}^2[T_g^{-1}J(R)]\rangle_R & \text{when } \lambda = 0 \\ 0 & \text{when } \tau = 0 \end{cases}.$$

Analogously to the preceding case, in the first order in $|\varepsilon|$, when

$$\nu^{-1}\nu_k(T_g) \gg 1, \quad k = 4, 6, \ldots, \tag{32}$$

one can obtain

$$\Lambda(\rho) = \nu\langle G\{i\rho\varkappa_2^{1/2}\,\mathrm{th}[T_g^{-1}J(R)]\}\rangle_R,$$
$$G(x) = \sum_{k=1}^{\infty}\frac{x^{2k}}{2k!!}, \qquad \varkappa_2 = \varkappa_2'(T_g)\,|\varepsilon|. \tag{33}$$

In the molecular-field approximation (4),

$$\Lambda(\rho) = \nu\left\langle G\left\{ i\rho\left(\frac{\varkappa_2}{\nu}\right)^{1/2}\frac{J(R)}{\langle J^2(R)\rangle_R^{1/2}}\right\}\right\rangle_R$$

§6. ALLOYS OF THE TYPE PdFe AND CuMn

We consider a model in which the spatial distribution of magnetic impurities is absolutely uniform, $P_R(R) = 1$.

Exponential interaction (1). The eigenvalues of the operator $\mathbf{K}(T/J_0)$ are shown in Fig. 1. The upper solid curve corresponds to the temperature of ferromagnetic ordering. Asymptotically (to within accuracy 0.1% at $\nu = 0.5$) it coincides with the Curie temperature $T_c = \nu\langle J(R)\rangle_R$ of the molecular-field approximation. Thus the Bethe-Peierls approximation gives a lower ordering temperature at all concentrations. The condition $\nu \gtrsim 0.5$ determines the range of applicability of the molecular-field approximation. For a Heisenberg ferromagnet, T_c is proportional to the mean energy of exchange interaction when $\nu \gg 1$.[1] When $T_c \gtrsim 4J_0$, the function $\varkappa_1'(T_c)/\varkappa_1''(T_c)$ and $\varkappa_2'(T_c)/\varkappa_2''(T_c)$ reach plateaus with values 1.1 and 0.6 for $\lambda = 0$ and values 5.3 and 1.4 for $\tau = 0$; that is, the second order (26) for $|\varepsilon|$ can be neglected, at least for the center of the distribution and for the dispersion when $|\tau| \ll 0.6$ and $|\lambda| \ll 1.4$. The range of applicability with respect to T_c of the generalized Poisson distribution is $T_c \gtrsim 1.2J_0$, $\nu \gtrsim 0.2$ $(\nu^{-1}\nu_2(T_c) \gtrsim 10)$.

At small impurity concentrations, the Curie temperature of a Heisenberg ferromagnetic, as was first shown by Korenblit and Shender,[1] is determined by the interaction at the mean distance, $T_c \sim J_9 \exp(-\mu\nu^{-1/3})$, where μ is a quantity of order unity. The condition (23), as was shown by Ginzburg,[4] gives the same concentration dependence. Analogously,[4] one can obtain for $\nu \ll 1$

$$\nu_k^{-1}(T) = \left(\frac{R_T}{r_0}\right)^3 + \frac{3}{k}\left(\frac{R_T}{r_0}\right)^2 - \frac{6}{k^2}\left(\frac{R_T}{r_0}\right) + \frac{6}{k^3},$$

where $R_T = r_0 \ln(T^{-1}J_0)$ is the distance at which the exchange energy is equal to the temperature. The equation $\nu_k(T_k) = \nu(T_1 \equiv T_c)$ gives

$$T_k = J_0 \exp\left[-\frac{R_T(\nu^{1/3})}{r_0}\right],$$

$$\frac{R_T(x)}{r_0} = \frac{1}{x}\left[1 - \frac{x}{k} + 3\left(\frac{x}{k}\right)^2 + \ldots\right].$$

Hence one easily obtains

$$w_{1k}(T_c) = [\nu^{-1}\nu_k(T_c) - 1]^{-1} = \frac{k}{3(k-1)}\frac{1}{x}(1 + 2x - 3x^2 + \ldots), \qquad (34)$$

$$x = \nu^{1/3},\ k = 2, 3, \ldots\ .$$

In contrast to (27), the values of w_{1k} are large at small concentrations. When $\nu \gtrsim 10^{-3}$, we shall restrict ourselves to the leading term of the expansion (34). In this range, all the quantities d_k and b_k, $k = 1, 2, \ldots$ are also of a single order:

$$d_k = \begin{cases} -3\nu^{1/3} & \text{when } \lambda = 0, \\ 1 & \text{when } \tau = 0, \end{cases} \qquad b_k = \begin{cases} 3\nu^{1/3} & \text{when } \lambda = 0, \\ 0 & \text{when } \tau = 0. \end{cases}$$

The asympotic behaviors of formulas (25) and (26) at $\nu \gtrsim 10^{-3}$ have the following form. When $\lambda = 0$, the cumulants are inversely proportional to the mean distance between impurities:

$$\varkappa_k' = \alpha_k \nu^{1/3},\ \varkappa_k'' = \beta_k \nu^{1/3}, \qquad (35)$$

where $\alpha_1 = 2.12$, $\alpha = 3.00$, $\alpha_3 = 9.55$, $\alpha_4 = 48.00$, $\beta_1 = 20.15$, $\beta_2 = 47.00$. For the center of the distribution and for the dispersion, the second order in $|\tau|$ can be neglected when $|\tau| \ll 0.06$. When $\tau = 0$,

$$\varkappa_k' = \alpha_k \nu^{1/3 - k/6},\ \varkappa_k'' = \beta_k \nu^{-k/6}, \qquad (36)$$

$\alpha_1 = 1.22$, $\alpha_2 = 1.00$, $\alpha_3 = 1.84$, $\alpha_4 = 5.33$, $\beta_1 = 3.78$, $\beta_2 = 5.06$. The second order in $|\lambda|$ of the center of the distribution and of the dispersion can be neglected when $|\lambda| \ll 0.20\nu^{1/3} \gtrsim 0.02$. Since the cumulants $\varkappa_k = \alpha = \alpha(|\tau|)\nu^{1/3}$ describe a discrete Poisson distribution, the distribution function $P_q(q)$ of molecular fields at small concentrations of magnetic atoms can evidently be approximated by a function with maxima at nonnegative integral values of the dimensionless molecular field q.

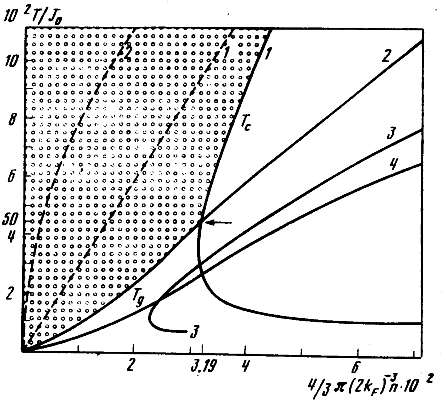

FIG. 3. Phase diagram for a model with the RKKY interaction (2). The circles denote the paramagnetic region, the solid curves the eigenvalues $\nu_k(T/J_0)$ ($k = 1, 2, 3, 4$) of the operator $\mathbf{K}(T/J_0)$; Curve 1 is the line of transition to the ferromagnetic phase ($\nu > 3.19 \cdot 10^{-2}$); Curve 2 is the line of transition to the spin-glass phase ($\nu < 3.19 \cdot 10^{-2}$). The dotted lines, corresponding to $\nu_1(T/J_0)$ and $\nu_2(T/J_0)$ in the molecular-field approximation, intersect at $\nu = 6.99 \cdot 10^{-2}$.

RKKY interaction (2). The eigenvalues of $\mathbf{K}(T/J_0)$ are shown in Fig. 3. The sign of the *RKKY* interaction at small distances, $2k_F R_{ij} \ll 1$, corresponds to ferromagnetism. The ferromagnetic state is generated at the bifurcation curve $T_c(\nu)$ (23) if the concentration n of the magnetic atoms exceeds $6.09 \cdot 10^{-2} k_F^2$. But if $n < 6.09 \cdot 10^{-2} k_F^2$, then on lowering of the temperature the paramagnetic phase $P_q(q) = \delta(q)$ transforms to a spin-glass phase with zero mean molecular field at a localized spin, but with nonvanishing dispersion and with cumulants of higher orders. Experimentally, the abrupt transition predicted for the classical Heisenberg model[14] and Ising model[11] is not always observed.[15] A bifurcational approach to the Ising model, different from from the method of a specific ensemble of exact copies (replicas) of the system at infinitely separated instants of time,[11] also shows an abrupt transition. From Fig. 3 it is evident that the transition temperature T_g is determined by the second characteristic value (29) of the operator $\mathbf{K}(T/J_0)$. In the range of small concentrations of magnetic atoms, $n \gtrsim 10^{-2} k_F^3$ ($\nu \gtrsim 0.5 \cdot 10^{-2}$), it varies linearly with the concentration, $T_g = 0.574 J_0 k_F^{-3} n$, since the mean distance between magnetic atoms is large, $2k_F R_{ij} \gg 1$, and $J(R_{ij}) \sim (2k_F R_{ij})^{-3}$. We note that the mean energy of exchange interaction is $\nu\langle J(R)\rangle_R = 1.572 J_0 k_F^{-3} n$. In the molecular-field approximation, $T_g = 0.574 J_0 k_F^{-3/2} n^{1/2}$.

Figure 4 shows graphs of the coefficients in the expansion of the low-order cumulants as series in $|\tau|$, as functions of T_c and T_g along the boundaries of the paramagnetic phase. The breakdown of the continuity of these coefficients at the point $T_c = T_g = 4.50 \cdot 10^{-2} J_0$ corresponds to the transition between ferromagnetic and

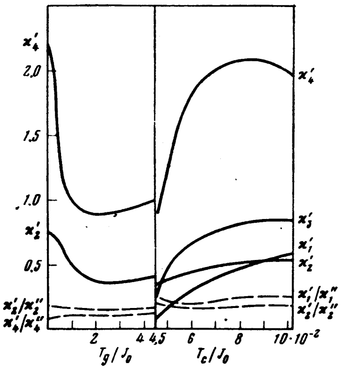

FIG. 4. Variation of the coefficients in the expansion of the cumulants as series in $|\tau|$ with the ordering temperature in the case of the RKKY interaction (2). Solid curves, first order $\varkappa_k'$; dotted, $\varkappa_k'/\varkappa_k''$; $k = 1, 2, 3, 4$.

spin glass. In the range of high concentrations of magnetic atoms, not shown in the figure, the high-order cumulants decrease rapidly with increase of the concentration. In this range, the distribution of molecular fields is nearly normal in agreement with formulas (28) and (33). But for the spin-glass phase near T_g, such concentrations are unattainable because of the transition to the ferromagnetic state. For example, near the phase boundary $T_g = 4.4 \cdot 10^{-2} J_0$ the coefficient of excess is $\varkappa_4'/\varkappa_2'^2 = 5.9$, whereas for the normal distribution it is zero. This means that for adequate description of the spin-glass phase, it is insufficient to consider only two equations of the system (8) (with $\varkappa_k = 0$ for $k \geqslant 3$); it is necessary to study the complete system or the integral equation (6).

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APPENDIX

Derivation of formula (A.1)

From the definition (22) of the kernel $N^{(i)}(\rho, y)$ we have

$$L_k - \mathbf{N}^{(i)} L_k = (1 - \nu_i \nu_k^{-1}) L_k + (L_i, L_k)_\varphi \varphi M_i.$$

We perform an inverse linear transformation:

$$L_k = (1 - \nu_i \nu_k^{-1})(L_k + \mathbf{R}^{(i)} L_k) + (L_i, L_k)_\varphi (\varphi M_i + \mathbf{R}^{(i)} \varphi M_i). \tag{A.2}$$

For $i = k$, formula (A.2) takes the form

$$L_i = \varphi M_i + \mathbf{R}^{(i)} \varphi M_i. \tag{A.3}$$

On substituting (A.3) in (A.2), we get for $i \neq k$ formula (A.1); see § 5.

[1] I. Ya. Korenblit and E. F. Shender, Usp. Fiz. Nauk **126**, 233 (1978) [Sov. Phys. Usp. **21**, 832 (1978)]; Zh. Eksp. Teor. Fiz. **62**, 1949 (1972) [Sov. Phys. JETP **35**, 1017 (1972)].
[2] M. W. Klein and R. Brout, Phys. Rev. **132**, 2412 (1963).
[3] M. W. Klein, Phys. Rev. **173**, 552 (1968); **B14**, 5008 (1976).
[4] S. L. Ginzburg, Zh. Eksp. Teor. Fiz. **74**, 236 (1978) [Sov. Phys. JETP **47**, 121 (1978)].
[5] M. M. Vaĭnberg and V. A. Trenogin, Teoriya vetvleniya resheniĭ nelineĭnykh uravneniĭ (Theory of Branching of Solutions of Nonlinear Equations), Nauka, 1969, p. 226 (transl., Noordhoff Intl. Pub., Leyden, 1974; German transl., Akademie-Verlag, Berlin, 1973).
[6] G. Marchal, Ph. Mangin, and Chr. Janot, Solid State Commun. **18**, 739 (1976).
[7] B. H. Verbeek, C. Van Dijk, C. J. Nieuwenhuys, and J. A. Mydosh, J. Phys. (Paris) **39**, Suppl., C6-918 (1978).
[8] D. E. Murnick, A. T. Fiory, and W. J. Kossler, Phys. Rev. Lett. **36**, 100 (1976).
[9] L. R. Walker and R. E. Walstedt, Phys. Rev. Lett. **38**, 514 (1977).
[10] C. Domb, Adv. Phys. **9**, 149 (1960).
[11] D. Sherrington and S. Kirkpatrick, Phys. Rev. Lett. **35**, 1792 (1975).
[12] S. Kirkpatrick and D. Sherrington, Phys. Rev. **B17**, 4384 (1978).
[13] N. G. Chebotarev, Sobranie Sochineniĭ (Collected Works), Vol. 3, Izd. Akad. Nauk SSSR, 1950, p. 47.
[14] S. F. Edwards and P. W. Anderson, J. Phys. F **5**, 965 (1975).
[15] A. P. Murani, J. Phys. (Paris) **39**, Suppl., C6-1517 (1978).

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Magnetic linear birefringence of light and dichroism in the region of the absorption band of the rare-earth ion in europium iron garnet

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Results are reported of a spectroscopic investigation, by a polarization magneto-optical procedure, of the optical transition ${}^7F_0 \rightarrow {}^7F_6$ of the Eu^{3+} ion in the structure of europium iron garnet. It is shown that in the employed Voigt geometry the magnetic linear birefringence and the dichroism reach values 10^{-3}, and have a strong dependence on the wavelength and a strong anisotropy. For a sample cut in the (110) plane, comparison shows that the spectra for the cases $\mathbf{I} \| [\bar{1}10]$, $\mathbf{e} \| [001]$ and $\mathbf{I} \| [001]$, $\mathbf{e} \| [\bar{1}10]$ differ noticeably. The known formulas for δn and δk (the contribution due to $\mathbf{I}$ to n and k), which describe well the magnetic birefringence and the dichroism of cubic crystals far from the absorption line, are invariant to interchange of the directions of $\mathbf{I}$ and $\mathbf{e}$. A model-based theory is proposed to explain the observed independence of the spectra as being due to the low local symmetry of the surrounding of magnetically active ions in the crystal.

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1. INTRODUCTION

Investigation of magnetic birefringence (MB) of light by magnetically ordered crystals is attracting much interest of late. A large value of the magnetic linear birefringence was first discovered by Dillon[1] in an yttrium iron garnet crystal, and used to observe the domain structure. It was shown in Ref. 2 that the mag-

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