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# Stability of the Bose spectrum of superfluid systems of the He$^3$ type

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We calculate the dispersion coefficient and investigate the stability to decays of the phonon modes of the Bose spectrum in the superfluid $A$, $B$, and $2D$ phases of a model system of the He$^3$ type. All the modes in the $B$ phase are stable, and in the $A$ and $2D$ phases the stability depends on the angle between the momentum of the collective excitation and the selected direction.

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## 1. INTRODUCTION

We have investigated, in a model Fermi system of the He$^3$ type, the stability of collective Bose excitations of the phonon type[1)] to decays of one excitation into several others. The Bose spectrum of He$^3$ was calculated in a number of studies.[1-5] Acoustic and spin waves were shown to exist in the $B$ phase, and acoustic and orbital waves in the $A$ and $2D$ phases.

The stability of the Bose spectrum can be considered with respect to various processes: to pair decay into initial fermions (see Ref. 5 for orbital waves), and to decay of a collective Bose excitation or several Bose excitations of the same type or into several Bose excitations of different types corresponding to different energy-spectrum dispersion laws.

In the isotropic $B$ phase, the decay of the phonon into individual fermions is forbidden, since the excitation energy is much lower than the binding energy $2\Delta$ of the Cooper pair. The decay of an excitation into two or several excitations of the same type is kinematically forbidden if $d^2E/dk^2 < 0$, and the $E(\mathbf{k})$ curve bends downward away from the tangent $uk$ (Fig. 1). This is equivalent to a positive dispersion coefficient $\gamma$ in the dispersion law $E(\mathbf{k}) = uk(1-\gamma k^2)$ (at small $k$). Therefore the question of the stability of phonon excitations is solved by calculating the corrections to the linear dispersion law. The calculation shows the stability of all the phonon branches in the $B$ phase.

In the anisotropic phases ($A, 2D$) the energy gap of the Fermi spectrum depends on the direction of the momentum and vanishes in the selected direction. Decay of the phonon into individual fermions is therefore energywise possible here. On the other hand, the question of the stability to decay into Bose excitations, just as in the $B$ phase, reduces to finding the corrections to the linear dispersion law. A calculation shows that the excitation is stable if its momentum lies within certain cones described around a preferred direction, and is unstable in the opposite case.

## 2. THE He$^3$ MODEL AND THE HYDRODYNAMIC ACTION FUNCTIONAL

We consider the model system of the He$^3$ type proposed by Alonso and Popov.[4] The collective Bose excitations in the system are described by a functional of the hydrodynamic action $S_h$, obtained after functional integration over the Fermi fields. The functional takes the form

$$S_h = -\frac{1}{g}\sum_{p,i,a} c_{ia}^+(p)c_{ia}(p) + \frac{1}{2}\ln\det\frac{\hat{M}(c,c^+)}{\hat{M}(0,0)}. \tag{2.1}$$

  

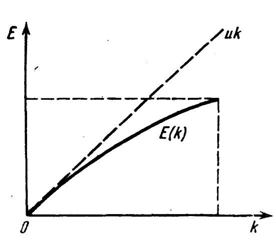


FIG. 1.

Here $c_{ia}(p)$ is the Fourier transform of the tensor field $c_{ia}(\mathbf{x}, \tau)$ with vector index $i$ and with isotopic index $a$, and $\hat{M}$ is the operator:

$$\hat{M} = \begin{pmatrix} Z^{-1}(i\omega-\xi+\mu H\sigma_3)\delta_{p_1p_2}, & (\beta V)^{-1/2}(n_{1i}-n_{2i})c_{ia}(p_1+p_2)\sigma_a \\ -(\beta V)^{-1/2}(n_{1i}-n_{2i})c_{ia}^+(p_1+p_2)\sigma_a, & Z^{-1}(-i\omega+\xi+\mu H\sigma_3)\delta_{p_1p_2} \end{pmatrix} \tag{2.2}$$

where $\xi = c_F(k-k_F)$, $n_i = k_i/k_F$, $H$ is the magnetic field, $\mu$ is the magnetic moment of the quasiparticle, $\sigma_a(a=1,2,3)$ are Pauli matrices, $Z$ is a normalization constant, $\beta^{-1}=T$, and $\omega=(2n+1)\pi T$ is the Fermi frequency. The negative constant $g$ in (2.1) is proportional to the scattering amplitude of two fermions near the Fermi sphere under the assumption that the amplitude is equal to $g(\mathbf{k}_1-\mathbf{k}_2, \mathbf{k}_3-\mathbf{k}_4)$, where $\mathbf{k}_1$ and $\mathbf{k}_2$ are the momenta of the incident fermions, and $\mathbf{k}_3$ and $\mathbf{k}_4$ are the momenta of the outgoing fermions. The method of obtaining the functional $S_h$ is described in greater detail in Ref. 4.

At the present stage it is important to us that this functional contains the entire information on the physical properties of the model system.

In the region $T_c - T \sim T_c$, we expand the functional ln det in (2.1) in powers of the deviation $c_{ia}(p)$ from the condensate value $c_{ia}^{(0)}(p)$, which is different for different phases. We apply the shift $c_{ia}(p) \to c_{ia}^{(0)}(p) + c_{ia}(p)$ and separate from $S_h$ the quadratic form

$$\sum_p c_{ia}^+(p)c_{jb}(p)A_{ijab}(p) + \frac{1}{2}\sum_p (c_{ia}(p)c_{jb}(-p) + c_{ia}^+(p)c_{jb}^+(-p))B_{ijab}(p). \tag{2.3}$$

It is this form which determines, in first approximation, the Bose spectrum obtained from the equation

$$\det Q = 0. \tag{2.4}$$

Here $Q$ is a matrix of quadratic form, determined by the coefficient tensors $A_{ijab}, B_{ijab}$ in (2.3). These quantities are proportional to the integrals of the products of the Green's functions of the fermions. Most effective in the calculation of these integrals is the Feynman procedure customarily used in relativistic quantum theory. In the present case the procedure is based on the identity

$$\frac{1}{(\omega_1^2+\xi_1^2+\Delta^2)(\omega_2^2+\xi_2^2+\Delta^2)} = \int_0^1 \frac{d\alpha}{[\alpha(\omega_1^2+\xi_1^2+\Delta^2)+(1-\alpha)(\omega_2^2+\xi_2^2+\Delta^2)]^2}. \tag{2.5}$$

It is easy to evaluate by this procedure the integrals with respect to the variables $\omega$ and $\xi$, and then with respect to the angle variables and the parameter $\alpha$.

Expanding the obtained expressions up to terms $\sim\omega^4$, $k^4$, and $\omega^2k^2$ inclusive and solving next Eq. (2.4), we obtain the sought corrections to the linear dispersion law in the $B$ phase. For the $A$ and $2D$ phases the coefficients of $\omega^2$ and $k^2$ have in general a logarithmic dependence of $p^2 = \omega^2 + c_F^2 k_\parallel^2$. For these phases, therefore, the corrections to the linear dispersion law can be obtained accurately by calculating only the terms $\sim\omega^2 \ln p^2$ and $k^2 \ln p^2$. We obtain below the phonon branches of the Bose spectrum in the limit as $T \to 0$, with the corrections to the linear dispersion law separately for each of the $B$, $A$, and $2D$ phases.

## 3. THE $B$ PHASE

The condensate function $c_{ia}^{(0)}(p)$ in the $B$ phase is given by[4]

$$c_{ia}^{(0)}(p) = (\beta V)^{1/2} c\,\delta_{p0}\delta_{ia}, \tag{3.1}$$

where $c$ is obtained from the equation

$$\frac{3}{g} + \frac{4Z^2}{\beta V}\sum_p (\omega^2+\xi^2+4c^2Z^2)^{-1} = 0. \tag{3.2}$$

Making the shift $c_{ia}(p) \to c_{ia}^{(0)}(p) + c_{ia}(p)$, we separate in $S_h$ a quadratic form in the new variables

$$\frac{1}{g}\sum_{p,i,a} c_{ia}^+(p)c_{ia}(p) - \frac{1}{4}\operatorname{Sp}(Gu)^2, \tag{3.3}$$

where

$$u_{p_1p_2} = (\beta V)^{-1/2}\begin{pmatrix} 0, & (n_1-n_2)_i\sigma_a c_{ia}(p_1+p_2) \\ -(n_1-n_2)_i\sigma_a c_{ia}^+(p_1+p_2), & 0 \end{pmatrix}, \tag{3.4}$$

$$G^{-1} = \begin{pmatrix} Z^{-1}(i\omega-\xi)\delta_{p_1p_2}, & 2c(\mathbf{n}\sigma)\delta_{p_1+p_2} \\ -2c\mathbf{n}\sigma\delta_{p_1+p_2}, & Z^{-1}(-i\omega+\xi)\delta_{p_1p_2} \end{pmatrix}. \tag{3.5}$$

Inverting $G^{-1}$, we get

$$G = \frac{Z}{M}\begin{pmatrix} -(i\omega+\xi)\delta_{p_1p_2}, & \Delta\mathbf{n}\sigma\delta_{p_1+p_2} \\ -\Delta\mathbf{n}\sigma\delta_{p_1+p_2}, & (i\omega+\xi)\delta_{p_1p_2} \end{pmatrix}, \tag{3.6}$$

where

$$M = \omega^2+\xi^2+\Delta^2, \quad \Delta = 2cZ. \tag{3.7}$$

It follows from (3.4) and (3.6) that

$$\frac{1}{4}\operatorname{Sp}(Gu)^2 = \frac{1}{4}\sum_{p_1,p_2,p_3,p_4} \operatorname{tr}(G_{p_1p_2}u_{p_2p_3}G_{p_3p_4}u_{p_4p_1})$$

$$= \frac{Z^2}{4\beta V}\sum_p \Big\{ c_{ia}^+(p)c_{jb}(p)$$

$$\times 2\sum_{p_1+p_3=p} (M_1M_3)^{-1}(i\omega_1+\xi_1)(i\omega_3+\xi_3)\operatorname{tr}(\sigma_a\sigma_b)(n_1-n_3)_i(-n_1+n_3)_j$$

$$+4\Delta^2(c_{ia}(p)c_{jb}(-p)+c_{ia}^+(p)c_{jb}^+(-p))$$

$$\times\sum_{p_3-p_1=p}(M_1M_3)^{-1}(n_1+n_3)_i(n_1+n_3)_j\operatorname{tr}(\mathbf{n}_1\sigma)\sigma_a(\mathbf{n}_3\sigma)\sigma_b\Big\}.$$

Here tr denotes the trace over the matrix indices.

Considering small $p$, we make the substitution $\mathbf{n}_3 \to -\mathbf{n}$ if $p_1+p_3=p$ and $\mathbf{n}_3 \to \mathbf{n}_1$ if $p_3-p_1=p$. Making also the substitution $p_1 \to -p_1$ in the sums over $p_3-p_1=p$ and taking the trace, we obtain the quadratic part of $S_h$ in the form

$$\sum_p c_{ia}^+(p)c_{ja}(p)\left[\frac{\delta_{ij}}{\sigma} + \frac{4}{\beta V}\sum_{p_1+p_2=p} n_{1i}n_{1j}\varepsilon(-p_1)\varepsilon(-p_2)G(p_1)G(p_2)\right]$$

$$-\frac{1}{2}\sum_p (c_{ia}(p)c_{jb}(-p)$$

$$+c_{ia}^+(p)c_{jb}^+(-p))\frac{4\Delta^2}{\beta V}\sum_{p_1+p_2=p} n_{1i}n_{1j}(2n_{1a}n_{1b}-\delta_{ab})G(p_1)G(p_2), \tag{3.8}$$

where

$$\varepsilon(p)=i\omega-\xi, \quad G(p)=Z(\omega^2+\xi^2+\Delta^2)^{-1}.$$

Equation (3.8) has the same form as (2.3). Following the substitutions

$$c_{ia}(p)=u_{ia}(p)+iv_{ia}(p), \quad c_{ia}^{+}(p)=u_{ia}(p)-iv_{ia}(p) \tag{3.9}$$

expression (3.8) breaks up into two independent forms, one of which depends on $u_{ia}$ and the other on $v_{ia}$:

$$-\sum_p(A_{ij}(p)u_{ia}u_{ja}+B_{ijab}u_{ia}u_{jb})-\sum_p(A_{ij}(p)v_{ia}v_{ja}-B_{ijab}v_{ia}v_{jb}). \tag{3.10}$$

In (3.10), the term that corresponds to $p=0$ is equal to[4]

$$\begin{aligned}&-A_{ij}(0)\,(u_{ia}u_{ja}+v_{ia}v_{ja})-B_{ijab}(0)\,(u_{ia}u_{jb}-v_{ia}v_{jb})\\&=-\frac{2Z^2k_F^2}{15\pi^2c_F}[u_{ia}u_{ja}+u_{aa}u_{bb}+u_{ia}u_{ai}+4v_{ia}v_{ia}-v_{aa}v_{bb}-v_{ia}v_{ai}],\end{aligned} \tag{3.11}$$

the $u$-form has three zero eigenvectors corresponding to the variables $u_{12}-u_{21}$, $u_{23}-u_{32}$, $u_{31}-u_{13}$, while the $v$ form has a single zero vector corresponding to the variable $v_{11}+v_{22}+v_{33}$. It is precisely these which are the "phonon" variables, and the corresponding spectral branches start out from zero.

We consider now the difference $A_{ij}(p)-A_{ij}(0)$. Applying to the denominators of the Green's functions $G(p_1)$ and $G(p_2)$ in (3.8) the Feynman procedure, we obtain

$$\begin{aligned}&A_{ij}(p)-A_{ij}(0)\\&=-\frac{4Z^2}{\beta V}\sum_{p_1+p_2=p}n_{1i}n_{1j}\int_0^1 d\alpha\left\{\frac{(\xi_1+i\omega_1)(\xi_2+i\omega_2)}{[\alpha(\omega_1^2+\xi_1^2)+(1-\alpha)(\omega_2^2+\xi_2^2)+\Delta^2]^2}\right.\\&\qquad\left.-\frac{\omega_1^2+\xi_1^2}{[\omega_1^2+\xi_1^2+\Delta^2]^2}\right\}.\end{aligned} \tag{3.12}$$

Considering the limit as $T\to 0$, we change in (3.12) to integration near the Fermi sphere in accordance with the rule

$$(\beta V)^{-1}\sum_{p_1}\to k_F^2(2\pi)^{-4}c_F^{-1}\int d\omega_1\,d\xi_1\,d\Omega_1,$$

where $\int d\Omega_1$ is the integral over the angle variables. We replace furthermore in (3.12) $\xi_2\to -\xi_2$ and make the shifts $\omega_1\to\omega_1+(1-\alpha)\omega$ and $\omega_2\to -\omega_1+\alpha\omega$, so that $\omega_1+\omega_2=\omega$, and the substitutions $\xi_1\to\xi_1+(1-\alpha)c_F\mathbf{n}_1\mathbf{k}$ and $\xi_2\to -\xi_1+\alpha c_F\mathbf{n}_1\mathbf{k}$, so that $\xi_1+\xi_2=c_F\mathbf{n}_1\mathbf{k}$. We then obtain for (3.12) the expression

$$\begin{aligned}&-\frac{4Z^2k_F^2}{(2\pi)^4c_F}\int_0^1 d\alpha\int d\omega_1\,d\xi_1\,d\Omega_1 n_{1i}n_{1j}\\&\times\left\{\frac{\omega_1^2+\xi_1^2-\alpha(1-\alpha)q^2}{[\omega_1^2+\xi_1^2+\Delta^2+\alpha(1-\alpha)q^2]^2}-\frac{\omega_1^2+\xi_1^2}{[\omega_1^2+\xi_1^2+\Delta^2]^2}\right\},\end{aligned} \tag{3.13}$$

where $q^2=\omega^2+c_F^2(\mathbf{n}_1\mathbf{k})^2$. Integrating with respect to the variable $r_1^2=\omega_1^2+\xi_1^2$, we obtain

$$\frac{Z^2k_F^2}{4\pi^3c_F}\int_0^1 d\alpha\int d\Omega_1 n_{1i}n_{1j}\left[\ln\left(1+\frac{\alpha(1-\alpha)q^2}{\Delta^2}\right)+\frac{\alpha(1-\alpha)q^2}{\Delta^2+\alpha(1-\alpha)q^2}\right]. \tag{3.14}$$

Expanding the integrand in powers of the small parameter $\alpha(1-\alpha)q^2\Delta^{-2}$ and integrating with respect to $\alpha$, we arrive at the formula

$$A_{ij}(p)-A_{ij}(0)\approx\frac{Z^2k_F^2}{4\pi^3c_F}\int d\Omega_1\left(\frac{q^2}{3\Delta^2}-\frac{q^4}{20\Delta^4}\right)n_{1i}n_{1j}. \tag{3.15}$$

Similar calculations yield for $B_{ijab}(p)-B_{ijab}(0)$

$$B_{ijab}(p)-B_{ijab}(0)\approx\frac{Z^2k_F^2}{4\pi^3c_F}\int d\Omega_1\left(\frac{q^2}{6\Delta^2}-\frac{q^4}{30\Delta^4}\right)(\delta_{ab}-2n_{1a}n_{1b})n_{1i}n_{1j}. \tag{3.16}$$

The integrals with respect to the angle variable in (3.15) and (3.16) can be easily calculated. Using (3.11), (3.15), and (3.16), we write down the explicit expression for (3.10) in the form

$$\begin{aligned}&-\frac{k_F^2Z^2}{\pi^2c_F}\sum_p\left\{\left(\frac{1}{3}\mp\frac{1}{5}\right)w_{ia}w_{ia}\pm\frac{2}{15}(w_{aa}w_{bb}+w_{ia}w_{ai})\right.\\&+w_{ia}w_{ja}\left[\frac{\delta_{ij}}{3}\left(\frac{\omega^2}{3\Delta^2}-\frac{\omega^4}{20\Delta^4}\right)+\frac{c_F^2}{15}(k^2\delta_{ij}+2k_ik_j)\left(\frac{1}{3\Delta^2}-\frac{\omega^2}{10\Delta^4}\right)\right.\\&\left.-\frac{c_F^4}{700\Delta^4}(k^4\delta_{ij}+4k^2k_ik_j)\right]\pm w_{ia}w_{jb}\delta_{ab}\left[\frac{\delta_{ij}}{3}\left(\frac{\omega^2}{6\Delta^2}-\frac{\omega^4}{30\Delta^4}\right)\right.\\&\left.+\frac{c_F^2}{15}(k^2\delta_{ij}+2k_ik_j)\left(\frac{1}{6\Delta^2}-\frac{\omega^2}{15\Delta^4}\right)-\frac{c_F^4}{1050\Delta^4}(k^4\delta_{ij}+4k^2k_ik_j)\right]\\&+(\delta_{ab}\delta_{ij}+\delta_{ai}\delta_{bj}+\delta_{aj}\delta_{bi})\left[\frac{2}{15}\left(-\frac{\omega^2}{6\Delta^2}+\frac{\omega^4}{30\Delta^4}\right)+\frac{2k^2}{105}\left(-\frac{c_F^2}{6\Delta^2}+\frac{\omega^2c_F^2}{15\Delta^4}\right)\right.\\&\left.+\frac{c_F^4k^4}{4725\Delta^4}\right]+\frac{4}{315}(\delta_{ij}k_ak_b+\delta_{ab}k_ik_j+\delta_{ai}k_bk_j+\delta_{bj}k_ak_i+\delta_{aj}k_bk_i+\delta_{bi}k_ak_j)\\&\left.\times\left(-\frac{c_F^2}{2\Delta^2}+\frac{\omega^2c_F^2}{5\Delta^4}+\frac{c_F^4k^2}{15\Delta^4}\right)+\frac{8c_F^4}{4725\Delta^4}k_ak_bk_ik_j\right\}.\end{aligned} \tag{3.17}$$

It is understood here that we first must substitute $w_{ia}=u_{ia}$ and take the upper sign in place of the symbols $\pm$ and $\mp$, and then substitute $w_{ia}=v_{ia}$ and take the lower sign.

Expression (3.17) determines the phonon branches of the spectrum of the system. Since the $B$ phase is spatially isotropic and has no preferred direction, it suffices to consider excitations propagating in any direction, say along the third axis. The quadratic form of the variables $w_{ia}$ in (3.17) breaks up after the substitutions $k_1=k_2=0$ and $k_3=k$ into a sum of four variables, of which the first depends on $w_{12}$ and $w_{21}$, the second on $w_{13}$ and $w_{31}$, the third on $w_{23}$ and $w_{32}$, and the fourth on $w_{11}$, $w_{22}$, and $w_{33}$.

For $w_{ia}=v_{ia}$ the phonon branch is determined by the form of the variables $v_{11}$, $v_{22}$, and $v_{33}$. This form is proportional to the expression

$$a(v_{11}^2+v_{22}^2)+bv_{33}^2+2cv_{11}v_{22}+2d(v_{11}+v_{22})v_{33}. \tag{3.18}$$

Here

$$\begin{gathered}a={}^4/_{15}+a_1, \quad b={}^4/_{15}+b_1, \quad c=-{}^2/_{15}+c_1, \quad d=-{}^2/_{15}+d_1,\\a_1={}^{11}/_{90}x^2+{}^{13}/_{630}y^2-{}^{17}/_{900}x^4-{}^{19}/_{3150}x^2y^2-{}^1/_{900}y^4,\\b_1={}^{11}/_{90}x^2+{}^{17}/_{210}y^2-{}^{17}/_{900}x^4-{}^9/_{350}x^2y^2-{}^{37}/_{3780}y^4,\\c_1={}^1/_{45}x^2+{}^1/_{315}y^2-{}^1/_{225}x^4-{}^2/_{1575}x^2y^2-{}^1/_{4725}y^4,\\d_1={}^1/_{45}x^2+{}^1/_{105}y^2-{}^1/_{225}x^4-{}^2/_{525}x^2y^2-{}^1/_{945}y^4,\end{gathered} \tag{3.19}$$

where $x=\omega/\Delta$, $y=c_Fk/\Delta$.

The third-order determinant of the form (3.18) is equal to $(a-c)\times(b(a+c)-2d^2)$. Since the difference $a-c$ does not vanish as $\omega\to 0$ and $k\to 0$, we obtain

$$b(a+c)-2d^2=0 \tag{3.20}$$

or

$${}^2/_{15}(b_1+2a_1+2c_1+4d_1)+b_1(a_1+c_1)-2d_1^2=0. \tag{3.21}$$

We have

$$b_1+2a_1+2c_1+4d_1={}^1\!/_2x^2+{}^1\!/_6y^2-{}^1\!/_{12}x^4-{}^1\!/_{18}x^2y^2-{}^1\!/_{60}y^4,$$
$$b_1(a_1-c_1)-2d_1^2={}^1\!/_{60}x^4+{}^{13}\!/_{945}x^2y^2+{}^{11}\!/_{6300}y^4.$$

Substitution in (3.21) leads to the equation

$$x^2+{}^1\!/_3y^2+{}^1\!/_{12}x^4+{}^2\!/_{21}x^2y^2+{}^1\!/_{140}y^4=0.$$

Its solution after returning to the variables $\omega$ and $k$ and making the substitution $i\omega\to E$ yields the acoustic branch of the spectrum

$$E=\frac{c_Fk}{3^{1/2}}\left(1-\frac{2c_F^2k^2}{45\Delta^2}\right). \tag{3.22}$$

It is stable with respect to decay of excitation into two or several excitations of the same type. It is interesting that the dispersion coefficient $\gamma=2cF^2/45\Delta^2$ turns out to be two times as large as for the Fermi-gas model with pointlike scalar interaction.[6]

We consider now the form of the variables $u_{ia}$ which decay at $k_1=k_2=0$ into four independent forms. Of importance to us are the forms of the variables $(u_{12},u_{21})$ and $(u_{13},u_{31})$, which are proportional to the expressions

$$\begin{gathered}{}^2\!/_{15}(u_{12}+u_{21})^2+(u_{12}^2+u_{21}^2)[{}^{13}\!/_{90}x^2+{}^{19}\!/_{630}y^2-{}^7\!/_{300}x^4-{}^{31}\!/_{3150}x^2y^2-{}^{41}\!/_{18900}y^4],\\ -2u_{12}u_{21}[{}^1\!/_{45}x^2+{}^1\!/_{315}y^2-{}^1\!/_{225}x^4-{}^2\!/_{1575}x^2y^2-{}^1\!/_{4725}y^4],\\ {}^2\!/_{15}(u_{13}+u_{31})^2+u_{13}^2({}^{13}\!/_{90}x^2+{}^1\!/_{42}y^2-{}^7\!/_{300}x^4-{}^{23}\!/_{3150}x^2y^2-{}^1\!/_{756}y^4)\\ +u_{31}^2({}^{13}\!/_{90}x^2+{}^{19}\!/_{210}y^2-{}^7\!/_{300}x^4-{}^{31}\!/_{1050}x^2y^2-{}^{41}\!/_{3780}y^4)\\ -2u_{13}u_{31}({}^1\!/_{45}x^2+{}^1\!/_{105}y^2-{}^1\!/_{225}x^4-{}^2\!/_{525}x^2y^2-{}^1\!/_{945}y^4).\end{gathered} \tag{3.23}$$

The form of the variables $(u_{23},u_{32})$ is obtained from the second form in (3.23) by making the substitution $(u_{13},u_{31})\to(u_{23},u_{32})$. Equating to zero the determinants of the forms (3.23), we obtain the branch of the longitudinal spin waves (the variables $u_{12}$ and $u_{21}$)

$$E=\frac{c_Fk}{5^{1/2}}\left(1-\frac{2c_F^2k^2}{105\Delta^2}\right) \tag{3.24}$$

and the doubly degenerate branch of transverse spin waves (the variables $u_{13},u_{31}$ and $u_{23},u_{32}$)

$$E=\left(\frac{2}{5}\right)^{1/2}c_Fk\left(1-\frac{173c_F^2k^2}{3360\Delta^2}\right). \tag{3.25}$$

Both branches, as well as the acoustic branch (3.22) are stable.

## 4. THE *A* PHASE

The $A$ phase is anisotropic and has a preferred direction along the common orbital angular momentum of the Cooper pair. The gap is given by

$$\Delta=\Delta_0\sin\theta,\quad \Delta_0=2cZ. \tag{4.1}$$

The quantity $c$ enters in the condensate wave function $c_{ia}^{(0)}$:

$$c_{ia}^{(0)}=c(\beta V)^{1/2}\delta_{p0}(\delta_{i1}+i\delta_{i2})\delta_{a3}, \tag{4.2}$$

and satisfies the equation

$$\frac{1}{g}+\frac{2Z^2}{\beta V}\sum_p\frac{\sin^2\theta}{\omega^2+\xi^2+4c^2Z^2\sin^2\theta}=0. \tag{4.3}$$

Performing a shift by an amount $c_{ia}^{(0)}$ in the functional $S_h$ and separating the quadratic form with respect to the new variables $c_{ia}$, we get

$$\begin{aligned}&\sum_p c_{ia}^+(p)c_{ja}(p)\left[\frac{\delta_{ij}}{g}+\frac{4}{\beta V}\sum_{p_1+p_2=p}G(p_1)G(p_2)(\xi_1+i\omega_1)(\xi_2+i\omega_2)n_{1i}n_{1j}\right]\\ &+\frac{1}{2}\sum_{p,a=1,2}(c_{ia}^+(p)c_{ja}^+(-p)\\ &+c_{ia}(p)c_{ja}(-p))\frac{4\Delta_0^2}{\beta V}\sum_{p_1+p_2=p}(n_1\pm in_2)^2n_{1i}n_{1j}G(p_1)G(p_2)\\ &-\frac{1}{2}\sum_p(c_{i3}^+(p)c_{j3}^+(-p)\\ &+c_{i3}(p)c_{j3}(-p))\frac{4\Delta_0^2}{\beta V}\sum_{p_1+p_2=p}(n_1\pm in_2)^2n_{1i}n_{1j}G(p_1)G(p_2),\end{aligned} \tag{4.4}$$

where $G(p)=Z(\omega^2+\xi^2+\Delta_0^2\sin^2\theta)^{-1}$. The plus or minus sign in (4.4) means that it is necessary to take the plus sign when multiplying by $c_{ia}^+c_{ja}^+$ and the minus sign when multiplying by $c_{ia}c_{ja}$.

The form (4.4) is a sum of three independent forms that differ in the value of the isotopic index $a$ and go over into one another when the variables are interchanged. Therefore the spectrum in the $A$ phase turns out to be triply degenerate and it suffices to consider one of the three forms (e.g., with $a=1$).

We take the form with $a=1$ and separate in it the terms corresponding to $p=0$:

$$-\frac{k_F^2Z^2}{2\pi^2c_F}(3v_{11}^2+3u_{21}^2-2u_{21}v_{11}+(u_{11}-v_{21})^2). \tag{4.5}$$

It is clear therefore that one can choose as the phonon variables

$$u=\mathrm{Re}\,c_{31},\quad v=\mathrm{Im}\,c_{31},\quad w={}^1\!/_2\mathrm{Re}(c_{11}-ic_{21}). \tag{4.6}$$

The Bose spectrum is determined by a form that depends on the phonon variables. It is possible to make in it the substitutions $A_{ij}(p)\to A_{ij}(p)-A_{ij}(0)$ and $B_{ij}(p)\to B_{ij}(p)-B_{ij}(0)$, since $A_{ij}(0)$ and $B_{ij}(0)$ are equal to zero for the phonon variables. Calculation of $A_{ij}(p)-A_{ij}(0)$ and $B_{ij}(p)-B_{ij}(0)$ is analogous to that carried out above for the $B$ phase. Using the Feynman procedure and integrating with respect to $\omega_1$ and $\xi_1$ (as $T\to0$), we obtain

$$\begin{aligned}A_{ij}(p)-A_{ij}(0)&=\frac{Z^2k_F^2}{4\pi^3c_F}\int_0^1d\alpha\int d\Omega_1n_{1i}n_{1j}\left[\ln\left(1+\frac{\alpha(1-\alpha)q^2}{\Delta_0^2\sin^2\theta_1}\right)+\frac{\alpha(1-\alpha)q^2}{\Delta_0^2\sin^2\theta_1+\alpha(1-\alpha)q^2}\right],\\ B_{ij}^\pm(p)-B_{ij}^\pm(0)&=\frac{Z^2k_F^2}{4\pi^3c_F}\int_0^1d\alpha\int d\Omega_1n_{1i}n_{1j}\\ &\quad\times\exp(\pm2i\varphi_1)\frac{\alpha(1-\alpha)q^2}{\Delta_0^2\sin^2\theta_1+\alpha(1-\alpha)q^2}.\end{aligned} \tag{4.7}$$

Here $q^2=\omega^2+c_F^2(\mathbf{n}_1\mathbf{k})^2$, $B_{ij}^+$ is the coefficient of $c_i^+c_j^+$ and $B_{ij}^-$ is the coefficient of $c_ic_j$.

If we expand under the integral signs in (4.7) in powers of $\alpha(1-\alpha)q^2/\Delta_0^2\sin^2\theta_1$, at small $\omega$ and $k$, and confine ourselves to the first term of the expansion, then we obtain in the calculation of $A_{33}(p)-A_{33}(0)$ a logarithmically diverging integral proportional to

$$\int d\Omega_1\frac{q^2n_3^2}{\Delta_0^2\sin^2\theta_1}=\frac{2\pi(\omega^2+c_F^2k_3^2)}{\Delta_0^2}\int_0^\pi\frac{d\theta_1}{\sin\theta_1}+\text{finite part}. \tag{4.8}$$

The cause of the divergence is that near the poles of the Fermi sphere ($\theta = 0, \pi$) the parameter $q^2\alpha(1-\alpha)/\Delta_0^2\sin^2\theta_1$, is no longer small and we cannot expand in its terms. What is needed here is an accurate calculation of the first integral of (4.7) at $i = j = 3$, the results of which is

$$A_{33}(p)-A_{33}(0)=\frac{k_F^2Z^2}{48\pi^2c_F\Delta_0^2}\left[2p^2\ln\frac{4\Delta_0^2}{p^2}+\frac{1}{3}p^2+\frac{2}{3}c_F^2k^2-2c_F^2k_3^2\right]. \tag{4.9}$$

at small $p^2 = \omega^2 + c_F^2k_3^2$.

This difficulty does not arise in the calculation of the remaining elements $A_{ij}(p) - A_{ij}(0)$, as well as $B_{ij}(p) - B_{ij}(0)$. However, it is precisely the appearance of logarithms which makes it possible to calculate the corrections to the linear dispersion law in the chosen approximation, where we confine ourselves to the phonon variables. For comparison we recall that in the $B$ phase the calculation of the dispersion coefficient called for allowance for the coupling between the photon and non-phonon modes.

As a result we arrive at the following matrix of the phonon variables:

$$Q=\frac{k_F^2Z^2}{48\pi^2c_F\Delta_0^2}\begin{pmatrix} a(p)+{}^1/_{12}c_F^2(k_1^2-k_2^2), & {}^1/_6\,c_F^2k_1k_2, & c_F^2k_1k_3 \\ {}^1/_6\,c_F^2k_1k_2, & a(p)+{}^1/_{12}c_F^2(k_2^2-k_1^2), & c_F^2k_2k_3 \\ c_F^2k_1k_3 & c_F^2k_2k_3, & 3\omega^2+c_F^2k^2 \end{pmatrix}, \tag{4.10}$$

where

$$a(p)=p^2\left(\ln\frac{4\Delta_0^2}{p^2}+\frac{1}{6}\right)+\frac{c_F^2}{3}(k^2-3k_3^2). \tag{4.11}$$

The equation $\det Q = 0$ can be written in the form

$$(a(p)-{}^1/_{12}c_F^2k_\perp^2)\left[(3\omega^2+c_F^2k^2)(a(p)+{}^1/_{12}c_F^2k_\perp^2)-c_F^4k_\perp^2k_\parallel^2\right]=0, \tag{4.12}$$

where $k_\parallel^2 = k_3^2$ and $k_\perp^2 = k_1^2 + k_2^2$. The equation gives three branches of the spectrum: one with $E^2 \approx c_F^2k^2/3$ and two with $E^2 \approx c_F^2k_\parallel^2$. From (4.12) we can obtain the corrections to the linear dispersion law and determine the region of stability of the Bose spectrum. The result of the solution of (4.12) is

$$\begin{aligned} E_1(\mathbf{k})&=\frac{c_Fk}{3^{1/2}}\left(1-\frac{\sin^2\theta\cos^2\theta}{2(\cos^2\theta-1/3)\ln(4\Delta_0^2/f_1(\theta,k))}\right),\\ E_2(\mathbf{k})&=c_Fk_\parallel\left(1-\frac{11\cos^2\theta-3}{24\cos^2\theta\ln(4\Delta_0^2/f_2(\theta,k))}\right)\\ E_3(\mathbf{k})&=c_Fk_\parallel\left(1-\frac{51\cos^4\theta-40\cos^2\theta+5}{72\cos^2\theta(\cos^2\theta-1/3)\ln(4\Delta_0^2/f_3(\theta,k))}\right), \end{aligned} \tag{4.13}$$

where

$$\begin{gathered} f_1(\theta,k)=c_F^2k^2(\cos^2\theta-1/3),\quad f_2(\theta,k)={}^1/_{12}c_F^2k^2(11\cos^2\theta-3),\\ f_3(\theta,k)=\frac{c_F^2k^2}{36}\frac{51\cos^4\theta-40\cos^2\theta+5}{\cos^2\theta(\cos^2\theta-1/3)}. \end{gathered} \tag{4.14}$$

The obtained equations show that the stability of the spectrum in the $A$ phase depends on the angle $\theta$ between the excitation momentum and the preferred direction. The first (acoustic) mode is stable inside the cones $\cos^2\theta > 1/3$, and the second inside the cones $\cos^2\theta > 3/11$, (Fig. 2). The third mode is stable in the regions

$$\cos^2\theta>\frac{20+(145)^{1/2}}{51},\quad \frac{1}{3}>\cos^2\theta>\frac{20-(145)^{1/2}}{51}.$$

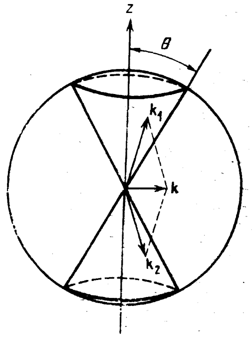


FIG. 2.

Outside the instability regions, the energy of the excitation becomes complex because of the imaginary parts of the logarithms in (4.13). Physically this is connected with the impossibility of the decay of the excitation into constituent fermions whose momenta are close to the preferred direction.

Equations (4.13) for the orbital waves $E_2(\boldsymbol{k})$ and $E_3(\boldsymbol{k})$ can be compared with the results of Ref. 5, where the following dispersion laws were obtained for them:

$$(\omega^2-k_z^2v_F^2)\left\{\ln\frac{4\delta^2}{k_z^2v_F^2-\omega^2}+\frac{1}{5}\right\}=\frac{1}{4}k_y^2v_F^2+\frac{1}{3}k_z^2v_F^2\left(\frac{1}{4}F_1^s-2\right), \tag{4.15}$$

$$(\omega^2-k_z^2v_F^2)\left\{\ln\frac{4\delta^2}{k_z^2v_F^2-\omega^2}+\frac{1}{6}\right\} =k^2v_F^2\left\{\frac{5}{12}\hat{k}_y^2-\frac{2}{3}\hat{k}_z^2-\hat{k}_y^2\hat{k}_z^2+\frac{F_1^S}{2}(\hat{k}_y^2-\hat{k}_z^2)^2\right\}, \tag{4.16}$$

where $\hat{k}_z = \cos\theta$, $\hat{k}_y = \sin\theta$, $\delta = \Delta_0$, and $v_F = c_F$. Although the excitation stability investigated in Ref. 5 was with respect to decay into fermions, we can use (4.15) and (4.16) to estimate the stability with respect to the decays of orbital excitations into two or several excitations of the same type (with respect to the sign of $\partial^2E/\partial k^2$).

If we neglect in (4.15) the Fermi-liquid correction $F_1^S$, we obtain an equation that differs from $a(p) - c_F^2k_\perp^2/12 = 0$ only in the addition to the logarithm [1/6 in (4.11) and 1/5 in (4.15)]. The stability region at $F_1^S$ turns out to be the same ($\cos^2\theta > 3/11$) as for the $E_2(\boldsymbol{k})$ branch in the model system considered here.

For the branch (4.16), the stability takes place inside the cones $\cos^2\theta > (25-(385)^{1/2})/24 \approx 0.22$, and the situation here is greatly different from that of $E_3(\boldsymbol{k})$, where we have two stability regions.

Thus, one of the modes of the orbital waves, obtained here by a method based on functional integration, coincides with the one obtained in Ref. 5 by the kinetic-equation method. As to the second orbital mode, the corrections to the linear spectrum $E = c_Fk_\parallel$ in Ref. 5 and in our paper are substantially different and lead to different stability regions. The possible reason is that in our case the mode $E_3(\boldsymbol{k})$ is coupled with the acoustic mode $E_1(\boldsymbol{k})$, whereas no such coupling is considered in Ref. 5.

## 5. THE 2D PHASE

The planar $2D$ phase is stable in a zero magnetic field.

However, as shown in Ref. 4, when the external magnetic field is increased in the considered model to $H = H_c$, the $B$ phase goes over into the $2D$ phase. Alonso and Popov[4] obtained an explicit formula for the critical magnetic field in the Ginzburg-Landau region

$$H_c^2 = 2\pi^2 T_c \Delta T / 7\zeta(3)\mu^2 \tag{5.1}$$

and advanced arguments favoring the possibility of a transition of real $He^3$ from the $B$ to the $2D$ phase in a magnetic field.

We investigate here the Bose spectrum of the $2D$ phase as $T \to 0$ and show that the phonon excitations are determined by the same equation (4.12) and are given by the same formulas (4.13) as in the $A$ phase, and differ only in the degeneracy multiplicity (2 in the $2D$ phase and 3 in the $A$ phase).

The preferred direction in the $2D$ phase is the direction of the external magnetic field. The condensate wae wave function is of the form

$$c_{ia}^{(0)}(p) = c(\beta V)^{1/2}\delta_{p0}(\delta_{i1}\delta_{a1} + \delta_{i2}\delta_{a2}), \tag{5.2}$$

where $c$ satisfies an equation that coincides with (4.3) for the $A$ phase. The form of the gap $\Delta = \Delta_0 \sin\theta$ also coincides with the one existing in the $A$ phase.

Following a shift $c_{ia} \to c_{ia}^{(0)} + c_{ia}$ and separation of the quadratic form with respect to the new variables, we obtain the expression

$$\begin{aligned}
&\sum_p c_{ia}^+(p) c_{jb}(p) \left[\frac{\delta_{ij}\delta_{ab}}{g} + \frac{2Z^2}{\beta V} \sum_{p_1+p_2=p} n_{1i} n_{1j} \operatorname{tr}_2 (A_1 - B_1\sigma_3)\sigma_a (A_2 + B_2\sigma_3)\sigma_b\right] \\
&-\sum_p c_{ia}^+(p) c_{jb}^+(-p) \frac{Z^2}{\beta V} \sum_{p_1+p_2=p} n_{1i} n_{1j} \operatorname{tr}_2 (C_{-1}\sigma_1 + D_{-1}\sigma_2)\sigma_a (C_2\sigma_1 + D_2\sigma_2)\sigma_b \\
&-\sum_p c_{ia}(p) c_{jb}(-p) \frac{Z^2}{\beta V} \sum_{p_1+p_2=p} n_{1i} n_{1j} \operatorname{tr}_2 (\bar C_{-1}\sigma_1 + \bar D_{-1}\sigma_2)\sigma_a (\bar C_2\sigma_1 + \bar D_2\sigma_2)\sigma_b .
\end{aligned} \tag{5.3}$$

Here $\operatorname{tr}_2$ denotes the trace of a second-order matrix, and the functions $A$, $B$, $C$, and $D$ are given by

$$\begin{aligned}
A &= M^{-1}[-(i\omega+\xi)(\omega^2+\xi^2+\mu^2H^2+\Delta_0^2(n_1^2+n_2^2)) + 2\xi\mu^2H^2], \\
B &= M^{-1}\mu H[\omega^2+\xi^2+\mu^2H^2+\Delta_0^2(n_1^2+n_2^2) - 2\xi(i\omega+\xi)], \\
C &= M^{-1}\Delta_0[n_1(\omega^2+\xi^2+\mu^2H^2+\Delta_0^2(n_1^2+n_2^2)) - 2i\xi\mu H n_2], \\
D &= M^{-1}\Delta_0[n_2(\omega^2+\xi^2+\mu^2H^2+\Delta_0^2(n_1^2+n_2^2)) + 2i\xi\mu H n_1], \\
M &= [\omega^2+\xi^2+\mu^2H^2+\Delta_0^2(n_1^2+n_2^2)]^2 - 4\xi^2\mu^2H^2.
\end{aligned} \tag{5.4}$$

Examination of those terms of (5.3) which correspond to $p = 0$ shows that the phonon variables of the system are

$$u = {}^1/_2(u_{12} - u_{21}), \quad v = {}^1/_2(v_{11} + v_{22}), \quad u_{31}, \quad u_{32}, \quad v_{31}, \quad v_{32}. \tag{5.5}$$

We set in (5.3) all the nonphonon variables equal to zero, making the substitution

$$\begin{aligned}
&u_{i3} = v_{i3} = 0, \quad u_{11} = u_{22} = 0, \quad v_{12} = v_{21} = 0, \quad u_{12} = -u_{21} = u, \\
&v_{11} = v_{22} = v, \quad u_{31} = u_{31}, \quad u_{32} = u_{32}, \quad v_{31} = v_{31}, \quad v_{32} = v_{32}.
\end{aligned} \tag{5.6}$$

Subtracting from the coefficient tensors their values at $p = 0$ and calculating the trace $\operatorname{tr}_2$, we obtain in place of (5.3)

$$\begin{aligned}
&\sum_p \Big\{ \sum_{a=1,2} c_{ia}^+ c_{ja} \frac{4Z^2}{\beta V} \sum_{p_1+p_2=p} n_{1i} n_{1j} (A_1A_2 + B_1B_2 - A_1A_{-1} - B_1B_{-1}) \\
&-(c_{i1}c_{j1} - c_{i2}c_{j2}) \frac{2Z^2}{\beta V} \sum_{p_1+p_2=p} n_{1i} n_{1j} (\bar C_{-1}\bar C_2 - \bar D_{-1}\bar D_2 - \bar C_{-1}\bar C_{-1} + \bar D_{-1}\bar D_{-1}) \\
&-(c_{i1}^+c_{j1}^+ - c_{i2}^+c_{j2}^+) \frac{2Z^2}{\beta V} \sum_{p_1+p_2=p} n_{1i} n_{1j} (C_{-1}C_2 - D_{-1}D_2 - C_{-1}C_{-1} + D_{-1}D_{-1}) \\
&-c_{i1}c_{j2} \frac{4Z^2}{\beta V} \sum_{p_1+p_2=p} n_{1i} n_{1j} (\bar C_{-1}\bar D_2 + \bar D_{-1}\bar C_2 - 2\bar C_{-1}\bar D_{-1}) \\
&-c_{i1}^+c_{j2}^+ \frac{4Z^2}{\beta V} \sum_{p_1+p_2=p} n_{1i} n_{1j} (C_{-1}D_2 + D_{-1}C_2 - 2C_{-1}D_{-1}).
\end{aligned} \tag{5.7}$$

It is understood that the substitutions (5.6) have been made in (5.7).

The expression $A_1A_2 + B_1B_2 - A_1A_{-1} - B_1B_{-1}$ can be represented in the form of a sum of two terms, one of which depends on $\xi_1 + \mu H$ and the other on $\xi_1 - \mu H$. Then, after replacing the integration variable $\xi_1 \to \xi_1 - \mu H$ in the first term and $\xi_1 \to \xi_1 + \mu H$ in the second term, we arrive at expressions that do not depend on $H$. We can therefore replace

$$\begin{gathered}
Z^2(A_1A_2 + B_1B_2 - A_1A_{-1} - B_1B_{-1}) \to (i\omega_1+\xi_1)(i\omega_2+\xi_2)G_1G_2 - (\omega_1^2+\xi_1^2)G_1^2, \\
G_i = Z(\omega_i^2 + \xi_i^2 + \Delta_0^2\sin^2\theta)^{-1}.
\end{gathered}$$

We similarly have

$$\begin{gathered}
Z^2(C_{-1}C_2 - D_{-1}D_2 - C_{-1}C_1 + D_{-1}D_1) \to \Delta_0^2(n_1^2 - n_2^2)(G_1G_2 - G_1^2), \\
Z^2(\bar C_{-1}\bar C_2 - \bar D_{-1}\bar D_2 - \bar C_{-1}\bar C_1 + \bar D_{-1}\bar D_1) \to \Delta_0^2(n_1^2 - n_2^2)(G_1G_2 - G_1^2), \\
Z^2(C_{-1}D_2 + D_{-1}C_2 - 2C_{-1}D_1) \to 2\Delta_0^2 n_1n_2(G_1G_2 - G_1^2), \\
Z^2(\bar C_{-1}\bar D_2 + \bar D_{-1}\bar C_2 - 2\bar C_{-1}\bar D_1) \to 2\Delta_0^2 n_1n_2(G_1G_2 - G_1^2).
\end{gathered}$$

We can now rewrite (5.7) in the form

$$\begin{aligned}
&\sum_p \Big\{ \sum_{a=1,2} (u_{ia}u_{ja} + v_{ia}v_{ja}) \frac{4}{\beta V} \\
&\times \sum_{p_1+p_2=p} n_{1i}n_{1j}[(i\omega_1+\xi_1)(i\omega_2+\xi_2)G_1G_2 - (\omega_1^2+\xi_1^2)G_1^2] \\
&-\frac{4\Delta_0^2}{\beta V} \sum_{p_1+p_2=p} n_{1i}n_{1j}(G_1G_2 - G_1^2)[(n_1^2 - n_2^2)(u_{i1}u_{j1} - u_{i2}u_{j2} - v_{i1}v_{j1} + v_{i2}v_{j2}) \\
&\qquad + 4n_1n_2(u_{i1}u_{j2} - v_{i1}v_{j2})].
\end{aligned} \tag{5.8}$$

We have obtained the sum of two forms, one of which depends only on $u$ and the only on $v$, while the $v$ form is obtained from the $u$ form by the substitutions $u \to v$, $u_{31} \to -v_{32}$, and $u_{32} \to v_{31}$. This shows that the spectrum is doubly degenerate.

Comparing the $v$ form in (5.8) with the form of the variables $c_{ij}$ in the $A$ phase (4.4), we see that they coincide if we replace $w = 1/2(u_{11} + v_{21})$ in (4.4) by $v = 1/2(v_{11} + v_{22})$, $u = u_{31}$ by $v_{31}$, and $v = v_{31}$ by $v_{32}$.

Thus, in the considered approximation (neglecting the coupling between the phonon and zero-phonon modes), the phonon modes of the spectrum in the $A$ and $2D$ phases coincide and differ only in the degeneracy multiplicity (3 in the $A$ phase and 2 in the $2D$ phase). This neglect does not influence the stability of the spectrum. We note that no such agreement obtains for the zero-phonon modes, which depend on $H$ in the $2D$ phase.

[1)] We call a Bose excitation a phonon (an excitation of the phonon type) if its spectrum starts out from zero, so that $\lim E(\mathbf{k}) = 0$ as $\mathbf{k} \to 0$.

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# New type of high-frequency instability in nematic liquid crystals

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The presence of inertia effects gives rise to the appearance of a new high-frequency electrohydrodynamic instability in nematic liquid crystals. The fundamental difference between this new instability and those previously known is considered theoretically. A qualitatively new behavior of the threshold characteristics of the instability is obtained analytically and numerically as a function of the field frequency, of the anisotropic parameters of the medium, and of the thickness of the liquid-crystal layer.

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## INTRODUCTION

It is known that there are two regimes of electrohydrodynamic (EHD) instability of nematic liquid crystals (NLC), namely the low-frequency conduction regime and the high-frequency dielectric regime.[1] In the conduction regime, which is realized at frequencies $\omega$ of the external electric field $E$ lower than the reciprocal $\tau_e^{-1}$ of the space-charge relaxation time, the orientation of the director n and the flow velocity $v$ of the NLC hardly change with time, and the volume electric charge $Q$ oscillates almost in phase with the electric field $E(t)$. In the dielectric regime at $\omega \gg \tau_e^{-1}$ the situation is reversed: the space charge oscillates weakly about a certain mean value, and the orientation and velocity vary with the frequency of the external field. These regimes differ substantially in the frequency dependence of the threshold characteristics—the voltage $U_c(\omega)$ and the wave numbers of the produced modulated structure $k_c(\omega)$. At $\omega \ll \tau_e^{-1}$ we have $U_c(\omega)=\text{const}$ and $k_c(\omega)\sim \pi d^{-1}$, where $d$ is the thickness of the NLC layer; at $\omega \gg \tau_e^{-1}$ we have $U_c(\omega)\sim d\omega^{1/2}$ and $k_c(\omega)\sim\omega^{1/2}$. We emphasize that in the dielectric regime the instability threshold depends strongly on the thickness $d$, while the wave number is independent of the latter.

These EHD instability regimes of NLC are described by the system of linear equations of nematodynamics,[1] in which one usually neglects the inertial terms connected with the derivative $dv/dt$ in the Navier-Stokes equations. In fact, this term is small at sufficiently low field frequency $\omega$ and layer thickness $d$.[2] The inertial effects were taken into account by us approximately in an earlier paper.[2] A solution branch was obtained, corresponding to oscillations of the space-charge density, of the flow velocity, and of the director orientation.

In the present paper we describe analytically and numerically a new EHD instability regime, which arises at $\omega \gg \tau_e^{-1}$ as a result of the presence of inertial effects. The fundamental difference between this regime and those mentioned earlier is that in the present case the space charge oscillates almost in counterphase with the electric field, whereas the orientation of the director and the flow velocity vary little about their mean values. A physical consequence of these solutions is a qualitatively new behavior of the threshold characteristics as functions of the field frequency, of the material parameters, and of the thickness of the NLC layer.

## ANALYTIC SOLUTION

We write down the complete system of equations of nematodynamics. We introduce a coordinate system $(x, y, z)$ with the $x$ axis directed along the preferred orientation of the molecules in the initial state in the substrate plane, the $z$ axis perpendicular to the substrate, and the $y$ axis along the direction of the domain structure. Assuming that the deviations of the director and the motion of the liquid occur in the $xz$ plane, we obtain in the linear approximation the following system of equations[2]:

$$\begin{aligned}&\frac{dv_z}{ds}+\frac{1}{\tau_v}v_z+\varkappa\psi+\delta E(t)Q=0,\\&\frac{d\psi}{ds}+\Gamma\psi+\Omega v_z+\frac{1}{\eta}E(t)Q=0,\\&\frac{dQ}{ds}+\frac{1}{\tau_e}Q+\Sigma E(t)\psi=0,\end{aligned}\tag{1}$$

where $\psi=\partial n_z/\partial x$, $Q$ is the space charge, $q=k_z/k_x$ is the ratio of the wave vectors of the deformation along the $z$ and $x$ axes (here, as in Ref. 2, the boundary conditions are taken into account approximately, and it is assumed

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