Thermomagnetic instability of a nonuniform plasma

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We consider an instability which may occur in a collisional nonuniform plasma with a heat flux. The magnetic field grows rapidly when this instability develops. We obtain the conditions for the occurrence of the instability. We evaluate the characteristic growth rates of the perturbations. The instability develops most effectively in a very hot and rarefied plasma when there are large temperature and density inhomogeneities. We apply the results of the theory to a laser plasma and to the plasma of the solar corona.

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It is well known that in various experiments about the interaction of laser radiation with matter after a negligibly short time ($\sim 10^{-9}$ s) huge magnetic fields may be generated in the plasma which is produced[1] occasionally reaching 10^6 gauss. Apparently, the so-called thermomagnetic instability may play an important role in the production of these fields.[2] A necessary condition for such an instability is first and foremost that the plasma must be inhomogeneous and non-isothermal. The development of the instability proceeds as follows. Small temperature perturbations lead to the occurrence of a heat current. This current produces a magnetic field which in turn affects the electron thermal conductivity and changes the regime of the heat transfer. The field leads to the appearance of a heat flux perpendicular to the temperature gradient and to the field itself. This flux supplies energy to the region with a higher temperature and thereby facilitates the growth of the initial perturbations. The thermomagnetic instability thus promotes the transfer of part of the energy of the heat flux to magnetic field energy. The strong magnetic fields which are then generated may affect in an essential way transfer processes in a plasma, the hydrodynamics of the disintegration, and a number of other phenomena.

A number of authors[2-4] have considered the thermomagnetic instability in a non-degenerate plasma in which electron thermal transfer predominates. We show that the effect of thermal effects on the field generation has been insufficiently taken into account in these papers. The conclusions by these authors[2-4] on the necessary conditions for the development of the instability are thus inexact. We shall in the present paper obtain an expression for the growth rate and we shall derive the necessary conditions for the development of the instability. The thermomagnetic instability may strongly affect processes occurring in a laser plasma and in the plasma of the solar corona.

1. EQUATIONS FOR SMALL PERTURBATIONS

We consider the instability in a completely ionized hydrogen plasma (the generalization to the case of an arbitrary charge Z of the ions can easily be made) with a non-degenerate electron gas. We assume that the plasma is non-uniform and non-isothermal in the unperturbed state and that there are no magnetic fields or hydrodynamic motions in it. For simplicity we consider a plane problem where all unperturbed quantities depend solely on a single Cartesian coordinate z (we shall, of course, not make any assumptions of this kind about the perturbations). We assume also that the heat transfer is mainly by the electrons and that the radiative heat conductivity is small. We shall merely be interested in the linear stage of the evolution of the perturbations when they are small compared to the unperturbed quantities. We shall in what follows indicate unperturbed quantities by an index zero and small perturbations by an index 1. By assumption $\mathbf{V}_0=0$ and $\mathbf{B}_0=0$ ($\mathbf{V}$ is the hydrodynamic velocity of the plasma motion, $\mathbf{B}$ the magnetic field).

The set of magnetohydrodynamic equations for a fully ionized plasma were derived by Braginskiĭ.[5] The equation of motion has the form

$$\rho\frac{\partial \mathbf{V}}{\partial t}+\rho(\nabla\mathbf{V})\mathbf{V}=-\nabla p+\frac{1}{4\pi}[[\nabla\times\mathbf{B}]\times\mathbf{B}].$$

Here p and ρ are the pressure and density of the plasma; we assume that there are no external forces. The radiation pressure often happens to be small and we shall neglect it. Moreover, practically always under astrophysical conditions and very often under laboratory conditions the effect of viscous stresses is unimportant and we have neglected them.

We consider the behavior of the very small-scale perturbations for which the wavelength λ is much smaller than the characteristic scale L of the change in the unperturbed quantities T and ρ (we assume that they are of the same order of magnitude). Linearizing the equation of motion we find

$$\rho_0\frac{\partial \mathbf{V}_1}{\partial t}=-\nabla p_1. \tag{1}$$

The linearized equation of continuity and equation of state for a gas are

$$\partial\rho_1/\partial t=-\nabla(\rho_0\mathbf{V}_1), \tag{2}$$

$$p_1/p_0=\rho_1/\rho_0+T_1/T_0. \tag{3}$$

Neglecting viscous stresses and the radiation pressure we can write Ohm's law in the following form[5]:

$$\mathbf{E}+\left[\frac{\mathbf{V}}{c}\times\mathbf{B}\right]+\frac{[\mathbf{B}\times\mathbf{j}]}{ecn_e}=-\frac{\nabla p_e}{n_e}+\frac{\hat{\alpha}\mathbf{j}}{(en_e)^2}-\frac{\hat{\beta}^{uT}}{en_e}\nabla T, \tag{4}$$

p_e and n_e are the electron pressure and density, $\mathbf{j}$ if the electric current density,

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$$\hat{\alpha}\mathbf{j}=\alpha_\parallel \mathbf{j}_\parallel+\alpha_\perp \mathbf{j}_\perp-\alpha_\wedge[\mathbf{b}\times\mathbf{j}],$$
$$\hat{\beta}^{uT}\nabla T=\beta_\parallel^{uT}\nabla_\parallel T+\beta_\perp^{uT}\nabla_\perp T+\beta_\wedge^{uT}[\mathbf{b}\times\nabla T],$$
$$\mathbf{b}=\mathbf{B}/|\mathbf{B}|.$$

The $\parallel$ and $\perp$ signs indicate that one must take the components of the corresponding vector which are parallel and perpendicular to the magnetic field. We shall not give the general expressions for the tensors $\hat{\alpha}$ and $\hat{\beta}^{uT}$ as they are very cumbersome (they are written down in Braginskiĭ's paper[5]), and we write them down only in the particular case $\omega_B\tau\ll 1$ (ω_B is the electron cyclotron frequency, τ the electron "free path" time):

$$\alpha_\parallel=\alpha_\perp=0.51m_e n_e/\tau,\quad \beta_\parallel^{uT}=\beta_\perp^{uT}=0.71k_B n_e,$$
$$\beta_\wedge^{uT}=0.81k_B n_e\frac{eB}{m_e c}\tau,\quad \tau=\frac{3m_e^{1/2}(k_BT)^{3/2}}{4(2\pi)^{1/2}e^4n_e\Lambda}. \tag{5}$$

Here k_B is Boltzmann's constant, m_e the electron mass, and Λ the Coulomb logarithm.

Taking the curl of Eq. (4) and using Maxwell's equations we can obtain the induction equation. Linearizing it we get

$$\frac{\partial \mathbf{B}_1}{\partial t}+\frac{c^2}{4\pi e^2}\left[\nabla\times\left[\frac{\alpha_\parallel}{n_e^2}\nabla\times\mathbf{B}_1\right]\right]+\frac{c}{e}\left(\left[\nabla p_{e1}\times\nabla\frac{1}{n_{e0}}\right]-\left[\nabla p_{e0}\times\nabla\frac{n_{e1}}{n_{e0}^2}\right]\right)$$
$$-0{,}81\frac{k_B}{m_e}[\nabla[\tau_0\mathbf{B}_1\times\nabla T_0]]=0. \tag{6}$$

Braginskiĭ[5] has also derived the equation for energy transfer in a plasma. Both in astrophysical and in laboratory conditions one can often neglect the work done by the viscous forces and by the heat current transferred by the ions (it is smaller by a factor $(m_p/m_e)^{1/2}$ than the electron heat current, if the ions are not magnetized, m_p is the proton mass). The energy transfer equation then has the form

$$\rho c_v\frac{dT}{dt}+p\nabla\mathbf{V}=-\nabla\left(\mathbf{q}_e-\frac{5k_BT}{2e}\mathbf{j}\right)+\mathbf{j}\left(\mathbf{E}+\left[\frac{\mathbf{V}}{c}\times\mathbf{B}\right]\right),$$
$$\mathbf{q}_e=-T\frac{\hat{\beta}^{uT}\mathbf{j}}{en_e}-\hat{\varkappa}\nabla T,\quad c_v=\frac{3k_B}{m_p},$$

$\hat{\varkappa}$ is the thermal conductivity tensor. When $\omega_B\tau\ll 1$ its components equal

$$\varkappa_\parallel\approx\varkappa_\perp=3{,}16\frac{n_ek_B^2T\tau}{m_e},\quad \varkappa_\wedge=5{,}75\frac{n_ek_B^2T\tau}{m_e}\frac{eB}{m_ec}\tau. \tag{7}$$

We linearize the heat transfer equation and compare the heat currents caused by the thermal conductivity and by the electric currents. One sees easily that we can neglect the latter, if the inequality $(k_BT/m_ec^2)\omega_p^2\tau^2\gg 1$ is satisfied ($\omega_p=(4\pi e^2n_e/m_e)^{1/2}$ is the electron plasma frequency). We shall assume that this relation holds. The linearized energy-transfer equation then becomes

$$\rho_0c_v\frac{\partial T_1}{\partial t}+\rho_0c_v(\mathbf{V}_1\nabla T_0)-p_0\nabla\mathbf{V}_1=\nabla(\hat{\varkappa}\nabla T)_1. \tag{8}$$

Equations (1) to (3), (6), and (8) completely determine the behavior of small perturbations.

2. DISPERSION EQUATION

The inhomogeneities of the unperturbed quantities are important for the thermomagnetic instability so that it is necessary to use the geometric optics approximation (WKB approximation) for its study. First of all we make clear how one can in that approximation construct a solution of the set (1), (2), (3), (6), (8) and obtain information about the behavior of small perturbations. We perform a Fourier transformation of the set with respect to the x and y coordinates and a Laplace transformation with respect to the time. After that we can write the set (1), (2), (3), (6), (8) schematically in the form

$$\sum_j L_{ij}\left(\frac{d^2}{dz^2},\frac{d}{dz},\omega,\mathbf{k}_\perp,z\right)\psi_{j\omega\mathbf{k}_\perp}=a_i(z),$$
$$L_{ij}=c_{ij}\frac{d^2}{dz^2}+d_{ij}\frac{d}{dz}+e_{ij}, \tag{9}$$

where L_{ij} is a linear differential operator, the coefficients c_{ij}, d_{ij}, and e_{ij} depend on ω, $\mathbf{k}_\perp=k_x\mathbf{e}_x+k_y\mathbf{e}_y$, and z; $(i,j)=1,2,\ldots,n$; n is the number of equations in the set; $\psi_j\omega\mathbf{k}_\perp$ are the Fourier transforms of the small perturbations,

$$\psi_{j\omega\mathbf{k}_\perp}=\int_{-\infty}^{+\infty}dx\,dy\int_0^\infty dt\,\psi_j(\mathbf{r},t)\exp(-i\omega t+ik_xx+ik_yy);$$

$a_i(z)$ are functions determined by the initial conditions.

We find first of all the solution of the homogeneous set of the form

$$\sum_j L_{ij}\left(-k_z^2-i\frac{dk_z}{dz},\ -ik_z,\ \omega,\ \mathbf{k}_\perp,\ z\right)b_j=0$$

(k_z are some unknown quantities) after that we use the b_j and k_z to construct a solution of the set (9). We get the condition for the homogeneous set to be soluble by putting the determinant

$$\left|L_{ij}\left(-k_z^2-i\frac{dk_z}{dz},\ -ik_z,\ \omega,\ \mathbf{k}_\perp,\ z\right)\right|=0$$

equal to zero. Hence we find the functions $k_z(\omega,\mathbf{k}_\perp,z)$ for which the set has a non-trivial solution. It is convenient to write the equations for the k_z in the form of dispersion relations

$$\omega=\omega_\alpha(\mathbf{k},z)=\Omega_\alpha(\mathbf{k},z)-i\gamma_\alpha(\mathbf{k},z), \tag{10}$$

$\mathbf{k}=\mathbf{k}_\perp+k_z\mathbf{e}_z$, α is the number of the root of the characteristic equation, i.e., the number of the oscillation mode (we assume that the number of modes of the set (9) is the same as the number of equations; if we use the equation of state (3) to eliminate the perturbation of the pressure p_1 from the other Eqs. (1), (2), (6), and (8) our system will satisfy that condition). We can determine $k_{z\alpha}$ as functions of ω, $\mathbf{k}_\perp$, and z for each mode α from the dispersion relations (10). We can also express for each mode the functions b_j in terms of an arbitrarily chosen single one, for instance, b_1: $b_j^{(\alpha)}=\chi_j^{(\alpha)}b_1^{(\alpha)}$, where the $\chi_j^{(\alpha)}$ are quantities which can be expressed in terms of the c_{ij}, d_{ij}, e_{ij}, and also of $k_{z\alpha}$ and $dk_{z\alpha}/dz$.

We shall look for the solution of the set (9) in the form

$$\psi_{j\omega\mathbf{k}_\perp}=\sum_\alpha B^{(\alpha)}b_j^{(\alpha)}\exp\left\{-i\int^z k_{z\alpha}\,dz'\right\};$$

the summation is over all modes, $B^{(\alpha)}$ are functions which we must find. We assume that $|k_{z\alpha}L|\gg 1$. Substituting $\psi_{j\omega\mathbf{k}_\perp}$ into Eq. (9) we get

$$\sum_{\alpha,j}\left\{c_{ij}\frac{d^2}{dz^2}(A^{(\alpha)}\chi_j^{(\alpha)})+(d_{ij}-2ik_{z\alpha}c_{ij})\frac{d}{dz}(A^{(\alpha)}\chi_j^{(\alpha)})\right\}\cdot$$
$$\cdot\exp\left(-i\int^z k_{z\alpha}\,dz'\right)=a_i(z),$$

where the $A^{(\alpha)}=B^{(\alpha)}b_1^{(\alpha)}$ are unknown functions. In transforming (9) we used the fact that the $b_j^{(\alpha)}$ are solutions of the homogeneous set.

We denote $d(A^{(\alpha)}\chi_j^{(\alpha)})/dz$ by $\varphi_j^{(\alpha)}$ so that

$$A^{(\alpha)}\chi_j^{(\alpha)}=\int^z\varphi_j^{(\alpha)}\,dz'.$$

We shall look for the $\varphi_j^{(\alpha)}$ in the form

$$\varphi_j^{(\alpha)}=x_j^{(\alpha)}\exp\left(i\int k_{z\alpha}\,dz'\right);$$

where the $x_j^{(\alpha)}$ must satisfy the equations

$$\sum_{\alpha,j}\left\{c_{ij}\frac{dx_j^{(\alpha)}}{dz}+(d_{ij}-ik_{z\alpha}c_{ij})x_j^{(\alpha)}\right\}=a_i(z).$$

We assume that initially at time $t=0$ the perturbations $a_i(z)$ are given in the form of wavepackets:

$$a_i(z)=a_i'\exp[-ik_0(z-z_0)-(z-z_0)^2/2b_0^2],$$

where $L\gg b_0\gg 2\pi/|k_0|$ while the characteristic scale of changes in a_i' is large ($\gtrsim L$), and z_0 is the center of the wavepacket at $t=0$. We note that perturbations of a more general form (differing from zero at distances $\sim b_0$ near the point z_0) can be written as a superposition of the packets considered by us. One can easily find for such $a_i(z)$ the $x_j^{(\alpha)}$ as expansions in powers of the small parameters b_0/L and λ/L. One shows easily that to lowest order (to which we can restrict ourselves) in these parameters we shall have, when the condition $b_0(b_0/\lambda)\ll L$ is satisfied

$$x_j^{(\alpha)}(z)\approx\sum_i a_i'(z_0)\lambda_{ji}^{(\alpha)}(z)\exp\left\{-ik_0(z-z_0)+i\int_{z_0}^{z}[k_{z\alpha}(\omega,\mathbf{k}_\perp,z')\right.$$
$$\left.-k_{z\alpha}(\omega,\mathbf{k}_\perp,z_0)]dz'-(z-z_0)^2/2b_0^2\right\};$$

$\lambda_{ij}^{(\alpha)}$ are functions with a scale of variation of the order L. One can thus write the solution of the set (9) in the form

$$\psi_{j\omega\mathbf{k}_\perp}\approx\sum_{\alpha,i}\int^z dz'\lambda_{ji}^{(\alpha)}(z')a_i'(z_0)\exp\left\{-ik_0(z'-z_0)\right.$$
$$+i\int_{z_0}^{z'}[k_{z\alpha}(\omega,\mathbf{k}_\perp,z'')-k_{z\alpha}(\omega,\mathbf{k}_\perp,z_0)]dz''$$
$$\left.+i\int_z^{z'}k_{z\alpha}(\omega,\mathbf{k}_\perp,z'')dz''-\frac{(z'-z_0)^2}{2b_0^2}\right\}$$

(compare the form of the solution for $|z-z_0|<L$ with p. 26 of Ref. 6). Since $k_{z\alpha}$ is a slowly changing function of z (characteristic scale $\sim L$) we have

$$\int_z^{z'}k_{z\alpha}\,dz''\approx\int_z^{z_0}k_{z\alpha}\,dz''+k_{z\alpha}(z_0)(z'-z_0),$$

if $|z'-z_0|\sim b_0\ll L$. Using this relation we can easily integrate over z' in $\psi_{j\omega\mathbf{k}_\perp}$ (under the condition $|z-z_0|\gg b_0$).

We now take the inverse Laplace transform with respect to the time and find the form of the perturbations $\psi_j(z,t)$ at time t at the point z. The x and y dependence of the perturbations can be found elementarily and in what follows we shall not be interested in it. We note also that all successive transformations are performed completely identically for each of the modes. For the sake of simplicity we shall therefore further drop the index α of the functions and the summation over the various modes. Then we have

$$\psi_j(z,t)\approx\int_{-\infty}^{+\infty}d\omega\,G_j(\omega,z_0)\exp\left\{i\omega t+i\int_z^{z_0}k_z(\omega,\mathbf{k}_\perp,z')dz'\right.$$
$$\left.-\frac{1}{2}b_0^2[k_0-k_z(\omega,\mathbf{k}_\perp,z_0)]^2\right\},$$

where $k_z(\omega,\mathbf{k}_\perp,z)$ is the solution of the dispersion Eq. (10) at the point z and the $G_j(\omega,z_0)$ are some functions (they can be expressed in terms of the initial conditions and of the unperturbed quantities).

In the integral we introduce instead of ω a new integration variable k_z' which is related to ω through the dispersion relation (10) at the point z_0: $\omega=\omega(k_z',z_0)$. The integration over k_z' is over some contour in the complex plane. For the cases of interest to us as $\omega\to\pm\infty$ we shall have $\mathrm{Re}k_z'\to\pm\infty$. If there are no singularities between the contour and the real axis, i.e., $d\omega/dk_z'\neq 0$ (which is, in particular, true for the thermomagnetic waves considered by us) we can change to an integration over real values of k_z':

$$\psi_j(z,t)\approx\int_{-\infty}^{+\infty}dk_z'\frac{d\omega}{dk_z'}G_j(k_z',z_0)\exp\left\{i\omega(k_z',z_0)t\right.$$
$$\left.+i\int_z^{z_0}k_z(k_z',z_0,z')dz'-\frac{1}{2}b_0^2(k_z'-k_0)^2\right\}.$$

We thus obtain expressions for the perturbations with frequencies which formally depend on the coordinates of the point where the perturbations originate. Gurevich and Gel'mont[7] were the first to propose such a representation for a study of an instability for the simple case of a medium with a permittivity that depended on a single coordinate.

For the wavepackets of the form considered by us we can perform the integration over k_z' in $\psi_j(z,t)$. If $|\mathrm{Re}\omega|\gg|\mathrm{Im}\omega|$, one sees easily that the motion of the point $z(t)$ at which the wavepackets has its maximum is described by the approximate equation

$$dz(t)/dt\approx V_g(k_0,z(t)),$$

where $V_g(k_z,z)=\partial\,\mathrm{Re}\omega(k_z,z)/\partial k_z$ is the local group velocity. The maximum of the absolute magnitude of the perturbation grows with time according to

$$|\psi_j|\propto\exp\left\{\gamma(k_0,z_0)t+\int_{z_0}^{z(t)}\mathrm{Im}\,k_z(\omega_0,\mathbf{k}_\perp,z')dz'\right\};$$

here $\omega_0=\Omega(k_0,z_0)-i\gamma(k_0,z_0)$, and $k_z(\omega_0,\mathbf{k}_\perp,z')$ satisfies the equation $\omega_0=\Omega(k_z,z')-i\gamma(k_z,z')$.

When $|\Omega|\gg|\gamma|$ one finds easily that

$$\mathrm{Im}\,k_z\approx[\gamma(k^{(0)},z')-\gamma(k_0,z_0)]/V_g(k^{(0)},z'),$$

where $k^{(0)}$ is determined from the relation $\Omega(k_0,z_0)=\Omega(k^{(0)},z')$. Substituting this expression into $|\psi_j|$ we get

$$|\psi_j|\propto\exp\left\{\gamma(k_0,z_0)\left[t-\int_0^t\frac{V_g(k_0,z(t'))}{V_g(k^{(0)},z(t'))}dt'\right]\right.$$
$$\left.+\int_0^t\gamma(k^{(0)},z(t'))\frac{V_g(k_0,z(t'))}{V_g(k^{(0)},z(t'))}dt'\right\}.$$

For thermomagnetic waves (as we shall see below) V_g is independent on k_z (since $\mathrm{Re}\,\omega \propto k_z$). In that case

$$|\psi_j| \infty \exp\left\{ \int_0^t \gamma[k_0 V_g(z_0)/V_g(z(t')), z(t')]\,dt' \right\}.$$

If, moreover, $|z(t)-z_0|<L$, we have

$$|\psi_j| \infty \exp\left\{ \int_0^t \gamma[k_0, z(t')]\,dt' \right\},$$

which is the same as Mikhaĭlovskiĭ's result on p. 29 of Ref. 6.

One can thus for a study of a local instability with a wavelength λ much shorter than L proceed as follows. One first substitutes into the original set of equations perturbations of the form

$$\psi_j \infty \exp\left\{ i\omega t - ik_x x - ik_y y - i\int k_z(z')\,dz' \right\}.$$

Then one writes down the dispersion equation and finds to lowest order in the small parameter λ/L both the real and the imaginary part of ω as functions of z, $\mathbf{k}_\perp$, and $k_z(z)$. If $\mathrm{Re}\,i\omega$ is positive this will mean that a local instability can develop.

We turn now directly to a study of the set of Eqs. (1), (2), (3), (6), (8). We note first of all that on the right-hand side of the equation of continuity (2) we can drop the term of order $1/L$ in comparison with $1/\lambda$. In the induction Eq. (6) we can neglect terms proportional to $\alpha_\parallel/\lambda L$ in comparison to $\alpha_\parallel/\lambda^2$ since the instability develops only if the Joule dissipation is sufficiently weak (we give below a quantitative criterion); in that case the term $\propto \alpha_\parallel/\lambda L$ gives merely a small correction to a small effect (dissipation). It is further convenient to use the induction equation, taking the scalar product of it with $[\mathbf{k}\times\mathbf{e}_z]$ as the magnetic field occurs in the energy transfer equation only as $\mathbf{B}_1[\mathbf{k}\times\mathbf{e}_z]$. The term $\rho_0 c_v(\mathbf{V}_1\nabla T_0)$ in Eq. (8) is smaller than $p_0\nabla\mathbf{V}_1$ by a factor L/λ and we can thus neglect it. We use the fact that $\varkappa_0 \propto T_0^{5/2}$ and $\varkappa_1 \propto 5\varkappa_0 T_1/2T_0$ (one can neglect the T and ρ dependence of the Coulomb logarithm) and we drop in Eq. (8) terms of order $\varkappa_0 T_1/L^2$ in comparison with $\varkappa_0 T_1/\lambda^2$. After all these transformations we can put the set of Eqs. (1), (2), (3), (5) to (8) in the following form:

$$\begin{gathered}
\omega\rho_1=\rho_0\mathbf{k}\mathbf{V}_1,\quad \omega\mathbf{k}\mathbf{V}_1=k^2p_1/\rho_0,\quad p_1/p_0=\rho_1/\rho_0+T_1/T_0,\\
\left\{i\omega+\frac{c^2k^2}{4\pi\sigma_0}+0{,}81\frac{k_B}{m_e}\left[i\tau_0k_z\frac{dT_0}{dz}-\frac{d}{dz}\left(\tau_0\frac{dT_0}{dz}\right)\right]\right\}\mathbf{B}_1[\mathbf{k}\times\mathbf{e}_z]\\
=-\frac{icm_p}{2e\rho_0^2}(k^2-k_z^2)\left(p_1\frac{d\rho_0}{dz}-\rho_1\frac{dp_0}{dz}\right),\\
\left[i\omega\rho_0c_v+\varkappa_0k^2+i\varkappa_0\frac{dk_z}{dz}+5i\varkappa_0k_z\frac{d\ln T_0}{dz}\right]T_1\\
-ip_0\mathbf{k}\mathbf{V}_1=1{,}82i\varkappa_0\frac{e\tau_0}{m_ec}\frac{dT_0}{dz}\mathbf{B}_1[\mathbf{k}\mathbf{e}_z],\\
\sigma_0=e^2n_{e0}\tau_0/0{,}51m_e.
\end{gathered}\tag{11}$$

Using the first three equations of this set we can easily express ρ_1, p_1, and $\mathbf{k}\cdot\mathbf{V}_1$ in terms of T_1:

$$\frac{\rho_1}{\rho_0}=-\frac{T_1}{T_0(1-\zeta)},\quad \frac{p_1}{p_0}=-\frac{\zeta T_1}{T_0(1-\zeta)},\quad \mathbf{k}\mathbf{V}_1=-\frac{\omega T_1}{T_0(1-\zeta)},\tag{12}$$

$\zeta=\omega^2\rho_0/k^2p_0$. Substituting the expressions for p_1 and ρ_1 into the induction equation and the expression for $\mathbf{k}\cdot\mathbf{V}_1$ into the heat transfer equation we get a set of two equations connecting the two unknown quantities $\mathbf{B}_1[\mathbf{k}\times\mathbf{e}_z]$ and T_1. Putting the determinant of that set to zero we get the dispersion equation for the thermomagnetic mode

$$\begin{gathered}
\left\{i\omega+\frac{c^2k^2}{4\pi\sigma_0}+0{,}81\frac{k_B}{m_e}\left[i\tau_0k_z\frac{dT_0}{dz}-\frac{d}{dz}\left(\tau_0\frac{dT_0}{dz}\right)\right]\right\}\\
\times\left\{i\omega\rho_0c_v\left[1+\frac{2}{3(1-\zeta)}\right]+\varkappa_0k^2+i\varkappa_0\frac{dk_z}{dz}+5i\varkappa_0k_z\frac{d\ln T_0}{dz}\right\}\\
=\frac{1{,}82k_B\tau_0\varkappa_0}{m_e(1-\zeta)}\frac{dT_0}{dz}(k^2-k_z^2)\left(-\zeta\frac{d\ln\rho_0}{dz}+\frac{d\ln p_0}{dz}\right).
\end{gathered}\tag{13}$$

It is rather difficult to obtain the general solution of this equation. We study the behavior of the solution in two special cases: $|\omega|\gg kc_s$ and $|\omega|\ll kc_s$ ($|\zeta|\gg 1$ and $|\zeta|\ll 1$), where c_s is the sound speed.

3. THERMOMAGNETIC MODE WHEN $|\omega|\gg kc_s$

One can use (12) to check easily that when $|\zeta|\gg 1$ the relative perturbation ρ_1/ρ_0 is much smaller than the relative perturbations in pressure and temperature:

$$\frac{\rho_1}{\rho_0}\approx\left(\frac{k^2p_0}{\omega^2\rho_0}\right)\frac{T_1}{T_0},\qquad \left|\frac{\rho_1}{\rho_0}\right|\ll\left|\frac{T_1}{T_0}\right|\approx\left|\frac{p_1}{p_0}\right|.$$

For the velocity perturbation we get approximately

$$\mathbf{k}\mathbf{V}_1=(k^2p_0/\omega\rho_0)(T_1/T_0).$$

Substituting this expression into the heat transfer equation we see that the term proportional to $\mathbf{k}\cdot\mathbf{V}_1$ can be neglected compared to $i\omega\rho_0c_vT_1$. This means that when $|\omega|\gg kc_s$ the convective energy transfer by the motions of the plasma is small compared to the transfer due to heat conduction. In this case the motion of the plasma affects neither the excitation of a magnetic field nor the heat transfer in the plasma. If besides $|\omega^2\rho_0/k^2p_0|\gg 1$ the inequality $|d\rho_0/dz|\gg|k^2/\omega^2\|dp_0/dz|$ is also satisfied (i.e., the characteristic scale for temperature changes is not much smaller than the scale for density changes), we obtain for ω instead of (13) a quadratic equation which can be solved elementarily. This equation has a solution with $\mathrm{Re}(i\omega)>0$, if

$$(k_BT_0/m_ec^2)\,\omega_p^2\tau_0^2\gg(L/\lambda)^2.\tag{14}$$

If inequality (14) is satisfied this means that the growth of the magnetic field due to a heat current turns out to be much faster than its damping due to Joule dissipation. Indeed, it is clear from the induction equation that the characteristic time for Joule dissipation of the field t_B (determined by the second term in the braces on the left-hand side) in (11) will be of the order $4\pi\sigma_0/3c^2k^2\sim4\pi\lambda^2e^2n_{e0}\tau_0/c^2m_e$. The characteristic time for field changes due to thermo-effects t_T (determined by the last term in the braces on the left-hand side) is of the order $m_eL^2/k_BT_0\tau_0$. One checks easily that when (14) is satisfied the dissipative term is small and $t_B\gg t_T$.

When (14) holds the root corresponding to the instability equals

$$\begin{gathered}
i\omega=-0.81ik_z\frac{k_B\tau_0}{m_e}\frac{dT_0}{dz}+\frac{k_B\tau_0}{m_e}\left\{\left(1.01-1.82\frac{k_z^2}{k^2}\right)\frac{dT_0}{dz}\right.\\
\left.\times\frac{d\ln\rho_0}{dz}+1.22\frac{dT_0}{dz}\frac{d\ln T_0}{dz}+0.81\frac{d^2T_0}{dz^2}\right\}
\end{gathered}\tag{15}$$

(when carrying out the calculations we assumed that

$\tau_0 \propto T_0^{3/2}/\rho_0$ and neglected the weak temperature and density dependence of the Coulomb logarithm). Apart from (14) a necessary condition for the development of the instability will be

$$\left(1.01-1.82\frac{k_z^2}{k^2}\right)\frac{dT_0}{dz}\frac{d\ln\rho_0}{dz}+1.22\frac{dT_0}{dz}\frac{d\ln T_0}{dz}+0.81\frac{d^2T_0}{dz^2}>0. \quad (16)$$

The characteristic time t_0 for the growth of the instability is given by $1/\mathrm{Re}(i\omega)$ and equals

$$t_0=\frac{m_e}{k_B\tau_0}\left[\left(1.01-1.82\frac{k_z^2}{k^2}\right)\frac{dT_0}{dz}\frac{d\ln\rho_0}{dz}+1.22\frac{dT_0}{dz}\frac{d\ln T_0}{dz}+0.81\frac{d^2T_0}{dz^2}\right]^{-1}. \quad (17)$$

It is clear that the growth time depends strongly on the magnitude of the gradients of the unperturbed temperature and density. The growth time decreases with increasing gradients of ρ_0 and T_0. Using the fact that $\tau_0 \propto T_0^{3/2}/\rho_0$ and estimating ∇T_0 and $\nabla\rho_0$ to be T_0/L and ρ_0/L we find that t_0 is approximately proportional to $\rho_0/T_0^{5/2}$, i.e., the instability develops most effectively in a very rarefied and hot plasma.

It is completely unnecessary for the development of the instability that $\nabla\rho_0$ and ∇T_0 should be in the same direction (see, e.g., Refs. 2 to 4) and in certain cases the perturbations can grow also when $\nabla\rho_0$ and ∇T_0 are in the opposite direction. The presence of an imaginary part of $i\omega$ leads to the fact that the instability turns out to be a drift instability, i.e., the initial perturbation will not only grow with time but also move in the plasma. The velocity of the perturbations decreases as the magnitude of $k_z(z)$ decreases. One checks easily that the drift of the perturbations proceeds in the direction in which the plasma temperature decreases.

4. THERMOMAGNETIC MODE WHEN $|\omega| \ll kc_s$

It follows from (12) that in this case the relative pressure perturbations are much smaller than the relative density and temperature perturbations:

$$\frac{p_1}{p_0}=-\left(\frac{\omega^2\rho_0}{k^2p_0}\right)\frac{T_1}{T_0},\quad \left|\frac{\rho_1}{\rho_0}\right|\approx\left|\frac{T_1}{T_0}\right|\gg\left|\frac{p_1}{p_0}\right|.$$

For the velocity perturbations we now get $\mathbf{k}\cdot\mathbf{V}_1=-\omega(T_1/T_0)$. If we substitute this expression into the heat-transfer equation one sees easily that in that case the convective transfer of the energy of the plasma motion has an appreciable value (the terms $i\omega\rho_0c_vT_1$ and $ip_0\mathbf{k}\cdot\mathbf{V}_1$ are of the same order of magnitude). The excitation of the magnetic field is basically guaranteed by the density perturbations, and not by the pressure perturbations, as when $|\zeta|\gg 1$. We shall assume that apart from $|\omega^2\rho_0/k^2p_0|\ll 1$ the inequality $|\omega^2/k^2||d\rho_0/dz|\ll|dp_0/dz|$ is satisfied; in that case we get for ω, as before, a quadratic equation which has a solution with $\mathrm{Re}(i\omega)>0$ only when condition (14) is satisfied.

Assuming that $\tau_0 \propto T_0^{3/2}/\rho_0$ we can write that solution in the following form:

$$i\omega=-0.81ik_z\frac{k_B\tau_0}{m_e}\frac{dT_0}{dz}+\frac{k_B\tau_0}{m_e}\left[\left(1.01-1.82\frac{k_z^2}{k^2}\right)\frac{dT_0}{dz}\frac{d\ln\rho_0}{dz}+\left(3.04-1.82\frac{k_z^2}{k^2}\right)\frac{dT_0}{dz}\frac{d\ln T_0}{dz}+0.81\frac{d^2T_0}{dz^2}\right]. \quad (18)$$

For the presence of the instability it is necessary that the condition

$$\left(1.01-1.82\frac{k_z^2}{k^2}\right)\frac{dT_0}{dz}\frac{d\ln\rho_0}{dz}+\left(3.04-1.82\frac{k_z^2}{k^2}\right)\frac{dT_0}{dz}\frac{d\ln T_0}{dz}+0.81\frac{d^2T_0}{dz^2}>0 \quad (19)$$

is satisfied. The characteristic growth time t_0 of the instability will when $|\zeta|\ll 1$ be equal to

$$t_0=\frac{m_e}{k_B\tau_0}\left[\left(1.01-1.82\frac{k_z^2}{k^2}\right)\frac{dT_0}{dz}\frac{d\ln\rho_0}{dz}+\left(3.04-1.82\frac{k_z^2}{k^2}\right)\frac{dT_0}{dz}\frac{d\ln T_0}{dz}+0.81\frac{d^2T_0}{dz^2}\right]^{-1}. \quad (20)$$

As when $|\zeta|\gg 1$ the growth time depends strongly on the magnitude of the density and temperature gradients. The growth time is approximately proportional to $\rho_0/T_0^{5/2}$, i.e., also in this case does the instability develop faster in a rarefied and hot plasma. As before, the perturbations drift to the region with a lower temperature.

5. DISCUSSION OF THE RESULTS

We compare first of all the results obtained by us with those of other authors. Tidman and Shanny[2] were the first to study the thermomagnetic instability in connection with experiments with a laser plasma. These authors considered the case $|\zeta|\gg 1$ and perturbations with $\mathbf{k}\perp\mathbf{e}_z$. They neglected hydrodynamic motions, convective energy transfer, and the presence of density perturbations which in the given case is justified. However, the effect of thermal effects on the magnetic field generation was incorrectly taken into account. The authors of Ref. 2 dropped the terms in the square brackets on the left-hand side of the induction Eq. (11). One of these terms, indeed, vanishes when $\mathbf{k}\perp\mathbf{e}_z$, but the second one is non-vanishing, even when $k_z=0$. They therefore obtained an incorrect expression for the growth rate [if we put $k_z=0$ in (15) the result of Ref. 2 will differ from ours by a term $0.81\nabla(\tau_0\nabla T_0)/\tau_0$ in the braces]. In Refs. 3, 4 the case $|\omega^2\rho_0/k^2p_0|\gg 1$ was also considered (although the authors themselves do not mention this). The authors took into account the effect of one of the thermal effects on the magnetic field generation (the so-called thermal-drift field, corresponding to the term $0.81ik_z(k_B\tau_0/m_e)(dT_0/dz)\mathbf{B}_1$ on the left-hand side of the induction equation) but they also neglected the second term in the square brackets of the left-hand side of (11). In those papers they therefore obtained a qualitatively incorrect conclusion that the instability develops only when the gradients of ρ_0 and T_0 are parallel. It is clear from Eq. (16) that this is altogether unnecessary. The case $|\omega|\ll kc_s$ which is of most interest for astrophysical applications has (as far as we know) not been considered earlier.

It is necessary for the development of the thermomagnetic instability considered by us that the electron heat transfer in the plasma dominates over the radiative transfer. Such a situation may occur, for instance, in a laser plasma or in the plasma of the solar corona. We estimate the characteristic times for the development of the instability in those cases.

A laser plasma drop is strongly inhomogeneous with a characteristic scale $L \sim 10^{-2}$ cm. The time of its dispersion t_H is of the order of 10^{-9} s. In order that the field may be created by the mechanism considered in the present paper it is necessary that the condition $t_H \gg t_0$ be satisfied. It is clear from (17) and (20) that as to order of magnitude $t_0 \approx L^2 m_e/k_B T_0 \tau_0$. The temperature of the plasma in the drop $T_0 \approx 10^7$ K, and the density $n_0 \approx 10^{21}$ cm^{-3}. Under those conditions $\Lambda \approx 8$, $\tau_0 \approx 10^{-12}$ s. For the growth time of the instability we get $t_0 \approx 6 \times 10^{-11}$ s, i.e., the thermomagnetic instability can fully guarantee the generation of a strong magnetic field during the time the drop exists, provided condition (16) is satisfied. It is rather complicated to establish whether there exists a region in the laser plasma drop in which (16) is valid. To do this it is necessary to know the detailed coordinate dependence of the temperature and density. It follows from the calculations by the authors of Ref. 8 that near the front of the heat wave the temperature and density gradients are in the opposite direction. However, one sees easily from Fig. 1a of Ref. 8 that there exists a region near the front where $|d \ln\rho_0/dz| < |d \ln T_0/dz|$ and $d^2T_0/dz^2 > 0$. In that region condition (16) can fully be satisfied and the instability will develop.

We make similar estimates for a plasma with parameters which are typical for the solar corona. Here $n_0 \sim 10^{10}$ cm^{-3}, $T_0 \sim 10^6$ K so that $\Lambda \approx 18$ and $\tau_0 \approx 1.5 \times 10^{-3}$ s. Assuming that the scale of the inhomogeneities is of the order of 10^{10} cm we get for the time of the development of the instability about 5 days. If, however, the instability develops in a region with a characteristic scale of 10^9 cm, the growth time will be of the order of one hour. Inequality (14) is satisfied with a large margin. We see thus that the thermomagnetic instability may occur also in the solar corona leading to the formation of fine-scaled magnetic fields and motions.

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Parametric absorption of electromagnetic radiation in a magnetized plasma

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A new mechanism of saturation of parametric instabilities in a magnetized plasma is revealed. The saturation level and the spectrum of the Langmuir turbulence excited by the pump wave in a homogeneous magnetoactive plasma are obtained. The question of the efficiency of parametric absorption of microwave energy in installations with magnetic containment of plasma is investigated.

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1. Much attention is being paid presently to the problem of additional heating of the plasma in magnetic-containment installations. One of the most promising is the use of electromagnetic radiation in the centimeter and millimeter bands for this purpose. The electric field intensities needed to attain the necessary heating rate are such that the plasma in the electromagnetic-radiation field turns out to be parametrically unstable.[1-6] Thus, Alikaev *et al.*[2] observed experimentally ion heating under decay-instability conditions. Grek and Porkolab[3,4] observed, besides anomalous absorption, a rise in the level of the turbulent pulsations, which likewise points to the presence of a correlation between the anomalous absorption and the development of parametric instabilities. Theoretical estimates of the efficiency of the parametric absorption were made by a number of workers.[5,6] Rubenchik *et al.*[5] have shown that parametric effects can impede the penetration of the electromagnetic radiation into the central regions of the plasma. Ramazashvili and Starodub proposed a new method