Nonlinear periodic motions in a gas of noninteracting particles

I. S. Veselovskiĭ

Nuclear Physics Research Institute of the Moscow State University
(Submitted 1 March 1979)
Zh. Eksp. Teor. Fiz. 77, 1352–1359 (October 1979)

An initial-value problem with periodic variation of the velocity in space is solved. The evolution of the density and velocity before and after the "toppling" of the velocity profile is investigated in detail. The formation of discontinuities in the density and in the average velocity after the onset of the multistream flow is traced. The number of mutually penetrating streams increases with time, but the system retains the smallest initial period and the number of density peaks does not increase. For a sinusoidal initial condition there is one density maximum for the period preceding the toppling of the profiled, and after this first toppling there are two density peaks per period, with the exception of individual instants of time when these peaks merge. At long times, the system tends to a homogeneous state throughout, with the exception of some rapidly narrowing regions that preserve the density singularity.

PACS numbers: 05.60. + w, 05.30. – d

1. A cold gas of noninteracting particles is one of the simplest classical systems suitable for simulation of nonlinear motion in continuous media. A qualitative study of this model reveals many processes that are typical also of a more complicated nonlinear system, say a plasma. In such a system, higher harmonics are generated, the slope of the leading fronts increases, "toppling" of the wave profile takes place, and multistream flow with density peaks sets in. The number of mutually penetrating streams increases in the course of the evolution of the process.

A simple and physically illustrative description of some of the main properties of nonlinear motions in such a system can be found in Refs. 1 and 2. However, as will be shown below, a more detailed quantitative treatment helps clarify certain details that refine and modify the picture of the evolution of the nonlinear periodic motions. We consider in detail the case of greatest interest, that of the toppling of the profile. From the qualitative point of view it becomes clear that the shortest period of the motion remains unchanged also after the toppling. After the formation of the multistream flow, the density has two peaks per period. The number of peaks does not increase in the course of the evolution. These two peaks converge into one at definite points of space only at certain individual instants of time. Thus, the purpose of the present paper is to analyze quantitatively the evolution of nonlinear periodic velocity perturbations in a cold gas of noninteracting particles.

We consider the initial-value problem for the plane case. Assume that at the initial instant of time $t=0$ there is specified the distribution function

$$f_0(x, v_x, 0)=n_0(x)\delta(v_x-V).$$

The velocity of the cold gas at the initial instant is

$$V=V_0(1+a\sin kx).$$

The evolution is along the characteristics

$$x=x_0+v_xt,\quad v_x=\text{const},\quad x_0=\text{const}$$

of the equation

$$\frac{\partial f}{\partial t}+v_x\frac{\partial f}{\partial x}=0.$$

At $t\geqslant 0$ the solution is the distribution function

$$f(x, v_x, t)=n_0(x-v_xt)\delta\{v_x-V_0[1+a\sin(kx-v_xt)]\}. \tag{1}$$

The particle concentration is

$$n=\int f\,dv_x=\sum_i n_i,$$

where

$$n_i=n_0(x-v_xt)\,|1+V_0akt\cos(kx-u_it)|^{-1}. \tag{2}$$

The flux is

$$j=\sum_i j_i=\sum_i u_in_i,$$

and the average velocity $u=j/n$. Here $u_i(x,t)$ are the roots of the equation

$$F(u)=u-V_0[1+a\sin(kx-ut)]=0. \tag{3}$$

At $0\leqslant t\leqslant(kaV_0)^{-1}$ Eq. (3) has a single root. The function $u(x,t)$ is in this case single-valued. Putting $(u-V_0)/aV_0=y$, $k(x-V_0t)=\varphi$, and $kaV_0t=p$, we get from Eq. (3)

$$y=\sin(\varphi-py), \tag{4}$$

whence

$$\varphi=\text{Arcsin}\,y+py=r\pi+(-1)^r\arcsin y+py,\quad r=0,\pm1,\pm2\dots. \tag{5}$$

Each value of y corresponds to an infinite denumerable set of values of φ at all p. The inverse function $y(\varphi)$ has a finite number of branches at any finite value of p. At $0<p<1$ the function $y(\varphi)$ is single-valued. Its derivative is

$$d\varphi/dy=(-1)^r(1-y^2)^{-1/2}+p.$$

Obviously $|y|\leqslant 1$.

The extremal values $y=\pm1$ are reached at

$$\varphi=r\pi\pm(-1)^r\pi/2\pm p.$$

The derivative is then $d\varphi/dy=\infty$. The toppling of the profile takes place at points where $d\varphi/dy=0$. This is possible only at odd values of r, when $y=\pm(p^2-1)^{1/2}/p$,

 0038-5646/79/100681-04$02.40

i. e., at the points

$$\varphi=r\pi\pm(-1)^r\arcsin\frac{(p^2-1)^{1/2}}{p}\pm(p^2-1)^{1/2},$$

where $r=2m+1$, m is any integer, and $p>0$. It is seen that at $0<p<1$ no toppling is possible. The first toppling occurs at $p=1+0$ at the points $\varphi=(2m+1)\pi$.

3. We introduce the variables $\tau=p-1$, $\psi=\varphi-(2m+1)\pi$ and consider the function $y(\varphi,p)$ near the first toppling, $\tau\ll1$, $y\ll1$. Taking into account the periodicity

$$y(\varphi)=y(\varphi+2\pi),$$

we put $\psi\ll1$. It follows then from (5) that

$$\psi\approx\tau y-{}^1/_6y^3+\ldots.$$

This cubic equation yields $y(\psi,\tau)$. The function $y(\psi)$ is odd and its plots for different values of τ are shown qualitatively in Fig. 1.

a) At $\tau<0$ the function $y(\psi)$ is single-valued and given by

$$y_1=2^{3/2}|\tau|^{1/2}\,\mathrm{sgn}\,\psi\,\mathrm{sh}\ln[\mu+(\mu^2+1)^{1/2}]^{1/3},$$

$$\mu=\frac{3\sqrt{2}}{4}|\psi||\tau|^{-3/2}. \tag{6}$$

b) At $\tau=0$ we have

$$y=(-6\psi)^{1/3}. \tag{7}$$

c) At $\tau>0$ the function $y(\psi)$ is single valued in the region $\mu<1$, where it is given by

$$y_1=2^{3/2}|\tau|^{1/2}\,\mathrm{sgn}\,\psi\,\mathrm{ch}\ln[\mu+(\mu^2-1)^{1/2}]^{1/3}. \tag{8}$$

Expression (8) tends to $\tau\to+0$ to (7) in the region $\mu>1$, and as $\mu\to1+0$ we have

$$y_1\to2^{3/2}|\tau|^{1/2}\,\mathrm{sgn}\,\psi.$$

In the region $\mu\leqslant1$ at $\tau>0$ the function $y(\psi)$ is triple-valued:

$$\begin{aligned}y_1&=-2^{3/2}|\tau|^{1/2}\,\mathrm{sgn}\,\psi\cos({}^1/_3\arccos\mu),\\ y_2&=2^{3/2}|\tau|^{1/2}\,\mathrm{sgn}\,\psi\cos{}^1/_3(\pi-\arccos\mu),\\ y_3&=2^{3/2}|\tau|^{1/2}\,\mathrm{sgn}\,\psi\cos{}^1/_3(\pi+\arccos\mu).\end{aligned} \tag{9}$$

In this region we have as $\mu\to0$

$$y_1=-(6\tau)^{1/2}\,\mathrm{sgn}\,\psi,\quad y_2=(6\tau)^{1/2}\,\mathrm{sgn}\,\psi,\quad y_3=0.$$

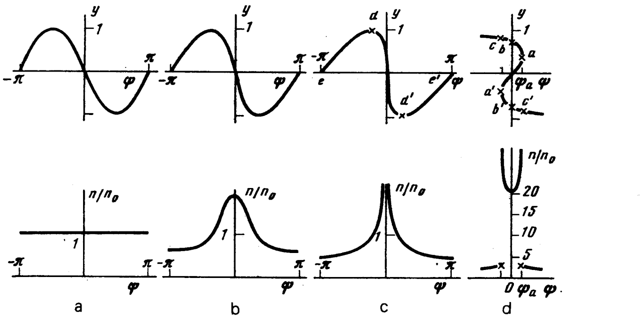

FIG. 1. Evolution of the velocity $y(\psi,\tau)$ and of the density $n(\psi,\tau)$ in the first toppling of the profile: a) $t=0$, b) $\tau=-0.5$, c) $\tau=0$, d) $\tau=0.1$.

At $\mu=1-0$ it follows from (9) that

$$y_1=-2^{3/2}\tau^{1/2}\,\mathrm{sgn}\,\psi,$$

the two other roots merge

$$y_2=y_3=(2\tau)^{1/2}\,\mathrm{sgn}\,\psi.$$

The branch y_1 at the point $\mu=1$ (point c in Fig. 1) is continuous and is determined by (9) at $\mu<1$ and by (8) at μ.

1. The branches y_2 and y_3 exist only at $\mu\leqslant1$. The sequence of the branches is shown in Fig. 1: $edcbb'c'd'e'$–the first branch (y_1), $a'b'ab$–second branch (y_2), $a'a$–third branch (y_3). The positions of the points a–c are determined by the approximate formulas (8) and (9). We have

$$\psi_a={}^1/_3(2\tau)^{3/2},\quad y_a=(2\tau)^{1/2},\quad \psi_b=0,\ y_b=(6\tau)^{1/2},$$
$$\psi_c=-{}^1/_3(2\tau)^{3/2},\quad y_c=2(2\tau)^{1/2}.$$

The coordinates of the points d and e are calculated from the exact formulas (4) and (5):

$$\psi_d=-\pi/2+1+\tau,\ y_d=1;\ \psi_e=-\pi,\ y_e=0.$$

The points a'–e' are mirror-symmetrical relative to a–e. The density is calculated from (2). Let $n_0=1$ in this formula. Then the concentration on each branch is

$$n_i=|1+p\cos(\varphi-py_i)|^{-1}=|1+p(-1)^r(1-y_i^2)^{1/2}|^{-1}. \tag{10}$$

The concentration on the branch ed decreases monotonically with time because of the dilatation of the volume. At the point e we have $n\sim|1+p|^{-1}$ and at the point d we have $n=1$. At the points a–c ($\psi\ll1$, $y\ll1$, $\tau\ll1$) we can use the expansion

$$n_i\approx|-\tau+{}^1/_2y_i^2+{}^1/_8y_i^4|^{-1}.$$

At the point 0 we have $y_2=0$ and on this branch $n_2\approx|\tau|^{-1}$. At the points c and b we have $y=2^{3/2}|\tau|^{1/2}$ and $y=(6\tau)^{1/2}$, and correspondingly $n\approx|3\tau|^{-1}$, $n\approx|2\tau|^{-1}$. At the toppling point a we have $\tau\approx\frac12y^2$ and the succeeding terms of the expansion must be retained when the concentration is calculated.

We turn therefore to the initial exact expression (5) and consider the vicinity of the toppling points in greater detail. At these points, as already noted,

$$y_*=\pm\frac{(p^2-1)^{1/2}}{p},\quad \varphi_*=r\pi\pm(-1)^r\arcsin\frac{(p^2-1)^{1/2}}{n}\pm(p^2-1)^{1/2}, \tag{11}$$

$r=2m+1$ is odd and $p\geqslant1$. The upper sign corresponds to the point a, and the lower to the point a'. At the toppling points $1-y_*^2=1/p^2$ we have

$$\begin{aligned}\left.\frac{d\varphi}{dy}\right|_{y_*}&=p+(-1)^r(1-y_*^2)^{-1/2}=0,\\ \left.\frac{d^2\varphi}{dy^2}\right|_{y_*}&=(-1)^ry_*(1-y_*^2)^{-3/2}=\pm(-1)^rp^2(p^2-1)^{1/2},\\ \left.\frac{d^3\varphi}{dy^3}\right|_{y_*}&=(-1)^r(1+2y_*^2)(1-y_*^2)^{-5/2}=(-1)^rp^3(3p^2-2),\\ \left.\frac{d^4\varphi}{dy^4}\right|_{y_*}&=(-1)^r3y_*(3-2y_*^2)(1-y_*^2)^{-7/2}=3(-1)^rp^4(p^2+2)(p^2-1)^{1/2}.\end{aligned} \tag{12}$$

We use the obtained values of the derivatives in the expansions

$$\varphi(y)=\varphi_* + \frac{1}{2}\frac{d^2\varphi}{dy^2}\bigg|_{y_*}(y-y_*)^2+\frac{1}{6}\frac{d^3\varphi}{dy^3}\bigg|_{y_*}(y-y_*)^3+\dots,$$

$$n^{-1}=|1-p(1-y^2)^{1/2}|=n_*^{-1}+\frac{dn^{-1}}{dy}\bigg|_{y_*}(y-y_*)+\frac{1}{2}\frac{d^2n^{-1}}{dy^2}\bigg|_{y_*}(y-y_*)^2+\dots .$$

Putting $y-y_*=\varepsilon$, $\varphi-\varphi_*=\chi$, we obtain near the point a

$$\chi=-{}^1/_2p^2(p^2-1)^{1/2}\varepsilon^2-{}^1/_6p^3(3p^2-2)\varepsilon^3+\dots, \tag{13}$$

$$n^{-1}=p^3(p^2-1)^{1/2}\varepsilon+{}^1/_2p^4(3p^2-2)\varepsilon^2+\dots . \tag{14}$$

The discarded terms are always small compared with the retained ones at $\varepsilon\ll 1$. In the derivation of (14) it was recognized that

$$n_*^{-1}=|1-p(1-y_*^2)^{1/2}|=0.$$

We note that χ is negative here.

In the case $\tau=p-1\ll 1$ we must distinguish between two regions: $|\chi|\ll\tau^{3/2}$ and $|\chi|\gg\tau^{3/2}$. In fact, if $\varepsilon\ll(2\tau)^{1/2}$, then the principal terms in (12) and (13) are the first ones

$$\varepsilon_{2,3}=\pm(-2\chi)^{1/2}(2\tau^{-1/4}),\quad n_{2,3}\approx(|\varepsilon|(2\tau)^{1/2})^{-1}=(-2\chi)^{-1/2}(2\tau)^{-1/4}.$$

These expressions are valid if $|\chi|\ll 2^{1/2}\tau^{3/2}$.

On the other hand if $|\varepsilon|\gg(2\tau)^{1/2}$, then the principal terms will be the second ones. Then $\varepsilon_1\approx(-6\chi)^{1/3}$, $n_1\approx 2\varepsilon_1^{-2}=2(-6\chi)^{-2/3}$. These expressions are valid at $|\chi|\gg 2^{1/2}\tau^{3/2}$, and in particular at $\tau=0$. We indicate here the subscript that numbers the branch.

Thus, at the instant of the first toppling at $\tau=0$, $\varphi=(2m+1)\pi$ the density has only one singularity of the type $n\sim\chi^{-2/3}$ over the entire period at the toppling point ($\chi=0$). In the subsequent instants of time $0<\tau<1$ there are two toppling points (a and a') in the period, as well as a density singularity of the type

$$n\sim|\chi|^{-1/2}\tau^{-1/4}$$

in the immediate vicinity on the left of the point a (on the right of a') at $|\chi|\ll 2^{1/2}\tau^{3/2}$. At the points e and e' the density equals $1/(2+\tau)$, while at the points d and d' the density is not distorted, $n=1$. At the toppling point $\psi_*=\psi_a$ on Fig. 1 the density has a discontinuity: On the right there is one branch $n\approx\frac{1}{3}\tau$. To the left of this point there are three and the density becomes infinite like

$$n\approx 1/3\tau+2(2\chi)^{-1/2}(2\tau)^{-1/4}$$

in the region

$$|\chi|\ll 2^{1/2}\tau^{3/2}.$$

The average velocity, determined by the ratio

$$\bar{u}=\sum_i u_in_i\Big/\sum_i n_i,$$

becomes discontinuous at the toppling points. It assumes successively the values V_0 at the point e, $(1\pm a)V_0$ at the point ψ_d, $V_0(1+2^{1/2}a\tau^{1/2})$ at the point ψ_c-0, $V_0(1-2^{1/2}a\tau^{1/2})$ at the point ψ_c+0, V_0 at the point 0, $V_0(1+2^{1/2}a\tau^{1/2})$ at the point ψ_a-0, $V_0(1-2^{1/2}a\tau^{1/2})$ at the point ψ_a+0, etc. (see Fig. 2). The jump of the average velocity at the toppling points is $\pm 3\times 2^{1/2}aV_0\tau$, $\tau\ll 1$. It increases with time and tends to the value $\pm aV_0$ as $t\to\infty$.

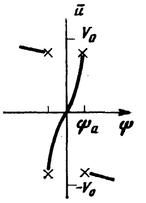

FIG. 2. Average velocity $u(\psi)$ after toppling ($a=0.1$, $\tau=0.1$, $V_0=1$, $\psi_a=3.10^{-2}$).

For each branch we can write everywhere, with the exception of the toppling points, the expansions

$$y(\varphi)=y(\varphi_a)+\frac{dy}{d\varphi}\bigg|_{\varphi_a}(\varphi-\varphi_a)+\frac{1}{2}\frac{d^2y}{d\varphi^2}(\varphi-\varphi_a)^2+\dots,$$
$$n(\varphi)=n(\varphi_a)+\frac{dn}{d\varphi}\bigg|_{\varphi_a}(\varphi-\varphi_a)+\frac{1}{2}\frac{d^2n}{d\varphi^2}(\varphi-\varphi_a)^2+\dots . \tag{15}$$

The functions $y(\varphi)$ and $n(\varphi)$ are implicitly specified here by Eqs. (4) and (10), and the derivatives are

$$\frac{dy}{d\varphi}=G^{-1}\cos(\varphi-py)=(-1)^r(1-y^2)^{1/2}G^{-1},$$
$$\frac{d^2y}{d\varphi^2}=G^3y,\quad \frac{dn}{d\varphi}=pyG^{-3}\,\mathrm{sgn}\,G, \tag{16}$$
$$\frac{d^2n}{d\varphi^2}=[p\cos(\varphi-py)G^{-4}+3p^2y^2G^{-5}]\,\mathrm{sgn}\,G.$$

To simplify the formulas, we have introduced the notation

$$G=1+p\cos(\varphi-py)=1+(-1)^rp(1-y^2)^{1/2},\quad n=G^{-1}$$

and have left out the subscript that numbers the branches.

We introduce the symbol $\chi=\varphi-\varphi_\alpha$ and consider the characteristic points anew. At the point e we have $\varphi_e=r\pi$, with r odd,

$$y_e=\sin(\varphi_e-py_e)=0,\quad \cos(\varphi_e-py_e)=1.$$

We obtain

$$y_e\approx(1+p)^{-1}\chi+O(\chi^3),\quad n\approx(1+p)^{-1}+{}^1/_2p(1+p)^{-4}\chi^2+\dots .$$

At the point d we have $y_d=1$, $\varphi_d=r\pi+(-1)^r\pi/2+p$. This point moves at constant (highest) velocity. About this point we have the expansion

$$y=1-{}^1/_2\chi^2+\dots,\quad n=1+p\chi+{}^3/_2p^2\chi^2+\dots .$$

At the points c and b we obtain a transcendental equation for $y(p)$. These points were considered at $\tau\ll 1$. At the toppling point a the expansion (15) is not suitable ($G=0$). At the point 0 the phase is $\varphi=r\pi$, r is odd, $y=0$, and $G=1-p$. The expansions at these points are given by

$$y=(p-1)\chi+O(\chi^3),$$
$$n=|p-1|^{-1}+{}^1/_2p(p-1)^4\,\mathrm{sgn}\,(p-1)\chi^2+O(\chi^3).$$

5. The behavior near the toppling points is described by Eqs. (13) and (14), from which we get as $t\to\infty$ the asymptotic expressions

$$\varepsilon=(2\chi)^{1/2}p^{-3/2},\ n=(2\chi)^{-1/2}p^{-5/2},$$

if $p^2\varepsilon\ll 1$, i. e., in the region $(\chi p)^{1/2}\ll 1$. On the other hand, if $p^2\varepsilon\gg 1$, then

$$\varepsilon=(2\chi)^{1/3}p^{-5/3},\ n={}^2/_3(2\chi)^{-2/3}p^{-8/3}$$

in the region $(2\chi p)^{1/3}\gg 1$. Thus, the density singularity continues to exist at all time when $p>1$, but narrows down rapidly.

As $t\to\infty$, $\varphi\approx -kV_0t\to\infty$ we get $y_r\approx(\varphi-r\pi)/p$. Since $-1<y<1$, we get for integer r the restriction $r_{\min}<r<r_{\max}$, where $r_{\min}=p(1-a)/\pi a$, $r_{\max}=p(1+a)/\pi a$. The number of mutually penetrating streams $N=r_{\max}-r_{\min}=2p/\pi$ increases linearly with time.

The concentration of each stream is

$$n_r=|1+p(-1)^r(1-y_r^2)^{1/2}|^{-1}\approx p^{-1}(1-y_r^2)^{-1/2}.$$

The density is calculated by summing over r, which can be replaced by integration over the equidistant spectrum y_r, with $\Delta y_r=y_{r+1}-y_r=\pi\Delta r/p$:

$$n=\sum_{r_{\min}}^{r_{\max}} n_r=\int_{r_{\min}}^{r_{\max}} n_r\,dr=\int_{-1}^{1} n(y)\frac{p}{\pi}\,dy=\frac{1}{\pi}\int_{-1}^{1}(1-y^2)^{-1/2}\,dy=1. \quad (17)$$

At $t\to\infty$ it is natural to regard the quantity

$$f(y)=\begin{cases}\pi^{-1}(1-y^2)^{-1/2}, & |y|<1\\ 0, & |y|>1\end{cases} \quad (18)$$

as the velocity distribution function with normalization (17). The average velocity is

$$\langle y\rangle=\int yf(y)\,dy=0,$$

The mean squared velocity of the "thermal" motion is

$$\langle y^2\rangle=\int y^2f(y)\,dy=\frac{1}{2}.$$

Thus, with the exception o narrow regions containing density singularities, the system evolves into a homogenous state with a distribution function (18), characterized by a density $n=1$, an average stream velocity $u=V_0$, and an effective temperature

$$T_{\text{eff}}=\frac{m}{2}\langle(u-V_0)^2\rangle=\frac{m}{4}(aV_0)^2.$$

Allowance for the thermal motion in the initial distribution (1) leads to a spreading of the inhomogeneities and to a finite amplitude of the density peaks. The equilibrium velocity distribution is established when account is taken of the collisions. Motions of the considered type are possible in a plasma consisting of cold ions and thermal electrons (see, e. g., Refs. 3 and 4), in which case the potential is

$$e\varphi=T_e\ln(n/n_0).$$

The author thanks V. I. Karpman and L. P. Pitaevskiĭ for helpful discussions.

[1]B. B. Kadomtsev, Kollektivnye yavleniya v plazme (Collective Phenomena in Plasma), Nauka, 1976, Ch. 3.
[2]B. B. Kadomtsev and V. I. Karpman, Usp. Fiz. Nauk 103, 193 (1971) [Sov. Phys. Usp. 14, 40 (1971)].
[3]I. A. Akhiezer and A. E. Borovik, Zh. Eksp. Teor. Fiz. 51, 1227 (1966) [Sov. Phys. JETP 24, 823 (1966)].
[4]A. V. Gurevich and L. P. Pitaevskiĭ, Zh. Eksp. Teor. Fiz. 56, 1778 (1969) [Sov. Phys. JETP 29, 954 (1969)].

Translated by J. G. Adashko

Self-action of electromagnetic waves in a plasma subject to modulational instability

A. G. Ltvak, V. A. Mironov, and A. M. Feĭgin

Institute of Applied Physics, Academy of Sciences of the USSR
(Submitted 5 April 1979)
Zh. Eksp. Teor. Fiz. 77, 1360–1366 (October 1979)

We study the nonlinear penetration of an electromagnetic wave into a plasma upon development of modulational instability. We use an averaging method which employs the substantial scale difference between the plasma and the electromagnetic waves to find the change in the refractive index of the incident wave when the plasma is layered on a fine scale. We consider in detail the stationary self-action of an s-wave in an initially uniform layer of an overdense plasma with sharp boundaries. We find the field distribution and the dependence of the transmission coefficient on the amplitude of the field of the incident wave.

PACS numbers: 52.35.Hr, 52.35.Py, 52.40.Db

INTRODUCTION

It is well known that the processes of the modulational instability of Langmuir waves, which lead to the formation of Langmuir solitons and of the corresponding inhomogeneities in the plasma density (cavitons), play an essential role when strong electromagnetic waves interact with a dense collisionless plasma. One usually studies the modulational instability in order to determine the magnitude of the effective collision frequency ν_{eff} which characterizes the additional electromagnetic-energy loss connected with the excitation of fine-scale electric fields. In that case one does not take into account the fact that appearance of cavitons (plasma stratification) also leads to a change in the real part of the refractive index of the electromagnetic wave and, thereby, to a change in the distribution of the large-scale electric fields in the plasma. It is clear that taking such "reactive" non-linear effects into account is important for the determination of the characteris-

