Hydrodynamics of defects in condensed media, using as examples vortices in rotating He II and disclinations in a planar magnet

G. E. Volovik and V. S. Dotsenko, Jr.

L. D. Landau Institute of Theoretical Physics, USSR Academy of Sciences
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The gauge-field formalism earlier by Dzyaloshinskii and Volovik [J. de Phys. **39**, 693 (1978); Ann. Phys. (1980)] for the description of continual dynamics of defects in condensed media is developed further. The Poisson brackets for hydrodynamic variables such as gauge fields are obtained from an analysis of the dynamics of an isolated defect. They turn out to be different for fields describing distributed vortices in He II and distributed disclinations in planar magnets. A complete system of hydrodynamic equations is obtained for a lattice of vortices in rotating He II. The spectrum of the Tkachenko waves is obtained, with dissipation taken into account. A generalization of the equations obtained for planar magnets to include the case of nonplanar magnets with linear defects and spin glasses is proposed. The fourth spin-glass low-frequency mode due to defect motion is obtained.

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INTRODUCTION

Interest has been evinced recently in the dynamics of various condensed media with structure defects. These include crystals with continuously distributed dislocations and disclinations (see, e.g., Ref. 1); superfluid HeII with a system of quantized vortices produced either when the vessel is rotated or in the presence of a chemical-potential gradient applied to the system as a result of an effect similar to the nonstationary Josephson effect in superconductors[2]; liquid crystals with systems of disclinations that can be produced by application of a difference between angular velocities (see, e.g., Ref. 3), etc. One can include among these systems also spin glasses, which can be regarded as the continual limit of a magnet with disclinations,[4,5] and two-dimensional systems with statistical defects.[6]

Although defects in condensed media are classified in accord with a general approach connected with homotopic groups,[7] no general approach to the description of the dynamics of distributed effects has yet been developed. Common to the derivation of equations of hydrodynamics for media with defects is apparently the method of Poisson brackets (see Refs. 5, 8, 9), in which the dynamic equations are obtained with the aid of the Liouville equations if the energy of the system is specified as a function of the variables that characterize the hydrodynamic motions of the system, and if the relative Poisson brackets for these variables are specified. So far, however, there is no general method of introducing the hydrodynamic variables responsible for the dynamics of the distributed defects. The method proposed by Dzyaloshinskii and one of us[4] for describing defects with the aid of gauge fields, while suitable for the case of defects such as singular lines, must be substantially modified if it is to describe systems with pointlike singularities and solitons.[10] In another paper, Dzyaloshinskii and one of us[8] consider systems with only linear topological defects, and therefore gauge fields are introduced as the hydrodynamic variables responsible for the defects, under the assumption that the Poisson brackets between these variables are equal to zero. As a result, the defects more together with the medium in the absence of dissipation, i.e., have no motion of their own. Therefore the equations obtained in Ref. 8 are valid only in the limit of a strong interaction between the defects and the medium.

The purpose of the present paper is to lift this restriction and find the Poisson brackets for gauge fields. As will be shown below, these Poisson brackets depend on the dynamics of the isolated defects and can be different for different condensed media, even if the defects are described by the same homotopic group. Therefore an individual derivation of the Poisson brackets is needed for each condensed medium with defects. We confine ourselves to two such systems: the vortex lattice in rotating HeII at $T \neq 0$ (the case $T = 0$ is described briefly by us earlier[11]), and a planar magnet in which, if the system is two-dimensional, there can exist a nonzero equilibrium concentration of defects—disclinations. In either case, the space of the internal states constitutes a one-dimensional unitary group U_1. In the first case this is the group of gauge transformations, and in the second it is a group of planar rotations. Therefore the defects in both systems are described by the same homotopic group $\Pi_1(U_1) = Z$.

In Sec. 1 we present the Poisson brackets for the variables that describe HeII without vortices. These brackets yield the known equations of the hydrodynamics of HeII, including the equation of motion for the density ρ^s of the superfluid component. In Sec. 2 we introduced, in place of the phase Φ of the condensate, a gauge-like vector field $\mathbf{v}^s$ whose solenoidal part describes the vortex density, and obtain the Poisson brackets for the components of $\mathbf{v}^s$ with one another and with other variables. In Sec. 3 are added variables that describe the displacements of the sites of the vortex lattice produced when a vessel with HeII is rotated. The result is the complete system of equations for the hydrodynamics of rotating HeII. In Sec. 4, starting from the derived equations, we investigate the spectrum of the Tkachenko waves. In Sec. 5 are obtained the Poisson brackets for variables describing a planar

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magnet with disclinations. These variables include, besides gaugelike fields, the density of the intrinsic momentum of the defects. The results are generalized in Sec. 6 to the case of a nonplanar magnet with disclinations and to spin glass.

1. SUPERFLUID HeII WITHOUT VORTICES

We consider first superfluid HeII in the absence of vortices. The variables that characterize the hydrodynamic motions in HeII are the mass density ρ, the momentum density p, the entropy density S, and the phase of the condensate (or the phase of the order parameter) Φ. The equations of motion for these variables in the absence of dissipation are the Liouville equations, for example

$$\frac{\partial\rho}{\partial t}+\{\rho,H\}=0,$$

with a Hamiltonian that constitutes the total energy of the system:

$$H=\int d^3r\,\varepsilon(\mathbf{p},\rho,S,\Phi).$$

To specify the equations concretely, it is necessary to know the relative Poisson brackets for the hydrodynamic variables and the dependence of the energy on these variables. To write down the Poisson brackets we repeat briefly the reasoning advanced in Ref. 8.

The hydrodynamic variables include the conserved quantities $\mathbf{p}$ and ρ, which are the densities of the generators of the symmetry transformations, namely: of the translation and of the gauge transformation. Let a small shift $\mathbf{u}(\mathbf{r})$ and a small change of phase $\varphi(\mathbf{r})$ be the infinitesimal parameters of these transformations. Then the Poisson brackets of any variable A with $\mathbf{p}$ and ρ take the form

$$\{\mathbf{p},A\}=\delta A(\mathbf{r}')/\delta\mathbf{u}(\mathbf{r}),\tag{1.1}$$
$$\{\rho(\mathbf{r}),A(\mathbf{r}')\}=\delta A(\mathbf{r}')/\delta\varphi(\mathbf{r}),\tag{1.2}$$

where δA in the right-hand sides stand for the changes, which depend on $\mathbf{u}$ and φ, of the value of A under the small transformations. We write down the changes of all the hydrodynamic quantities under these transformations:

$$\delta p_k=-\delta u^i\nabla_i p_k-p_i\nabla_k\delta u^i-p_k\nabla_i\delta u^i-\rho\nabla_k\delta\varphi,\tag{1.3}$$
$$\delta\rho=-\delta u^i\nabla_i\rho-\rho\nabla_i\delta u^i,\tag{1.4}$$
$$\delta S=-\delta u^i\nabla_i S-S\nabla_i\delta u^i,\tag{1.5}$$
$$\delta\Phi=-m\hbar^{-1}\delta\varphi-\delta u^i\nabla_i\Phi.\tag{1.6}$$

Using (1.1)–(1.6), we obtain the following nonzero Poisson brackets:

$$\{p_i(\mathbf{r}),p_k(\mathbf{r}')\}=(p_k(\mathbf{r})\nabla_i-p_i(\mathbf{r}')\nabla_k')\delta(\mathbf{r}-\mathbf{r}'),\tag{1.7}$$
$$\{p_i(\mathbf{r}),\rho(\mathbf{r}')\}=\rho(\mathbf{r})\nabla_i\delta(\mathbf{r}-\mathbf{r}'),\tag{1.8}$$
$$\{p_i(\mathbf{r}),S(\mathbf{r}')\}=S(\mathbf{r})\nabla_i\delta(\mathbf{r}-\mathbf{r}'),\tag{1.9}$$
$$\{p_i(\mathbf{r}),\Phi(\mathbf{r}')\}=-\nabla_i\Phi\delta(\mathbf{r}-\mathbf{r}');\tag{1.10}$$
$$\begin{gathered}\{\rho(\mathbf{r}),p_k(\mathbf{r}')\}=-\rho(\mathbf{r}')\nabla_k'\delta(\mathbf{r}-\mathbf{r}')=-\{p_k(\mathbf{r}'),\rho(\mathbf{r})\},\\ \{\rho(\mathbf{r}),\Phi(\mathbf{r}')\}=-m\hbar^{-1}\delta(\mathbf{r}-\mathbf{r}').\end{gathered}\tag{1.11}$$

In some situations (for example, near the transition temperature), the system of hydrodynamic equations must be supplemented by one more quantity—the super-hyperfluid component density ρ^s, which is the modulus of the order parameter (see Ref. 12), and as such is canonically conjugate to the order-parameter phase Φ. Therefore

$$\{\rho^s(\mathbf{r}),\Phi(\mathbf{r}')\}=-m\hbar^{-1}\delta(\mathbf{r}-\mathbf{r}'),\tag{1.12}$$

and, since ρ^s is transformed in accordance with the formula

$$\delta\rho^s=-\delta u^i\nabla_i\rho^s-\rho^s\nabla_i\delta u^i,$$

it follows that

$$\{p_i(\mathbf{r}),\rho^s(\mathbf{r}')\}=\rho^s(\mathbf{r})\nabla_i\delta(\mathbf{r}-\mathbf{r}').\tag{1.13}$$

The remaining Poisson brackets are equal to zero. The Hamiltonian of the system depends also on ρ^s:

$$H=\int d^3r\,\varepsilon(\mathbf{p},\rho,\rho^s,\Phi,S).\tag{1.14}$$

With the aid of the Liouville equations we obtain from the Hamiltonian (1.14) and the Poisson brackets (1.7)–(1.13) all the equations of the nondissipative dynamics of HeII. The dissipation can be introduced in standard fashion in terms of the dissipation function R (see Ref. 13), which depends on variables that are the thermodynamic conjugates of the variables $\mathbf{p}$, ρ, ρ^s, S, and Φ. Let A be any of these variables, except the entropy; then the equation for this variable, with allowance for dissipation, is

$$\frac{\partial A}{\partial t}+\{A,H\}=-\frac{\delta R}{\delta(\delta H/\delta A)}.\tag{1.15}$$

It is necessary to add to the equation for the entropy a term that describes the entropy production. In the case when R is a quadratic form of its variables, this term is equal to $2R/T$:

$$\frac{\partial S}{\partial t}+\{S,H\}=-\delta R/\delta T+2R/T,\tag{1.16}$$

where $T=\partial\varepsilon/\partial S$ is the temperature.

To write down the equations we must specify the forms of H and R. Let the energy density ε be given by

$$\varepsilon=\frac{\rho^s(\mathbf{v}^s)^2}{2}+\frac{(\mathbf{p}-\rho^s\mathbf{v}^s)^2}{2(\rho-\rho^s)}+\varepsilon_0(\rho,\rho^s,S)+\frac{1}{2}\alpha(\rho,\rho^s,S)(\nabla\rho^s)^2,\tag{1.17}$$

where $\mathbf{v}^s=(\hbar/m)\nabla\Phi$ is the superfluid velocity. Then

$$\frac{\delta H}{\delta\mathbf{p}}=\frac{\partial\varepsilon}{\partial\mathbf{p}}=\frac{\mathbf{p}-\rho^s\mathbf{v}^s}{\rho^n}\equiv\mathbf{v}^n,\quad\rho^n\equiv\rho-\rho^s,\tag{1.18}$$
$$\frac{\delta H}{\delta\Phi}=-\frac{m}{\hbar}\nabla\frac{\partial\varepsilon}{\partial\mathbf{v}^s}=-\frac{m}{\hbar}\nabla(\rho^s(\mathbf{v}^s-\mathbf{v}^n)),\tag{1.19}$$
$$\frac{\delta H}{\delta\rho}=\frac{\partial\varepsilon}{\partial\rho}\equiv\mu-\frac{(\mathbf{v}^n)^2}{2},\tag{1.20}$$
$$\frac{\delta H}{\delta\rho^s}=\frac{\partial\varepsilon}{\partial\rho^s}-\nabla(\alpha\nabla\rho^s)\equiv\mu^s+\frac{(\mathbf{v}^s-\mathbf{v}^n)^2}{2}-\nabla(\alpha\nabla\rho^s).\tag{1.21}$$

The quadratic form for the dissipative function R is chosen as follows (see Refs. 14 and 12):

$$\begin{aligned}R=&\frac{1}{2}\eta\left(\frac{\partial v_i^n}{\partial x^k}+\frac{\partial v_k^n}{\partial x^i}-\frac{2}{3}\delta_{ik}\nabla\mathbf{v}^n\right)^2+\frac{1}{2}\zeta_2(\nabla\mathbf{v}^n)^2\\&+\frac{1}{2}\zeta_3(\nabla(\rho^s(\mathbf{v}^s-\mathbf{v}^n)))^2+\zeta_1(\nabla\mathbf{v}^n)\nabla(\rho^s(\mathbf{v}^s-\mathbf{v}^n))\\&+\frac{1}{2}\varkappa(\nabla T)^2+\frac{1}{2}\gamma(\mu^s-\nabla(\alpha\nabla\rho^s)+\frac{1}{2}(\mathbf{v}^s-\mathbf{v}^n)^2)^2.\end{aligned}\tag{1.22}$$

In this case the hydrodynamics equations (1.15) and (1.16) take the form

$$\dot\rho+\nabla(\rho^s\mathbf{v}^s+\rho^n\mathbf{v}^n)=0,\tag{1.23}$$
$$\partial p_i/\partial t+\nabla_k\Pi_{ik}=0,\tag{1.24}$$

where the momentum flux density tensor is

$$\Pi_{ik}=p\delta_{ik}+\rho^n v_i^n v_k^n+\rho^s v_i^s v_k^s+\alpha\nabla_i\rho^s\nabla_k\rho^s$$
$$-\eta\left(\nabla_k v_i^n+\nabla_i v_k^n-\frac{2}{3}\delta_{ik}\nabla \mathbf{v}^n\right)-\delta_{ik}(\zeta_1\nabla(\rho^s(\mathbf{v}^s-\mathbf{v}^n))+\zeta_2\nabla\mathbf{v}^n). \quad (1.25)$$

Here p is the pressure

$$p=-\varepsilon+TS+\rho\frac{\partial\varepsilon}{\partial\rho}+\rho^s\frac{\partial\varepsilon}{\partial\rho^s}+\mathbf{p}\mathbf{v}^n. \quad (1.26)$$

Next

$$\frac{\partial\Phi}{\partial t}+\frac{m}{\hbar}\left(\mu+\mu^s-\nabla(\alpha\nabla\rho^s)+\frac{1}{2}(\mathbf{v}^s)^2-\zeta_3\nabla(\rho^s(\mathbf{v}^s-\mathbf{v}^n))-\zeta_1\nabla\mathbf{v}^n\right)=0, \quad (1.27)$$

$$\frac{\partial\rho^s}{\partial t}+\nabla(\rho^s\mathbf{v}^s)=-\gamma\left(\mu^s-\nabla(\alpha\nabla\rho^s)+\frac{(\mathbf{v}^s-\mathbf{v}^n)^2}{2}\right), \quad (1.28)$$

$$\frac{\partial S}{\partial t}+\nabla(S\mathbf{v}^n)=-\nabla(\varkappa\nabla T)+\frac{2R}{T}. \quad (1.29)$$

These equations coincide with those given in the review of Ginzburg and Sobyanin[12] and in Khalatnikov's book,[14] if the function α is chosen in a definite manner ($\alpha=\hbar^2/8m^2\rho^s$). This confirms the statement that ρ^s and Φ are canonically conjugate variables, which dictates the form of the Poisson bracket (1.12). In this connection, we call attention to the substantial difference between the approaches used by Dzyaloshinskii and one of us[8] and by Lebedev and Khalatnikov[9] to obtain the Poisson brackets. In contrast to the former, the latter determine not completely, the density of the generators, but only their action on the order parameter. Therefore in principle the Poisson brackets in Ref. 9 must be supplemented with those missing parts of the generated densities, which act on the normal part of the system, for otherwise contradictions will arise. For example, the momentum density $\mathbf{j}$ is defined in Ref. 9 in terms of the order parameter and its canonically conjugate variable, i.e., in our notation

$$\mathbf{j}=\rho^s\hbar m^{-1}\nabla\Phi=\rho^s\mathbf{v}^s;$$

and at the same time it is implicitly assumed [see Eq. (24) of Ref. 9] that $\mathbf{j}=\rho\mathbf{v}^s$.

The derived equations can be easily generalized also to the case of a charged liquid, for example, a superfluid electron liquid in metals. In this case two new variables are added: the vector potential of the electromagnetic field A and the electric induction D. The only additional nonzero Poisson bracket takes, according to Dirac, the form (c is the speed of light)

$$\{A_i(\mathbf{r}),D_k(\mathbf{r}')\}=4\pi c\delta_{ik}\delta(\mathbf{r}-\mathbf{r}').$$

It should only be recognized that the energy must be written in a gauge-invariant form

$$H=\int d^3r\,\varepsilon\left(\rho,\mathbf{p}-\frac{e}{c}\rho\mathbf{A},\ \nabla\Phi-\frac{2e}{c}\mathbf{A},\rho^s,\operatorname{rot}\mathbf{A},\mathbf{D}\right).$$

2. SUPERFLUID He II WITH VORTICES

We now explain how to supplement the system of Poisson brackets in the presence of vortices in He II. We are interested in a continual description of the vortices when each element of the volume contains many vortices, and the velocity $\mathbf{v}^s$ is assumed equal to the average of the velocities of the individual vortex filaments over the fields. Now on the average curl $\mathbf{v}^s\neq 0$, so that the relation $\mathbf{v}^s=(\hbar/m)\nabla\Phi$ is meaningless, although it retains its meaning locally in the interval between vortices. Therefore in place of Φ we must introduce into the system of variables the vector $\mathbf{v}^s$, all three components of which are independent. It is easy to write the transformation of $\mathbf{v}^s$ under translations and gradient transformation:

$$\delta v_i^s=-\frac{\partial v_i^s}{\partial x_k}\delta u^k+v_k^s\nabla_i\delta u^k-\nabla_i\delta\varphi. \quad (2.1)$$

We call attention to the fact that under the gradient transformation $\mathbf{v}^s$ transforms like the gauge field that corresponds to this group of symmetry transformations. In the general case (see Ref. 8) on going to the continuous description of a system with linear topological defects one introduces as the hydrodynamic variables the gaugelike fields in place of the order parameter.

Using (2.1), (1.1), and (1.2) we get the following Poisson brackets

$$\{p_i(\mathbf{r}),v_k^s(\mathbf{r}')\}=v_i^s(\mathbf{r})\nabla_k\delta(\mathbf{r}-\mathbf{r}')+(\nabla_k v_i^s-\nabla_i v_k^s)\delta(\mathbf{r}-\mathbf{r}'), \quad (2.2)$$
$$\{\rho(\mathbf{r}),\mathbf{v}^s(\mathbf{r}')\}=\nabla\delta(\mathbf{r}-\mathbf{r}'). \quad (2.3)$$

In addition, since locally $\mathbf{v}^s$ is connected with the gradient of the phase, we obtain from (1.12)

$$\{\rho^s(\mathbf{r}),\ \mathbf{v}^s(\mathbf{r}')\}=\nabla\delta(\mathbf{r}-\mathbf{r}'). \quad (2.4)$$

It remains to find the Poisson brackets between the components of $\mathbf{v}^s$ themselves. To this end we must consider the dynamics of an isolated vortex. Let the vortex line Γ be specified in parametric form $x_i(\sigma)$, where σ can be chosen to be the length of the vortex. In this case $dx_i/d\sigma$ is a unit vector tangent to the vortex line. Since the phase Φ changes by $2\pi\nu$ (ν is an integer) on going around the vortex, the circulation of the superfluid velocity around the vortex line is

$$\oint d\mathbf{l}\,\mathbf{v}^s=2\pi\hbar\nu/m=\varkappa\nu. \quad (2.5)$$

Outside the vortex, curl $\mathbf{v}^s=0$, and on the vortex line itself curl $\mathbf{v}^s$ has a δ-function singularity:

$$\operatorname{rot}\mathbf{v}^s=\varkappa\nu\int_\Gamma d\sigma\frac{\partial\mathbf{x}}{\partial\sigma}\delta(\mathbf{r}-\mathbf{x}(\sigma)). \quad (2.6)$$

It is easily seen that (2.5) follows from (2.6). If the number of vortices is large, it is necessary to sum (2.6) over all the vortex lines:

$$\operatorname{rot}\mathbf{v}^s=\varkappa\sum_a\nu_a\int_{\Gamma_a}d\sigma\frac{\partial\mathbf{x}_a}{\partial\sigma}\delta(\mathbf{r}-\mathbf{x}_a(\sigma)). \quad (2.7)$$

Therefore the problem of finding the Poisson brackets for the components of curl $\mathbf{v}^s$ reduces to the problem of finding the Poisson brackets for the components of $\mathbf{x}(\sigma)$. The latter are obtained from the equations of motion of an isolated vortex.

There are disagreements in the literature concerning the dynamics of an isolated vortex in superfluid HeII at $T\neq 0$ (see Putterman's book,[15] on the one hand, and the papers by Iordanskiĭ[16] and Sonin,[17] on the other). There is, however, a region of parameter values where the differences are reconciled, and the equations of motion take the form

$$-\rho^s\varkappa\nu\left[\frac{\partial\mathbf{x}}{\partial\sigma},\frac{\partial\mathbf{x}}{\partial t}\right]+\frac{\delta H}{\delta\mathbf{x}}=-\lambda\left(\frac{\partial\mathbf{x}}{\partial t}-\mathbf{v}_\perp^n\right). \quad (2.8)$$

If extraneous forces act on the core of the vortex, then

$$\delta H/\delta\mathbf{x}=\rho^s\varkappa\nu[\partial\mathbf{x}/\partial\sigma,\mathbf{v}_0^s],$$

where $\mathbf{v}_0^s$ is the superfluid velocity produced at the location of the given vortex element by the remaining vortex elements and by the other vortices. Therefore the left-hand side of (2.8) is the Magnus force acting on the vortex element and is produced when the vortex velocity $\partial \mathbf{x}/\partial t$ differs from the velocity $\mathbf{v}_0^s$ of the superfluid component. The right-hand side in (2.8) is the dissipative force that compensates the Magnus force and arises because of the motion of the vortex relative to the viscous normal component.

According to Putterman,[15] this equation is valid whenever the two-velocity dynamics equations are valid. The friction coefficient λ is connected with the second-viscosity coefficient ζ_3 by the relation $\lambda = \zeta_3 \rho_s^2 \varkappa \nu$. According to Iordanskiĭ and Sonin[16,17] λ is connected with the first viscosity η:

$$\lambda = 4\pi\eta/\ln(r_m/l),$$

and the equation holds only in the low-viscosity limit $\eta \ll \rho^n \varkappa$. We shall assume at any rate that this condition is satisfied and limit ourselves by the same token to the temperature $1.6\,\mathrm{K} < T < T_\lambda$. If furthermore the condition $\lambda \ll \rho^s \varkappa$ is satisfied, then $\partial \mathbf{x}/\partial t$ in the right-hand side of (2.8) can be replaced by

$$\frac{1}{\rho^s \varkappa \nu}\left[\frac{\delta H}{\delta \mathbf{x}}, \frac{\partial \mathbf{x}}{\partial \sigma}\right].$$

The equation can then be rewritten in standard form (1.15):

$$\frac{\partial \mathbf{x}}{\partial t} + \{\mathbf{x}, H\} = -\frac{\delta R}{\delta(\delta H/\delta \mathbf{x})}, \tag{2.9}$$

where

$$R = \frac{\lambda}{2}\left(\left[\frac{\partial x}{\partial \sigma}, \mathbf{v}^n\right] - \frac{1}{\rho^s \varkappa \nu}\frac{\delta H}{\delta \mathbf{x}}\right)^2 \tag{2.10}$$

under the condition that the Poisson brackets for the components x take the form

$$\{x_i(\sigma), x_k(\sigma')\} = -\frac{1}{\rho^s \varkappa \nu} e_{ikl} \frac{\partial x_l}{\partial \sigma} \delta(\sigma - \sigma'). \tag{2.11}$$

We point out that a similar Poisson bracket was obtained by Rasetti and Regge[16] from the Lagrangian description of the motion of a vortex with the aid the canonical Dirac formalism, but instead of ρ^s they obtained ρ, since they considered the case $T = 0$, when $\rho^s = \rho$.

Using (2.11) and (2.6), we calculate

$$\{(\operatorname{rot} \mathbf{v}^s)_i, (\operatorname{rot} \mathbf{v}^s)_k\} = -e^{inl}\nabla_n e^{kml}\nabla_m{}' \left(\frac{\nabla_i v_l^s - \nabla_l v_i^s}{\rho^s}\delta(r - r')\right). \tag{2.12}$$

Integrating this equation, we can find the Poisson brackets for the components of $\mathbf{v}^s$:

$$\{v_i^s(\mathbf{r}), v_k^s(\mathbf{r}')\} = -\frac{1}{\rho^s}(\nabla_i v_k^s - \nabla_k v_i^s)\delta(\mathbf{r} - \mathbf{r}'). \tag{2.13}$$

It must be noted that integration can produce in (2.13) additional terms whose curl is zero. The criterion for the selection of these terms is provided by the Jacobi identities for the system of Poisson brackets (1.7)–(1.9), (2.2)–(2.4), and (2.13), which require that these terms vanish. Thus, in contrast to the assumptions made in the preceding papers[8,9] we have obtained nonzero Poisson brackets (2.13).

3. HYDRODYNAMICS OF A VORTEX LATTICE IN ROTATING He II

We now use the obtained Poisson brackets to derive the equations of the hydrodynamics of He II in a rotating vessel. It must be recognized here that the vortices in the rotating vessel form a lattice, and consequently the symmetry of the state of the system with respect to translations is violated, so that an additional hydrodynamic variable appears, namely the displacement of a vortex from the equilibrium position. To describe the two displacement components in a plane perpendicular to the vortex lines, we have introduced in our earlier paper[11] two functions $X_1(\mathbf{r})$ and $X_2(\mathbf{r})$. We recall that these functions specify two sets of surfaces $X_1(\mathbf{r}) = C_1$ and $X_2(\mathbf{r}) = C_2$, the lines of intersection of which are the vortex lines. Therefore the gradients of these functions are perpendicular to the direction of the vortices, i.e.,

$$\nabla X_\mu \operatorname{rot} \mathbf{v}^s = 0, \quad \mu = 1, 2. \tag{3.1}$$

The constants C_1 and C_2 are assumed discrete values, for example, integer N_1 and N_2, which number the vortices in the lattice. In the continual description, C_1 and C_2 take on a continuous set of values, and X_μ become continuous hydrodynamic variables.

The Poisson brackets between the quantities X_1 and X_2 can be obtained by going to the continual limit in (2.11), in which ν must be set equal to +1, since the lattice consists of identical vortices. We shall show how to do this using as an example straight vortices directed along the rotation axis z [then $x_i = (x, y)$]. Generalization to the case of curvelinear vortices will be obvious.

We rewrite (2.11), replacing σ with z and recognizing that the coordinates of different vortices commute with one another, in the form

$$\{x_i(z, N_1, N_2), x_k(z', N_1', N_2')\} = -\frac{e_{ikl}\hat{z}_l}{\varkappa\rho^s}\delta(N_1 N_1')\delta(N_2 N_2')\delta(z - z').$$

In the continual limit we make the substitutions $N_1 \to \tilde{X}_1$ and $N_2 \to \tilde{X}_2$ (we have assumed the particular case of functions X_μ that take on integer values on the vortices):

$$\{x_i(z, \tilde{X}_1, \tilde{X}_2), x_k(z', \tilde{X}_1', \tilde{X}_2')\} = -\frac{e_{ikl}\hat{z}_l}{\varkappa\rho^s}\delta(z - z')\delta(\tilde{X}_1 - \tilde{X}_1')\delta(\tilde{X}_2 - \tilde{X}_2').$$

We now change from the variables $x_i(z, X_\mu)$ to the inverse variables $\tilde{X}_1(z, x, y)$ and $\tilde{X}_2(z, x, y)$:

$$\{\tilde{X}_1(\mathbf{r}), \tilde{X}_2(\mathbf{r}')\} = \left(\frac{\partial \tilde{X}_1}{\partial x}\frac{\partial \tilde{X}_2}{\partial y} - \frac{\partial \tilde{X}_1}{\partial y}\frac{\partial \tilde{X}_2}{\partial x}\right)\{x, y\}$$
$$= \left(\frac{\partial \tilde{X}_1}{\partial x}\frac{\partial \tilde{X}_2}{\partial y} - \frac{\partial \tilde{X}_1}{\partial y}\frac{\partial \tilde{X}_2}{\partial x}\right)\frac{\delta(z - z')\delta(\tilde{X}_1 - \tilde{X}_1')\delta(\tilde{X}_2 - \tilde{X}_2')}{-\varkappa\rho^s}$$
$$= -\frac{\delta(z - z')\delta(x - x')\delta(y - y')}{\varkappa\rho^s} = -\frac{\delta(\mathbf{r} - \mathbf{r}')}{\varkappa\rho^s}.$$

In the general case, transforming to arbitrary functions $X_\mu(\tilde{X}_\nu)$, we obtain

$$\{X_1(\mathbf{r}), X_2(\mathbf{r}')\} = -\frac{J}{\varkappa\rho^s}\delta(\mathbf{r} - \mathbf{r}'), \tag{3.2}$$

where J is the Jacobian of the transformation from the variables X_μ to $\tilde{X}_\nu$ and is equal to the ratio of the areas

$$J = |[\nabla X_1, \nabla X_2]| / |[\nabla \tilde{X}_1, \nabla \tilde{X}_2]|.$$

The area $|[\nabla \tilde{X}_1, \nabla \tilde{X}_2]|$ is equal to the area of the cell in the reciprocal two-dimensional lattice, which coincides with the density of the vortex filaments $|\text{curl}\,\mathbf{v}^s|/\varkappa$. Therefore, taking the condition (3.1) into account, we rewrite the Poisson bracket (3.2) in final form:

$$\{X_1(\mathbf{r}),X_2(\mathbf{r}')\}=\frac{(\text{rot}\,\mathbf{v}^s,[\nabla X_1,\nabla X_2])}{\rho^s(\text{rot}\,\mathbf{v}^s)^2}\delta(\mathbf{r}-\mathbf{r}'). \tag{3.3}$$

We obtain similarly

$$\{X_\mu(\mathbf{r}),\mathbf{v}^s(\mathbf{r}')\}=\frac{1}{\rho^s}\nabla X_\mu\delta(\mathbf{r}-\mathbf{r}'). \tag{3.4}$$

Finally, the last nonzero Poisson bracket follows from the law of transformation of the variables X_μ by the rotating

$$\delta X_\mu(\mathbf{r})=X_\mu(\mathbf{r}-\delta\mathbf{u})-X_\mu(\mathbf{r})=-(\delta\mathbf{u}\nabla)X_\mu.$$

We obtain then

$$\{\mathbf{p}(\mathbf{r}),X_\mu(\mathbf{r}')\}=-\nabla X_\mu\delta(\mathbf{r}-\mathbf{r}'). \tag{3.5}$$

Thus, Eqs. (1.7)–(1.9), (2.2)–(2.4), (2.13) and (3.3)–(3.5) yield a complete system of Poisson brackets for the hydrodynamic variables in rotating He II. The equations of hydrodynamics take the form (1.15), (1.16) with a Hamiltonian that can be written, for HeII in a vessel rotating with angular velocity $\boldsymbol{\Omega}$, in the following general form:

$$H=\int d^3r\left(\tfrac{1}{2}\rho[\boldsymbol{\Omega}\times\mathbf{r}]^2+\varepsilon(\mathbf{p}-\rho[\boldsymbol{\Omega}\times\mathbf{r}],\mathbf{v}^s-[\boldsymbol{\Omega}\times\mathbf{r}],\rho,\rho^s,S,g_{ik}-g_{ik}^0)\right). \tag{3.6}$$

Here

$$g_{ik}=\sum_{\mu=1,2}\frac{\partial X_\mu}{\partial x_i}\frac{\partial X_\mu}{\partial x_k}$$

is the metric tensor and g_{ik}^0 is the value of this tensor in the undeformed lattice.

We do not present the equations of motion in general form, since they follow automatically from the Poisson brackets, and confine ourselves to a linear approximation with the concrete forms of the energy ε and of the dissipation function R in this approximation. In the linear approximation we introduce in place of X_μ the displacements of the lattice from the equilibrium position in accord with the formula

$$u=u_1\hat{x}+u_2\hat{y},\quad u_1=-X_1+x,\quad u_2=-X_2+y. \tag{3.7}$$

We express the energy ε in the form

$$\begin{aligned}\varepsilon={}&\tfrac{1}{2}\rho^s(\mathbf{v}^s-[\boldsymbol{\Omega}\times\mathbf{r}])^2+(\mathbf{p}-\rho^s\mathbf{v}^s-(\rho-\rho^s)[\boldsymbol{\Omega}\times\mathbf{r}])^2/2(\rho-\rho^s)\\&+\varepsilon_0(\rho,\rho^s,S)+\tfrac{1}{2}\rho^s(K_1(\nabla_{\perp}\mathbf{u})^2+K_2(\nabla\mathbf{u})^2+K_3(\partial\mathbf{u}/\partial z)^2),\end{aligned} \tag{3.8}$$

and the dissipation function in the form

$$\tilde{R}=R+\frac{\lambda}{4\Omega\varkappa(\rho^s)^2}\left(\left[\frac{\partial\varepsilon}{\partial\mathbf{v}^s},2\boldsymbol{\Omega}\right]-\frac{\delta\varepsilon}{\delta\mathbf{u}}\right)^2. \tag{3.9}$$

Here R is given by Eq. (1.22), and the increment to it describes the friction of the vortices against the viscous normal component. This increment is none other than the dissipation function (2.10) for an isolated vortex, multiplied by the density of the vortices

$$|\text{rot}\,\mathbf{v}^s|/\varkappa\approx2\Omega/\varkappa.$$

In fact, recognizing that

$$\partial\varepsilon/\partial\mathbf{v}^s=\rho^s(\mathbf{v}^s-\mathbf{v}^n),$$

and that the force

$$\rho^s[\mathbf{v}^s\times2\boldsymbol{\Omega}]-\delta\varepsilon/\delta\mathbf{u}$$

is the force acting on $2\Omega/\varkappa$ vortices, i.e., is equal to

$$-\frac{2\Omega}{\varkappa}\frac{\delta H}{\delta\mathbf{x}},$$

we obtain Eq. (2.10) multiplied by $2\Omega/\varkappa$. In principle it is possible to take into account in R also the anisotropy, but it is usually small to the extent that the ratio of the interatomic distance to the distance between vortices is small.

To obtain linear equations it is convenient to rewrite the Poisson brackets (2.13) and (3.3)–(3.5) in the following approximate form:

$$\begin{aligned}&\{v_i^s,v_k^s\}=-\frac{2\Omega}{\rho^s}e_{ikl}\hat{z}_l\delta(\mathbf{r}-\mathbf{r}'),\qquad \{u_i,u_k\}=-\frac{e_{ikl}\hat{z}_l}{2\Omega\rho^s}\delta(\mathbf{r}-\mathbf{r}'),\\&\{u_i,v_k^s\}=-\frac{1}{\rho^s}(\delta_{ik}-\hat{z}_i\hat{z}_k)\delta(\mathbf{r}-\mathbf{r}'),\qquad \{p_i,u_k\}=(\delta_{ik}-\hat{z}_i\hat{z}_k)\delta(\mathbf{r}-\mathbf{r}').\end{aligned} \tag{3.10}$$

As a result we have the following linear equations:

$$\frac{\partial\mathbf{u}}{\partial t}=\tilde{\mathbf{v}}_\perp^s+\frac{1}{2\Omega\rho^s}\left[\frac{\delta\varepsilon}{\delta\mathbf{u}}\times\hat{z}\right]-\frac{\lambda}{2\Omega\varkappa\rho_s^2}\left(\frac{\delta\varepsilon}{\delta\mathbf{u}}-\left[\frac{\partial\varepsilon}{\partial\mathbf{v}^s}\times2\boldsymbol{\Omega}\right]\right), \tag{3.11}$$

$$\begin{aligned}\frac{\partial\mathbf{v}_s}{\partial t}+\nabla\left(\frac{\partial\varepsilon}{\partial\rho}+\frac{\partial\varepsilon}{\partial\rho^s}\right)={}&[\tilde{\mathbf{v}}^s\times2\boldsymbol{\Omega}]-\frac{1}{\rho^s}\frac{\delta\varepsilon}{\delta\mathbf{u}}\\&-\frac{\lambda}{\Omega\varkappa\rho_s^2}\left[\boldsymbol{\Omega}\times\left[\frac{\partial\varepsilon}{\partial\mathbf{v}^s}\times2\boldsymbol{\Omega}\right]-\frac{\delta\varepsilon}{\delta\mathbf{u}}\right],\end{aligned} \tag{3.12}$$

$$\frac{\partial\mathbf{p}}{\partial t}+\nabla p=-\frac{\delta\varepsilon}{\delta\mathbf{u}}+\eta\Delta\mathbf{v}^n+\left(\zeta_2+\frac{\eta}{3}\right)\nabla(\nabla\mathbf{v}^n)+\rho^s\zeta_1\nabla(\nabla\mathbf{v}^s-\nabla\mathbf{v}^n), \tag{3.13}$$

$$\partial\rho/\partial t+\nabla(\rho^s\tilde{\mathbf{v}}^s+\rho^n\tilde{\mathbf{v}}^n)=0, \tag{3.14}$$

$$\partial S/\partial t+S(\nabla\mathbf{v}^n)=-\varkappa\nabla T, \tag{3.15}$$

$$\partial\rho^s/\partial t+\rho^s(\nabla\mathbf{v}^s)=-\gamma\partial\varepsilon/\partial\rho^s. \tag{3.16}$$

$\tilde{\mathbf{v}}^s$ and $\tilde{\mathbf{v}}^n$ denote here the velocities in the rotating coordinate system:

$$\tilde{\mathbf{v}}^s=\mathbf{v}^s-[\boldsymbol{\Omega}\times\mathbf{r}],\quad \tilde{\mathbf{v}}^n=\mathbf{v}^n-[\boldsymbol{\Omega}\times\mathbf{r}],$$

and the variational derivative of the energy with respect to the displacement is given by

$$\frac{1}{\rho^s}\frac{\delta\varepsilon}{\delta\mathbf{u}}=-K_1\Delta_\perp\mathbf{u}-K_2\nabla_\perp(\nabla\mathbf{u})-K_3\frac{\partial^2}{\partial z^2}\mathbf{u}.$$

4. LATTICE MODES IN ROTATING He II

Equations (3.11)–(3.16) yield the spectrum of all the long-wave modes [with q smaller than the reciprocal distance between the vortices $(\Omega/\varkappa)^{1/2}$] in rotating He II. We consider now the modes connected with the lattice. For simplicity we confine ourselves to the particular case when: 1) $\mathbf{v}^n=0$; 2) the liquid can be regarded as incompressible, so that $\nabla\mathbf{v}^s=0$; 3) the wave vector $\mathbf{q}$ is perpendicular to the rotation axis $\boldsymbol{\Omega}=\Omega\hat{\mathbf{z}}$; 4) we retain from among all the rigidity coefficients only K_1; 5) the temperature variation can be neglected. We are then left with three variables u_x, u_y, and $\tilde{v}_y^s$, for which we have three scalar equations: the two components of (3.11) and one scalar equation obtained from (3.12) by taking the curls of both halves:

$$\frac{\partial\mathbf{u}}{\partial t}=\tilde{\mathbf{v}}^s+\frac{K_1}{2\Omega}q^2[\mathbf{u}\times\hat{\mathbf{z}}]+\gamma\left[\mathbf{v}^s+\frac{K_1}{2\Omega}q^2[\mathbf{u}\times\hat{z}],\hat{\mathbf{z}}\right], \tag{4.1}$$

$$\frac{1}{2\Omega}\left[\mathbf{q}\times\frac{\partial\tilde{\mathbf{v}}^s}{\partial t}\right]=\left[\mathbf{q},\left[\tilde{\mathbf{v}}^s+\frac{K_1}{2\Omega}q^2[\mathbf{u}\times\hat{z}],\hat{z}\right]-\gamma\left(\tilde{\mathbf{v}}^s+\frac{K_1}{2\Omega}q^2[\mathbf{u}\times\hat{z}]\right)\right]. \tag{4.2}$$

Here $\gamma=\lambda/\varkappa\rho^s$, $\mathbf{q}\parallel\hat{\mathbf{x}}$, $\tilde{\mathbf{v}}^s\parallel\hat{\mathbf{y}}$. These equations yield three modes. The first two are obtained from the equation

$$i\omega(i\omega-2\gamma\Omega)=-K_1q^2. \tag{4.3}$$

At $\omega \gg 2\gamma\Omega$ we obtain the spectrum of the Tkachenko waves (for more details on Tkachenko waves see Ref. 19):

$$\omega^2 = K_1 q^2. \tag{4.4}$$

At $\omega \ll 2\gamma\Omega$ the Tkachenko waves are transformed into the diffusion mode

$$\omega = -iK_1 q^2/2\gamma\Omega. \tag{4.5}$$

[We call attention to the fact that (4.5) was obtained in the simplest case $\mathbf{v}^n = 0$. In the general case it is necessary to take into account the motion of the normal component that is dragged by the motion of the vortices. This, as shown by Tkachenko[20] decreases the damping of the waves and may prevent the diffusion regime (4.5) from being realized for them.]

The third mode has a frequency identically equal to zero. To determine the origin of this mode, we note that Eq. (4.2) is linearly dependent on (4.1). In fact, we have the condition (3.1), which takes in the linear approximation the form

$$\operatorname{rot} \tilde{\mathbf{v}}^s = -2\boldsymbol{\Omega}(\nabla\mathbf{u}) + 2\Omega\partial\mathbf{u}/\partial z. \tag{4.6}$$

At $q_z = 0$ (i.e., $\partial\mathbf{u}/\partial z = 0$) this condition has a clear physical meaning: lattice expansion or compression, described by the quantity $\nabla\mathbf{u}$, leads to a relative change in the vortex density $|\operatorname{curl}\tilde{\mathbf{v}}^s|/2\Omega = -\nabla\mathbf{u}$. Equation (4.2) is obtained by differentiating the condition (4.6) with respect to time and by using Eq. (4.1). This gives rise to the extra unphysical mode $\omega \equiv 0$.

This mode, however, becomes physical in the presence of vacancies, i.e., lattice sites not occupied by vortex lines. In this case, as in an ordinary crystal, the change of the vortex density is no longer connected with the displacement of the lattice sites. In a solid the frequency $\omega \equiv 0$ is then transformed into a mode that describes the diffusion of the vacancies (see Ref. 21). To obtain this mode in our case we must change the form of the dissipation function $\tilde{R}$ and add to it a term connected with the diffusion of the vacancies. The second term in (3.9) is a definite combination of $\delta\varepsilon/\delta\mathbf{u}$ and $\partial\varepsilon/\partial\mathbf{v}^s$ and takes into account the ideal character of the lattice and, by the same token, the condition (12.6). To take the vacancies into account it is necessary to disturb this combination by adding, for example, the term

$$\tfrac{1}{2}\beta(\delta\varepsilon/\delta\mathbf{u})^2. \tag{4.7}$$

Equations (4.1) and (4.2) now become independent, and we obtain from them a diffusion vacancy mode

$$\omega = -i\beta K_1 \rho^s q^2. \tag{4.8}$$

5. HYDRODYNAMICS OF A PLANAR MAGNET WITH DISCLINATIONS

We examine now the hydrodynamic variables that must be used to describe a planar magnet with defects and what are the Poisson brackets for these variables. For simplicity we confine ourselves to the case $T = 0$. The order parameter in a planar magnet in the absence of defects is the angle Φ between the direction of the spontaneous moment lying in the plane (we choose the z axis perpendicular to this plane) and some chosen direction in this plane. In addition to the order parameter, the hydrodynamic variable is the density of the generator of the rotations in spin space about the z axis: $\mathbf{M} = M\hat{\mathbf{z}}$. The vector $\mathbf{M}$ has the meaning of the magnetization along the z axis. The energy of the system is written in the simplest case in the form

$$H = \int d^3r\left(\tfrac{1}{2}\rho^s(\nabla\Phi)^2 + M^2/2\chi\right), \tag{5.1}$$

where ρ^s is the spin rigidity and χ is the magnetic susceptibility.

The Poisson brackets for M and Φ follow from the transformation of Φ upon rotation through an angle $\delta\varphi$, generated by the quantity M:

$$\{M(\mathbf{r}), \Phi(\mathbf{r}')\} = \delta\Phi(\mathbf{r}')/\delta\varphi(\mathbf{r}) = \delta(\mathbf{r}-\mathbf{r}'). \tag{5.2}$$

From (5.1) and (5.2) we obtain with the aid of the Liouville equations the following relation

$$\partial M/\partial t = \nabla(\rho^s\nabla\Phi), \quad q\Phi/\partial t = M/\chi.$$

We call attention to the fact that the spin system has a momentum that generates the translations of the spin subsystem. In fact, we consider a combination of variables in form

$$\mathbf{p} = -M\nabla\Phi. \tag{5.3}$$

It is easily seen that this combination has all the properties of the translation—generator density, namely:

$$\{\mathbf{p}(\mathbf{r}), M(\mathbf{r}')\} = \delta M(r')/\delta\mathbf{u}(r) = M(\mathbf{r})\nabla\delta(\mathbf{r}-\mathbf{r}'), \tag{5.4}$$

$$\{\mathbf{p}(\mathbf{r}), \Phi(\mathbf{r}')\} = \delta\Phi(\mathbf{r}')/\delta\mathbf{u}(r) = -\delta(\mathbf{r}-\mathbf{r}')\nabla\Phi, \tag{5.5}$$

$$\{p_i(\mathbf{r}), p_k(\mathbf{r}')\} = (p_k(\mathbf{r})\nabla_i - p_i(\mathbf{r}')\nabla_k')\delta(\mathbf{r}-\mathbf{r}'). \tag{5.6}$$

The variable $\mathbf{p}$ is not a new hydrodynamic variable, since it is expressed in terms of M and Φ. The situation changes in the presence of vortices.

In the presence of vortices or disclinations (singular line, circuiting around which causes Φ to change by $2\pi\nu$), the angle Φ ceases to be definite and its space is taken by the gaugelike hydrodynamic variable $\mathbf{A}$, which is transformed under rotations in accordance with the rule

$$\delta\mathbf{A} = \nabla\delta\varphi. \tag{5.7}$$

The curl of this variable determines the density of the vortices [cf. (2.7)]:

$$\operatorname{rot}\mathbf{A} = \sum_a 2\pi\nu_a \int_{\Gamma_a} d\sigma \frac{\partial\mathbf{x}_a}{\partial\sigma}\delta(\mathbf{r}-\mathbf{x}_a(\sigma)). \tag{5.8}$$

The Poisson brackets between A and M are obtained from (5.7):

$$\{M(\mathbf{r}), \mathbf{A}(\mathbf{r}')\} = -\nabla\delta(\mathbf{r}-\mathbf{r}'). \tag{5.9}$$

To find the Poisson brackets for the components of A we must, as before, consider the dynamics of an isolated vortex, which, just as in He II (see Ref. 17) follows from the momentum conservation law

$$\partial p_i/\partial t + \nabla_k\Pi_{ik} = 0, \tag{5.10}$$

where Π_{ik} is the momentum-flux tensor. The force acting on the vortex is equal to an integral of Π_{ik} with respect to a surface surrounding the vortex in a coordinate frame moving together with the vortex. Let v_a be the velocity of the a-th vortex. Then the momentum-flux tensor in the frame moving together with the vortex is

$$\Pi_{ik} = \rho^s\nabla_i\Phi\nabla_k\Phi + \delta_{ik}\left((M-\chi\mathbf{v}_a\nabla\Phi)^2/2\chi - \tfrac{1}{2}\rho^s(\nabla\Phi)^2\right). \tag{5.11}$$

If there is no spin flux in the system, i.e., Φ consists only of the field of the vortex proper, and also if there is no magnetization, then the force exerted on the vortex by the spin subsystem, $F_i = \int dS_k \Pi_{ik}$, is equal to zero. This means that the vortex can move at any constant velocity and is not frozen into the medium or into the superfluid component, as is a vortex in He II. Therefore the vortex in a magnet, having motion of its own, has also momentum of its own.

The momentum of the vortex is obtained from (5.3), in which it must be taken into account the $M = \chi\dot{\Phi}$, and the dependence of Φ on the time is connected with the motion of the vortex. The momentum per unit length, of a vortex with ν_a circulation quanta, is then

$$\mathbf{p}_a = -\frac{1}{L}\int M\nabla\Phi d^3r = \frac{\chi}{L}\int d^3r(\mathbf{v}_a\nabla\Phi)\nabla\Phi = \varepsilon_a\frac{\mathbf{v}_a}{c^2}, \tag{5.12}$$

where ε_a is the energy of the vortex per unit length. For a vortex at rest $\varepsilon_a^0 = 2\pi\rho^s\nu_a^2\ln(R/\xi)$ (R is the characteristic distance between vortices and ξ is the radius of the vortex core), in the general case $\varepsilon_a = \varepsilon_a^0(1 - v_a^2/c^2)^{-1/2}$, c is the spin-wave velocity, $c^2 = \rho^s/\chi$.

If a spin flux $\rho^s\nabla\Phi_0$ and a magnetization M are present in the system, the vortex is acted upon by a force exerted by the spin subsystem, similar to the Magnus force, which leads to a change of the vortex momentum with time, in accordance with Newton's second law:

$$\frac{\partial \mathbf{p}_a}{\partial t} = -\frac{\delta H}{\delta \mathbf{x}_a} + 2\pi\nu_a M\left[\frac{\partial \mathbf{x}_a}{\partial\sigma}, \frac{\partial \mathbf{x}_a}{\partial t}\right], \quad \frac{\partial \mathbf{x}_a}{\partial t} = \frac{\partial H}{\partial \mathbf{p}_a} = \mathbf{v}_a. \tag{5.13}$$

The first term in the right-hand side of (5.13) is that part of the Magnus force which is due to the spin flux produced in the location of the given vortex element, the remaining elements of the vortex, and other vortices, as well as the external spin superflux. The second term is connected with the magnetization.

In He II, in contrast to a magnet, if a vortex moves relative to the superfluid component then, in a coordinate frame that moves with the vortex, there is always a mass superflux $\rho^s\nabla\Phi_0$ that flows around the vortex and produces the Magnus force. Therefore in the absence of any forces to compensate the Magnus force a vortex cannot move in He II with a constant relative velocity. This difference between the dynamics of an isolated vortex in He II and in a magnet is precisely the cause of the different forms of the Poisson brackets for these systems.

Equations (5.13) can be written in Hamiltonian form by introducing the following Poisson brackets:

$$\{p_a^i(\sigma), x_{a'}^k(\sigma')\} = -\delta^{ik}\delta(\sigma-\sigma')\delta_{aa'}, \tag{5.14}$$

$$\{p_a^i(\sigma), p_{a'}^k(\sigma')\} = e^{ikl}\frac{\partial x_a^l}{\partial\sigma}\delta(\sigma-\sigma')2\pi\nu_a M\delta_{aa'}, \tag{5.15}$$

$$\{x_a^i(\sigma), x_{a'}^k(\sigma')\} = 0. \tag{5.16}$$

It follows from them, first, that the Poisson brackets for the components of the gaugelike $\mathbf{A}$, as seen from (5.8), are equal to zero:

$$\{A_i(\mathbf{r}), A_k(\mathbf{r}')\} = 0. \tag{5.17}$$

Second, the system momentum density $\mathbf{p}$, which was previously the momentum of the field (5.3), now consists of both the momentum of the field $-M\mathbf{A}$ and of the momentum of the vortices

$$\mathbf{p} = -M\mathbf{A} + \sum_a \int_{\Gamma_a} d\sigma \mathbf{p}_a(\sigma)\delta(\mathbf{r}-\mathbf{x}_a(\sigma)). \tag{5.18}$$

Using (5.9) and (5.14)–(5.16), it is easy to verify that (5.18) indeed satisfies all the properties of the translation-generator density:

$$\{p_i(\mathbf{r}), A_k(\mathbf{r}')\} = A_i\nabla_k\delta(\mathbf{r}-\mathbf{r}') + (\nabla_k A_i - \nabla_i A_k)\delta(\mathbf{r}-\mathbf{r}'), \tag{5.19}$$

$$\{p_i(\mathbf{r}), p_k(\mathbf{r}')\} = (p_k(\mathbf{r})\nabla_i - p_i(\mathbf{r}')\nabla_k)\delta(\mathbf{r}-\mathbf{r}'), \tag{5.20}$$

$$\{p_i(\mathbf{r}), M(\mathbf{r}')\} = M(\mathbf{r})\nabla_i\delta(\mathbf{r}-\mathbf{r}'). \tag{5.21}$$

The momentum density (5.18) is now an independent hydrodynamic variable.

To the set of hydrodynamic variables M, $\mathbf{A}$, and $\mathbf{p}$ it remains to add the vortex mass density ρ:

$$\rho(\mathbf{r}) = \sum_a \frac{\varepsilon_a^0}{c^2}\int_{\Gamma_a} d\sigma\delta(\mathbf{r}-\mathbf{x}_a(\sigma)) \tag{5.22}$$

and the corresponding nonzero Poisson bracket:

$$\{\mathbf{p}(\mathbf{r}), \rho(\mathbf{r}')\} = \rho(\mathbf{r})\nabla\delta(\mathbf{r}-\mathbf{r}'). \tag{5.23}$$

The equations of the dynamics of a planar magnet in the absence of dissipation are determined by the Liouville equations with the aid of the Poisson brackets (5.9), (5.19)–(5.21) and (5.23) (the remaining Poisson brackets are equal to zero) and with the aid of the Hamiltonian, which in the simplest case is given by

$$H = \int d^3r(M^2/2\chi + \rho^s\mathbf{A}^2/2 + \mathbf{K}^2/2\rho + \varepsilon_0(\rho)). \tag{5.24}$$

The third term in the Hamiltonian is the kinetic energy of the vortices ($\mathbf{K} = \mathbf{p} + M\mathbf{A}$).

Hopefully, these equations, supplemented with dissipative terms, are applicable for a two-dimensional planar magnet (the x,y model), where an equilibrium vortex concentration exists as a result of the statistics.

6. SPIN DYNAMICS OF NONPLANAR MAGNET WITH DISCLINATIONS AND OF SPIN GLASS

We now attempt to generalize the obtained Poisson brackets to include the case of a nonplanar magnet with defects, for example a multisublattice antiferromagnet with a system of disclinations, or else spin glass, which can be represented as the continual limit of an antiferromagnet having a nonzero equilibrium disclination density.[4,5] Here M is replaced by the density M^α of the generator of the three-dimensional rotations, and the field $\mathbf{A}$ is replaced by three fields that transform under three-dimensional rotations like gauge fields. The quantities having the meaning of the disclination density take the form of the intensities of the guage fields:

$$F_{ik}^\alpha = \nabla_i A_k^\alpha - \nabla_k A_i^\alpha + e^{\alpha\beta\gamma}A_i^\beta A_k^\gamma. \tag{6.1}$$

In analogy with planar magnets, we must introduce the defect momentum density $\mathbf{K}$ and the density ρ of the inertial mass of the defects. To find the Poisson brackets for K we must recognize that the sum of the field and disclination momenta $\mathbf{p} = \mathbf{K} - M^\alpha\mathbf{A}^\alpha$ is the total momentum of the spin system [cf. (5.18)] and consequently has all the properties of a translation generator. Using this circumstance, and also finding the changes of the hydrodynamic variables under the influence of the rotation transformation generated by the quantity M^α, we obtain a closed set of nonzero Poisson brackets for the variables M^α, $\mathbf{A}^\alpha$, $\mathbf{K}$, and ρ:

$$\{M^\alpha(\mathbf{r}), M^\beta(\mathbf{r}')\} = -e^{\alpha\beta\gamma}M^\gamma\delta(\mathbf{r}-\mathbf{r}'), \tag{6.2}$$

$$\{M^\alpha(\mathbf{r}), \mathbf{A}^\beta(\mathbf{r}')\} = -\delta^{\alpha\beta}\nabla\delta(\mathbf{r}-\mathbf{r}') - e^{\alpha\beta\gamma}\mathbf{A}^\gamma\delta(\mathbf{r}-\mathbf{r}'), \tag{6.3}$$

$$\{K_i, A_k{}^\alpha\} = -F_{ik}{}^\alpha\delta(\mathbf{r}-\mathbf{r}'), \tag{6.4}$$

$$\{K_i, K_k\} = (K_k(\mathbf{r})\nabla_i - K_i(\mathbf{r}')\nabla_k' + M^\beta F_{ik}{}^\beta)\delta(\mathbf{r}-\mathbf{r}'), \tag{6.5}$$

$$\{K_i, \rho\} = \rho\nabla_i\delta(\mathbf{r}-\mathbf{r}'). \tag{6.6}$$

All the Jacobi identities are satisfied.

The equations of motion take the form (we introduce $A_4^\alpha = \delta H/\delta M^\alpha$ and $\mathbf{v} = \delta H/\delta\mathbf{K}$):

$$\frac{\partial\mathbf{M}}{\partial t} = [\mathbf{A}_4\times\mathbf{M}] + \left[\frac{\delta H}{\delta\mathbf{A}_k}\times\mathbf{A}_k\right] + \nabla_k\frac{\delta H}{\delta\mathbf{A}_k} - v_k\mathbf{F}_{4k} - \frac{\delta R}{\delta\mathbf{A}_4}, \tag{6.7}$$

$$\mathbf{F}_{i4} + v_k\mathbf{F}_{ik} = \frac{\delta R}{\delta(\delta H/\delta\mathbf{A}_i)}, \quad \mathbf{F}_{i4} = \nabla_i\mathbf{A}_4 - \frac{\partial\mathbf{A}_i}{\partial t} + [\mathbf{A}_i\times\mathbf{A}_4], \tag{6.8}$$

$$\frac{\partial K_i}{\partial t} = -\nabla_l(v^l K_i) - K_l\nabla_k v^l + \mathbf{M}\mathbf{F}_{ki}v_k + \frac{\delta H}{\delta\mathbf{A}_k}\mathbf{F}_{ik} - \frac{\delta R}{\delta v_i} - \rho\nabla_i\frac{\delta H}{\delta\rho}, \tag{6.9}$$

$$\frac{\partial\rho}{\partial t} + \nabla(\rho\mathbf{v}) = -\frac{\delta R}{\delta(\delta H/\delta\rho)}. \tag{6.10}$$

It is seen from (6.8) that v_k is indeed the macroscopic disclination velocity, since $\mathbf{F}_{i4}$ is the disclination flux,[4,5] and $\mathbf{F}_{ik}$ is the density of the disclinations.

We write down the Hamiltonian and the dissipation function in the form:

$$H = \int d^3r\left(\rho^s\frac{(A_i{}^\alpha)^2}{2} + \frac{\mathbf{M}^2}{2\chi} + \frac{\mathbf{K}^2}{2\rho} + \varepsilon_0(\rho)\right), \tag{6.11}$$

$$R = \int d^3r\left(\frac{1}{2}\alpha(A_i{}^\alpha)^2 + \frac{1}{2}\beta(\nabla_i\mathbf{A}_4)^2 + \frac{1}{2}\gamma\mathbf{v}^2 + \frac{1}{2}\lambda\left(\frac{\partial\varepsilon}{\partial\rho}\right)^2\right). \tag{6.12}$$

The third term in (6.12) is due friction of the disclinations against the medium, while the fourth term is due to the possibility of a change in the disclination mass on account of annihilation of the disclinations. If we neglect the last term, then Eqs. (6.7)–(6.10) yield four low-frequency modes: the three spin-diffusion modes (which go over into spin waves at higher frequencies[4,5]), and one diffusion mode connected with the motion of the disclinations,

$$\omega_4 = -iq^2\frac{\rho}{\gamma}\frac{\partial\varepsilon_0}{\partial\rho}. \tag{6.13}$$

It is possible that this mode has a bearing on the fourth low-frequency mode obtained in a numerical experiment with spin glass.[22] In the limit as $\gamma\to 0$ it goes over into an acoustic mode in the system of defects.

CONCLUSION

We have considered two cases of a continual description of defects. These cases, in the sense of topology, are the simplest ones, since both the vortices in He II and the vortices in a planar magnet are described by the homotopic group $\Pi_1 = Z$ and consequently are specified by analytic expressions with the aid of δ functions [see (2.7) and (5.8)]. Even in this case the hydrodynamic variables that describe the distributed vortices have entirely different Poisson brackets and consequently a different dynamics. In other systems with $\Pi_1 = Z$, for example in a system of vortices in a superconductor and in a system of dislocations in a crystal, one should also expect the defects to have different continual dynamics peculiar to these systems only. It appears that this dynamics can be easily obtained by the method described above from the dynamics of an isolated defect.

The situation is much more complicated if $\Pi_1 \neq Z$, for example $\Pi_1 = Z_2$ or $\Pi_1 = Z_4$. This takes place, for example, in systems in which the symmetry of the states relative to three dimensional rotations has been violated, such as liquid nematic crystals, nonplanar magnets, the superfluid phases of He^3, etc. The difficulty lies here in the fact that for the linear defects that are described by such homotopic groups there are generally speaking no analytic expressions. Therefore, when considering in this paper nonplanar magnets and spin glasses, we have simply drawn an analogy with planar magnets. Further research is needed here. In addition, it is necessary to elucidate the dynamics of the distributed defects of other topologic sorts, described by the groups Π_2, Π_3, as well as of solitons described by relative homotopic groups (see Ref. 23).

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Saturation of nuclear magnetic resonance under conditions of large dynamic frequency shift

V. A. Tulin

Institute of Solid State Physics, Academy of Sciences of the USSR
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In magnetically ordered crystals with strong hyperfine interaction there is a large observed frequency shift of the nuclear magnetic resonance (NMR), proportional to the magnetization of the nuclei. Under these conditions the NMR has strongly nonlinear properties. The process of locking of the nuclear magnetic resonance withn the detuning is varied in the easy-plane antiferromagnetics $MnCO_3$ and $CsMnF_3$ is investigated. It is shown that in $CsMnF_3$ the results can be described in terms of excitation of NMR on the wing of a line of the Lorentz form. For $MnCO_3$, besides this mechanism, the possibility of locking via excitation of nuclear spin waves is also assumed.

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INTRODUCTION

In magnetically ordered substances with strong hyperfine interaction and large density of magnetic nuclei, coupled modes of electron-nucleus magnetic resonance appear. The nuclearlike branch of these modes, like that which has been investigated[1-5] for antiferromagnetics with small crystalline anisotropy, has the form

$$\omega=\omega_n\left[1-\frac{B\langle m\rangle}{\Omega_e^2+B\langle m\rangle}\right]^{1/2},\tag{1}$$

where ω_n is the frequency of nuclear resonance in the hyperfine field of the electron when there is no coupling, Ω_e is the frequency of antiferromagnetic resonance in the absence of the nuclear magnetic system, and $B\langle m\rangle$ is a quantity which characterizes the coupling. Here the dependence of the coupling on the value of the mean nuclear magnetization $\langle m\rangle$ has been distinguished. The formula used for this quantity is

$$B\langle m\rangle=2\gamma_e^2H_EH_N,\tag{2}$$

where γ_e is the electronic magnetomechanical ratio, H_E is the effective exchange field of the antiferromagnetic system, and H_N is the hyperfine field on the electrons which the nuclei produce:

$$H_N=A\langle m\rangle/\gamma_e,\tag{3}$$

(A is the hyperfine interaction constant). From this we have

$$B=2\gamma_eH_EA.\tag{4}$$

The form in which the nuclearlike resonance modes are written shows that the frequency depends on the mean nuclear magnetization, which can get changed easily under the action of a high-frequency field, a phenomenon described as saturation of NMR.[6] This leads to a strong nonlinearity of nuclear magnetic resonance in the case under consideration. This nonlinearity was pointed out in the very first papers on NMR research in substances with a strong dynamic shift.[1]

If we fix the external parameters—the temperature, the external magnetic field that determines the frequency Ω_e, and the frequency of the exciting field in the range from ω, given by Eq. (1) to ω_n—then the nuclear magnetic system can be in two stable states, viz., at small power levels $\langle m\rangle\approx\langle m\rangle_T$, where $\langle m\rangle_T$ is determined by the thermostat temperature, and at a power $\langle m\rangle\approx\langle m\rangle_0$ sufficient to saturate the NMR, where $\langle m\rangle_0$ is the solution of Eq. (1) for a frequency ω equal to the frequency ω_0 of the exciting field. If $|\omega_0-\omega|>\delta\omega$, where $\delta\omega$ is the linewidth of the NMR, there is a decided difficulty in the system's making the transition from the state $\langle m\rangle_T$ to the state $\langle m\rangle_0$. This problem was considered in the paper of de Gennes and others[1] for a Gaussian shape of the NMR line. They found an exponential dependence of the amplitude of the exciting field required to take the system from the state $\langle m\rangle_T$ to the states $\langle m\rangle_0$ on the detuning $\omega_0-\omega$. Experiments made by double resonance on the saturation of NMR in $KMnF_3$ (Ref. 7) and $MnCO_3$ (Ref. 8) showed a decidedly weaker dependence. It was suggested that this dependence could be described on the basis of the defect structure in the crystals.[7,8] This point of view was maintained, in spite of existing deviations of the experimental results from the calculated dependence.

Kurkin[9] calculated the critical power necessary for the transition of the nuclear system into the saturated state $\langle m\rangle_0$ in the case of a Lorentz shape of the NMR line. He also showed that the Lorentz line shape is closer to a realistic description of the situation. As could be expected in this case, the dependence of the critical power on the detuning is of a power-law type:

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