Quantum restrictions on the measurement of the parameters of motion of a macroscopic oscillator

M. B. Menskiĭ

All-Union Research Institute of Physicotechnical and Radio Engineering Measurements
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The quantum restrictions imposed by the uncertainty principle of quantum mechanics on measurements of the parameters of motion of a macroscopic oscillator are estimated by means of the Feynman path integral. Two regimes of measurement are considered; continuous tracking of the oscillator coordinate, and measurement of the individual spectral components of motion of the oscillator. It is shown that a quantum threshold exists in each regime. Increasing the accuracy of the instrument above the quantum threshold does not improve the accuracy of the estimate of the force acting on the oscillator (due to the presence of quantum "measurement noise") in the case of continuous tracking and even impairs it in the case of spectral measurements. It is shown that an optimal measurement (from the viewpoint of quantum errors) is one performed with such accuracy that the measurement is intermediate between the purely classical and purely quantum measurements. The perturbation of the quantum system state in this case is insignificant, so that the measurement is a nonperturbing or almost nonperturbing one. It is shown that to observe a force with a known frequency band the spectral measurements have an advantage over those involving continuous tracking of the coordinate. An exception is the case when the force acts in a very narrow frequency band that includes the natural frequency of the oscillator.

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1. INTRODUCTION

The old problem of the description of measurements in quantum mechanics has recently acquired a more specific meaning and has been considered in a more practical light, since precise measurements, even in mechanical macroscopic systems have come close to those limits set on them by the quantum uncertainty principle. A number of papers have appeared in this connection, in which the quantum restrictions on the accuracy of the measurements, and also the possibility of measurements that do not perturb the state of the system subject to the measurements are discussed.[1-10] As a rule, a model is constructed for the measurement system and with the help of a quantum calculation of this model conclusions are drawn as to the magnitude of the quantum noise and by the same token on the quantum restrictions on the accuracy of the measurements.

To this problem can be applied[11] the method of Feynman path integrals.[12] For this one should express the transition amplitude of the quantum system from one point to another in the form of an integral over the classical path connecting these points and restrict the integral to only those paths which correspond to the definite result of measurement. The amplitude thus obtained characterizes the probability of obtaining this result in the measurement process. Analysis of the resultant distribution of probabilities allows us to draw conclusions on the quantum restrictions on the measurement and to work out a method for estimating of the forces acting on the system from the results of measurements carried out on the parameters of motion of the system. The advantage of such an approach is that it does not require the fixing of a concrete model of the measurement system. The only thing that is required in the given case is that we settle on the class of measurements, i.e., that we indicate which observables are measured, and which, on the other hand, cannot be assessed from the measurement results. The restrictions on the measurement accuracy obtained in this approach are extremal, i.e., they cannot be improved by choice of a concrete measurement setup within the limits of the given class.

The most direct application of this methodology is[11] in the analysis of continuous measurements of a coordinate x. In this case, the integral is carried out along paths in configuration space and integration over all paths is replaced by integration over the paths lying in a given corridor. Specification of the corridor is none other than the approximate specification of the function $x(t)$. Just this result is produced by any measurement apparatus that continuously tracks some coordinate. The width of the corridor corresponds here to the accuracy with which the measurement apparatus determined the instantaneous value of the coordinate.

In the simplest case when the investigated system is a harmonic oscillator, the integral over paths lying in a given corridor can be calculated approximately. The result allows us to estimate the quantum restrictions on the measurement in such a regime.[11] It turns out that the results of the tracking of the oscillator coordinate give the usual classical estimate for the force acting on the oscillator, so long as the error in this tracking Δa remains greater than the quantum threshold Δa_q. If Δa becomes less than Δa_q then the classical picture is destroyed. Decrease of the error Δa below the limit Δa_q does not increase the accuracy of the estimate of the force acting on the oscillator. This estimate remains the same as if the coordinate tracking error were equal to Δa_q. Here $\Delta a_q = (\hbar/m\omega)^{3/2}$ if the measurements interval is much greater than the period of the oscillator, $\omega t \gg 1$, and $\Delta a_q = (\hbar\tau/m)^{1/2}$ if the measurement interval is much less than the period, $\omega\tau \ll 1$.

There is interest in the consideration of another class

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of measurement systems, in which, instead of tracking of the motion of the oscillator, a spectral analysis of this motion is carried out. The most widespread are systems in which the signal, proportional to the coordinate, passes through a filter which isolates a narrow band of frequencies, and in the limit a single spectral component (harmonic). In this case, the information on how the coordinate changes with time, i.e., on the function $x(t)$, is almost all lost and there remains only information on a small number of spectral components of this function.

In the present work, the estimate of the quantum restrictions in the spectral regime of measurements of the oscillator is found with the help of the path integral. For this purpose, we use the expression for the path integral in terms of the integrals over the amplitudes of the individual harmonics that enter into the expansion of the function $x(t)$. For the analysis of the measurement procedure, the limits of integration over each harmonic are so chosen that they express the result of the measurement. In particular, the length of the interval of integration corresponds to the accuracy with which the harmonic is determined in the measurements. The resultant infinite-multiplicity integral represents the probability amplitude of obtaining a given measurement result. By analyzing the probability distribution, we can estimate the quantum restrictions on the measurement in the spectral regime.

Calculations show that the optimal error of measurement of the n-th harmonic, which has the frequency $\Omega_n = n\pi/\tau$, is equal to

$$\Delta a_n^{opt} = (\hbar/m\tau|\omega^2-\Omega_n^2|)^{1/2}.$$

The measurement result gives an estimate of the n-th harmonic f_n in the expansion of the force $f(t)$ which acts on the oscillator. The accuracy with which we can determine f_n in the case of optimal measurement is equal to

$$\Delta f_n^{opt} = (\hbar m|\omega^2-\Omega_n^2|/\tau)^{1/2},$$

and in the non-optimal regime, the accuracy is even worse.

Of interest is the degree to which the estimate of the quantum restrictions on the measurement depends on the measurement procedure. For example, it is not clear beforehand whether the accuracy of measurement of the n-th harmonic worsens if the other harmonics are measured simultaneously. Calculations show that it is not made worse. The measurements of the various harmonics are completely independent of one another. This recalls the frequently used procedure of expansion of a complicated motion of a system into normal modes and the consideration of them as independent quantum oscillators. However, in our case we are dealing with a different expansion. The first difference is that the expansion into harmonic depends on the duration of the measurement and not on the properties of the measured system (dimensions, elastic properties, etc.). The second is connected with the fact that we are analyzing harmonics under specific conditions of measurement and not under the natural conditions of motion of the system under the action of external forces. It turns out that under these conditions the different harmonics make independent contributions to the error of measurement.

This allows us to give for the quantum restrictions on the measurement a simple interpretation whose possibility is not obvious *a priori*. This interpretation is that a quantum noise arises in the measurements. The noises at different harmonics are independent of one another. The noise at the n-th harmonic depends on the error with which this harmonic is measured, and is equal to

$$\Delta a_n^q = (\Delta a_n^{opt})^2/\Delta a_n.$$

It is clear that the sum of the error of measurement Δa_n and of the quantum noise Δa_n^q becomes minimal if $\Delta a_n = a_n^{opt}$. Then the quantum noises become equal to the measurement error and their sum are of the same order of magnitude, and the same order of magnitude determines the error of the estimate of the force acting on the oscillator.

In the case in which all the harmonics are measured and the error in each case is optimal, the total quantum noise turns out to be exactly the same as in the measurement in the regime of continuous coordinate tracking:

$$\langle\Delta x^2\rangle = \sum_{n=1} (\Delta a_n^{opt})^2 \approx \Delta a_q^2.$$

This gives a still firmer basis for interpreting the quantum restrictions on the measurement as independently existing "measurement quantum noises," although, naturally, we must use this concept with caution. If the measurements are carried out in a narrow band of frequencies $\Delta\Omega$, then the error is due only to those quantum noises whose frequencies lie in this band. In a large interval of measurement, $\omega\tau \gg 1$, and in the optimal measurement regime, the total quantum noise is equal to

$$\langle\Delta x^2\rangle = \sum_{n=\Omega/\Omega_1}^{(\Omega+\Delta\Omega)/\Omega_1} (\Delta a_n^{opt})^2 \approx \frac{\hbar\Delta\Omega}{m|\omega^2-\Omega^2|},$$

where it is assumed that the frequency band $\Delta\Omega$ is larger than or of the order of τ^{-1}.

The last two formulas allow us to draw conclusions about the optimal regime of measurement in the case in which we must detect or measure a small force acting on the oscillator from the response of the oscillator. If there is no *a priori* information on the acting force, then the optimal regime is the continuous tracking of the coordinate or the measurement of all the spectral components with optimal error Δa_n^{opt}. If it is known beforehand that the force which must be measured has a spectrum in a limited band of frequencies, then it turns out to be convenient to carry out the optimal measurement of the spectral components in this band, filtering out all the unnecessary parts of the spectrum.

The structure of the paper is as follows. The problem of the determination of the quantum restrictions on the measurement is formulated in Sec. 2 within the framework of the method of path integrals and results obtained in Ref. 11 are given for measurements in the regime of continuous coordinate tracking. In Sec. 3, the spectral measurements of an oscillator, namely measurement of the amplitude of a single harmonic, are analyzed within the framework of the path-integral method. In Sec. 4, the general case of spectral measurements is considered, in which all the harmonics are measured in a definite frequency band, and different harmonics measured, generally speaking with different errors. A comparison is made of the quantum restrictions that arise in measurements in the spectral regime and in the regime of continuous coordinate tracking. In Sec. 5, we discuss the results and several prospects of the investigation are noted.

In the calculation process, we do not take into account thermal noise, i.e., we consider such conditions of measurement in which the thermal noise is negligibly small in comparison with the quantum indeterminacies. The quantum errors in all the calculations are estimated only in order of magnitude, a fact not specifically stipulated.

2. CONTINUOUS MEASUREMENT OF THE COORDINATE

According to the Feynman theory,[12] the probability amplitude of transition of a quantum system from one point of configuration space to another during a definite time interval τ is equal to the path integral

$$K=\int d[x]\exp\left\{\frac{i}{\hbar}S[x]\right\}. \tag{1}$$

Here the integral is carried out over all paths $[x]=\{x(t)\}$ that lead from the initial point to the final one, and the classical action, calculated for the path $[x]$, is denoted by $S[x]$. The definition of the path integral can be found in the book of Feynman and Hibbs.[12] Briefly, it reduces to the following. The time interval corresponding to the transition is broken up into equal intervals, each path is approximated by a broken line, which is a straight line segment on each of the intervals of the breakup, and the path integral is replaced by a multiple integral over the nodes of these broken lines. Then the limit of the resultant expression is taken as the lengths of the intervals approach zero. This limit is, by definition, the path integral. We can formulate all the premises of quantum mechanics in terms of such integrals and this allows us to make clear to a maximum extent the connection between quantum mechanics and classical mechanics. We apply this apparatus to the solution of the problem of the quantum restrictions on the measurement.

For definiteness, we shall speak of a one-dimensional harmonic oscillator. Then the integral (1) takes the form

$$K=\int d[x]\exp\left\{\frac{i}{\hbar}\int_0^\tau dt\left[\frac{m}{2}\dot{x}^2-\frac{m\omega^2}{2}x^2+fx\right]\right\}. \tag{2}$$

Here the integral is taken over paths with fixed ends $x(0)$ and $x(\tau)$. Since we shall not be interested in "edge effects," i.e., we shall not take into account the state of the oscillator immediately before the measurement and immediately after the measurement, we set $x(0)=x(\tau)=0$. Then the path integral (2) with such end points gives the probability amplitude that the oscillator, having zero coordinate at the initial instant of time, will have zero coordinate also at the time τ. Here it is assumed that the force $f(t)$ acts on the oscillator in the interval $[0,\tau]$.

We now assume that measurement is made of certain parameters of motion of the oscillator during the interval $[0,\tau]$. The resultant information can be formulated in terms of paths. For example, if the measurement consists of continuous tracking of the coordinate of the oscillator, then, as a result of the measurement we shall know precisely the path along which it travels from the point $x=0$ at time 0 to the point $x=0$ at time τ. If the measurement of the coordinate is carried out with finite accuracy, then we shall know exactly that the path of the oscillator lies in a definite corridor, but we shall not know the exact form of the peak within the limits of the corridor. The probability amplitude that the measurement give a result described by such a corridor can be expressed by the integral (2) taken not over all paths but only over those which lie in the given corridor. The calculation of such an integral gives the probability distribution over all possible results of measurement.

Such a calculation was made in Ref. 11. Here we formulate only its result. First of all, we must define more precisely the concept of the corridor, which is the result of the measurement. We shall say that the path $x(t)$ lies in the corridor $\{x_0(t),\Delta a\}$ if

$$\frac{1}{\tau(\Delta a)^2}\int_0^\tau dt[x(t)-x_0(t)]^2\leqslant 1. \tag{3}$$

It would be more accurate in this case to say that the mean square departure of the curve $x(t)$ from the path $x_0(t)$ does not exceed the value Δa. The integral (2) taken over paths satisfying the condition (3) gives the probability amplitude that the continuous tracking of the coordinate of the oscillator gives the result $\{x_0(t),\Delta a\}$. It is clear that the probability distribution obtained in such fashion depends on what force $f(t)$ acts on the oscillator.

We shall characterize the force f by the trajectory along which the classical oscillator would have moved under the action of such a force, i.e., the function $\xi(t)$, which satisfies the condition

$$m\ddot{\xi}+m\omega^2\xi=f,\quad \xi(0)=\xi(\tau)=0. \tag{4}$$

The probability distribution does not depend on whether the measurement error Δa exceeds the quantum threshold

$$\Delta a_q=\begin{cases}(\hbar/m\omega)^{1/2} & \text{for } \omega\tau\gg 1\\ (\hbar\tau/m)^{1/2} & \text{for } \omega\tau\ll 1\end{cases}. \tag{5}$$

If $\Delta a\gg\Delta a_q$, then the situation turns out to be completely classical. In this case, the probability of obtaining the result $\{x_0(t),\ \Delta a\}$ turns out to be approximately the same for any path $x_0(t)$ lying in the corridor $\{\xi(t),\Delta a\}$ and vanishes if $x_0(t)$ lies outside this corridor. It can be said that the path obtained as a result of the measurement is identical, to within experimental error, with that predicted by classical theory (4). If the measurement error becomes less than the quantum threshold, $\Delta a < \Delta a_q$, then the accuracy of prediction ceases to increase. In this case, the probability of the result of the measurement $\{x_0(t), \Delta a\}$ turns out to be approximately the same for any path $x_0(t)$ lying in the limits of the corridor $(\xi, \Delta a_q)$, and is equal to zero if $x_0(t)$ lies outside this corridor.

The result obtained in this fashion has a very illustrative interpretation. The motion of the oscillator contains quantum noise, which is equal to Δa_q in amplitude, and can be of arbitrary form. If the measurement error is much greater than the amplitude of this noise, $\Delta a \gg \Delta a_q$, then the quantum noise, naturally, does not affect the results of the measurements. If $\Delta a \lesssim \Delta a_q$, then the quantum noise itself begins to play the decisive role and further increase in the accuracy of measurement becomes meaningless, since in no way does it decrease the scatter of the measurement results in the case of a given external force. It should be noted, however, that the formulated results were obtained for a given class of measurements, that is, for measurements whose result is the path corridor (3). Therefore, the interpretation of the results in terms of "quantum noise" is valid beforehand only for this class of measurements and its application to another class of measurements can lead to errors.

The probability of the various measurement results allow us to solve the problem of the estimate of the external force from the given measurement result. Let the measurement give the result $\{x_0(t), \Delta a\}$. Then at $\Delta a \gg \Delta a_q$, we should draw the conclusion that the function $\xi(t)$ (which characterizes the force) lies in the corridor $\{x_0(t), \Delta a\}$ and that at $\Delta a \lesssim \Delta a_q$, the function $\xi(t)$ lies in the corridor $\{x_0(t), \Delta a_q\}$. It is obvious that the estimate of the external force is also naturally formulated in terms of the quantum noise of amplitude Δa_q.

3. MEASUREMENT OF A SINGLE HARMONIC

We now consider a measurement of a different class, as a result of which information is obtained only about a single spectral component of the function $x(t)$ and consequently, only about a single spectral component of the force $f(t)$. In order to find the quantum restrictions for the measurement system of this class, we can also use the path integral method. However, in this case, we must use another (equivalent) definition of the path integral (2). Instead of calculating this integral as the limit of a multiple integral with integration over the nodes of broken lines, we can calculate the same integral by expanding the function $x(t)$ in harmonics and integrating over the amplitudes of all the harmonics.[12]

Since we are interested only in the interval $[0, \tau]$ on the time axis, we expand the functions $x(t)$ and $f(t)$ in this interval in Fourier series:[1)]

$$x(t)=\sum_{n=1}^{\infty} a_n \sin \Omega_n t, \quad f(t)=\sum_{n=1}^{\infty} f_n \sin \Omega_n t, \tag{6}$$

where $\Omega_n = n\pi/\tau$. Then the action for the oscillator can be expressed in terms of the spectral components a_n and f_n in the following fashion:

$$S=\frac{1}{4}\tau m \sum_{n=1}^{\infty}\left[a_n^2(\Omega_n^2-\omega^2)+\frac{2}{m}a_n f_n\right].$$

It was shown in Ref. 12 that an integral analogous to (2), but at $f=0$, taken over paths with zero boundary conditions, can be expressed as an infinitely multiple integral over the variables a_n. By analogy with this, we write, in the case of a non-zero force,

$$K=J\int da_1 \int da_2 \ldots \exp\left\{i\frac{m\tau}{4\hbar}\sum_{n=1}^{\infty}\left[a_n^2(\Omega_n^2-\omega^2)+\frac{2}{m}a_n f_n\right]\right\}, \tag{7}$$

where J is a normalizing factor. We now verify the correctness of this formula.

Integration over all paths $x(t)$ means that the integration over each of the variables a_n should be taken in the limits from $-\infty$ to $+\infty$. As a result, we obtain integrals of the Gaussian type,[12] which can be calculated explicitly and yield

$$K=J_1\prod_{n=1}^{\infty}\left(\frac{4\pi i\hbar}{\tau m|\Omega_n^2-\omega^2|}\right)^{1/2}\exp\left\{-i\frac{\tau f_n^2}{4m\hbar(\Omega_n^2-\omega^2)}\right\}, \tag{8}$$

where J_1 is some new normalizing constant. We use the formula[2)] (Ref. 12)

$$\prod_{n=1}^{\infty}\left(1-\frac{\omega^2}{\Omega_n^2}\right)=\frac{\sin\omega\tau}{\omega\tau},$$

and also the fact that

$$\frac{\tau}{2m}\sum_{n=1}^{\infty}[f_n^2/(\Omega_n^2-\omega^2)]=\int_0^{\tau} f\xi dt,$$

where $\xi(t)$ is the solution of the problem (4). If we take these equations into account and choose J_1 in appropriate fashion, then the infinite product (8) can be put in the form

$$K=\left(\frac{m\omega}{2\pi i\hbar\sin\omega\tau}\right)^{1/2}\exp\left\{\frac{i}{\hbar}S[\xi]\right\}, \tag{9}$$

in which the transition amplitude of the oscillator is expressed in terms of a functional of the action on the classical trajectory (4). According to problem 3.11 of Ref. 12, the transition amplitude of the oscillator in a strong field should actually be expressed by Eq. (9). Nevertheless, it is shown that the spectral representation of the path integral (7) is valid even if $f \neq 0$. Therefore, we can use it for the solution of the problem of the quantum restrictions on the spectral measurements.

We assume that a measurement is carried out in the time interval $[0, \tau]$ and determines the quantity a_n with error Δa_n. Then the probability amplitude of obtaining the result a_n can be found with the help of the path integral (7). The integrals over the variables a_n' and $n' \neq n$ should be taken here in the limits from $-\infty$ to $+\infty$, and the integral over a_n in the limits from $a_n - \Delta a_n$ to $a_n + \Delta a_n$. Lumping all the integrals except the integral over a_n in the normalizing constant, we obtain for the desired probability amplitude

$$A(a_n, \Delta a_n)=J_2\int_{-\infty}^{\infty} du\, \rho(u)\exp\left\{i\frac{m\tau}{4\hbar}\left[u^2(\Omega_n^2-\omega^2)+\frac{2}{m}uf_n\right]\right\}, \tag{10}$$

where $\rho(u)$ is a weighting factor, which is approximately constant in the range $[a_n - \Delta a_n,\ a_n + \Delta a_n]$ and falls off rapidly outside this range. In order that the resultant integral be of Gaussian type, it is convenient to choose this weighting factor proportional to $\exp\{-(u-a_n)^2/4\Delta a_n^2\}$. Calculating the resultant Gaussian integral, we obtain an explicit expression for the desired probability amplitude. The square of the modulus of this amplitude is the probability that the measurement, carried out with error Δa_n, gives the result a_n;

$$P(a_n, \Delta a_n)=J_3 \exp\left\{-\frac{1}{2}\frac{(a_n-a_n^{cl})^2}{\Delta a_n^2+(\Delta a_n^{opt})^4/\Delta a_n^2}\right\}, \tag{11}$$

where

$$\Delta a_n^{opt}=(\hbar/\tau m|\omega^2-\Omega_n^2|)^{1/2}, \tag{12}$$

$$a_n^{cl}=f_n/m(\omega^2-\Omega_n^2). \tag{13}$$

The probability (11) reaches a maximum at $a_n=a_n^{cl}$, which corresponds to the classical connection between a spectral component of the force and the spectral component of the response of the oscillator. The probability falls off. With increasing deviation of the measurement result a_n from this classical value. The more rapidly it falls off, the more accurately can we estimate the quantity f_n from this classical value. It is obvious that the fastest fall-off, i.e., optimal measurement conditions is achieved at $\Delta a_n=\Delta a_n^{opt}$. Here the probability distribution takes the form

$$P(a_n, \Delta a_n^{opt})=J_3 \exp\left\{-\frac{(a_n-a_n^{cl})^2}{4(\Delta a_n^{opt})^2}\right\}. \tag{14}$$

This probability distribution is concentrated almost completely in the interval $[a_n-\Delta a_n^{opt},\ a_n+\Delta a_n^{opt}]$. Consequently, the results of measurement of a_n will reproduce in this case the classical value of this quantity a_n^{cl} accurate to Δ_n^{opt}. Conversely, if the result of measurement is equal to a_n, we can then guarantee that a_n^{cl} differs from a_n by not more than a quantity of the order of Δa_n^{opt}.

From formula (13) we can easily transform to the estimate of the spectral component of the force. If the result of measurement of the spectral component of the coordinate of the oscillator is equal to a_n, then the optimal estimate for the corresponding spectral component of the force is given by the formula (13) with the replacement of a_n^{cl} by a_n. The error of this estimate under optimal conditions ($\Delta a_n=\Delta a_n^{opt}$) is of the order of

$$\Delta f_n^{opt}=\left(\frac{\hbar m|\omega^2-\Omega_n^2|}{\tau}\right)^{1/2}. \tag{15}$$

If the error Δa_n of measurement of the quantity a_n differs from Δa_n^{opt}, then the conditions of measurement are not optimal and the error in the estimate of the force is worse than (15). In this case, according to (11), the scatter of the resultant measurement of a_n is determined by the classical error of measurement, Δa_n, and the quantum error is equal, in order of magnitude, to

$$\Delta a_n^q=(\Delta a_n^{opt})^2/\Delta a_n. \tag{16}$$

If $\Delta a_n>\Delta a_n^{opt}$, then the classical error Δa_n predominates and in fact determines the scatter of results. If $\Delta a_n<\Delta a_n^{opt}$, i.e., the apparatus error of measurement is less than optimal, then the quantum error begins to predominate, $\Delta a_n^q>\Delta a_n$. Under both conditions, the scatter of the measurement results is greater than in the optimal case, and because of this, the error in the estimate of the force acting on the oscillator is also non-optimal:

$$\Delta f_n=m|\omega^2-\Omega_n^2|[(\Delta a_n)^2+(\Delta a_n^q)^2]^{1/2}. \tag{17}$$

In order to apply formula (15) to the case in which the measurements are carried out over the resonance frequency $\Omega_n\approx\omega$, it must be recalled that ω cannot coincide exactly with any of the frequencies Ω_n (see footnote 2). Since the interval between Ω_n and Ω_{n+1} is equal to $\Omega_1\sim\tau^{-1}$, it must be assumed that the measurement at the resonance frequency corresponds to a choice of Ω_n such that $|\omega-\Omega_n|\sim\tau^{-1}$. Here the formulas (12) and (15) give

$$\Delta a_{res}^{opt}=(\hbar/m\omega)^{1/2},\quad \Delta f_{res}^{opt}=(\hbar m\omega)^{1/2}/\tau.$$

We thus see that in spectral measurements at the resonance frequency the quantum error is essentially no different from that which arises in the regime of continuous tracking of the coordinate (Sec. 2).

We compare these two regimes (spectral measurement and coordinate tracking) in the case of non-resonant interaction. If a periodic force $f(t)+F\sin\Omega t$ acts on the oscillator, then its response has the amplitude $F/m\ |\omega^2-\Omega^2|$. According to Eq. (5), this response can be measured in the tracking regime only with accuracy $(\hbar\omega/m)^{1/2}$ (we assume that the measurement is of long duration, $\omega\tau\gg1$). Consequently, the force can be measured by this method with accuracy

$$\Delta F=|1-\Omega^2/\omega^2|(\hbar m\omega^3)^{1/2}.$$

At the same time, by measuring a single spectral component of the oscillator response, we can attain the accuracy (15), which, in the same notation, is equal to

$$\Delta F_{spectr}=|1-\Omega^2/\omega^2|^{1/2}(\hbar m\omega^2/\tau)^{1/2}.$$

If the ratio Ω/ω is less than or of the order of unity, then

$$\Delta F_{spectr}/\Delta F=(\omega\tau)^{-1/2},$$

which makes the spectral measurement more suitable. The advantage comes essentially from the fact that in the spectral regime of measurement, we use *a priori* information on the frequency of the force acting on the oscillator.

On the other hand, continuous tracking of the coordinate has the advantage over the spectral measurements in that it allows us to observe a force of arbitrary frequency. Furthermore, within the limits of quantum errors, the shape of the curve describing the change in force with passage of time is reproduced. It is true that the spectral measurements also can be organized in such a way as to reproduce a force about which there is no *a priori* information. For this purpose, it is necessary to measure all the spectral components of the oscillator response. However, as will be shown in the following section, the estimate of the quantum errors in this case turns out to be exactly the same as in continuous coordinate tracking. In other words, in the absence of *a priori* information on the frequency of the force the spectral measurements offer no advantage.

4. GENERAL CASE OF SPECTRAL MEASUREMENTS

We now can consider the case of an arbitrary spectral measurement on an oscillator. We assume that the signal, which is proportional to the oscillator coordinate, is subjected to a spectral analysis and the value of each harmonic a_n is then measured; the errors of measurement Δa_n are different for different harmonics. The result of the measurement is expressed by a sequence of numbers $\{a_1, a_2, \ldots, a_n, \ldots\}$ which are determined with the respective errors $\{\Delta a_1, \Delta a_2, \ldots, \Delta a_n, \ldots\}$. The probability amplitude of obtaining such a result is given by the integral (7), in which the n-th integral (for arbitrary n) is taken over an interval of the order Δa_n around the value a_n. It is clear that the result of such an integration can be expressed as a product of amplitudes of the form (10). Transforming to probabilities, we obtain a product of expressions of the form (11):

$$P\{a_1, a_2, \ldots\} = J_1 \exp\left\{-\frac{1}{2}\sum_{n=1}^{\infty} \frac{(a_n - a_n^{cl})^2}{\Delta a_n^2 + (\Delta a_n^{opt})^4/\Delta a_n^2}\right\}. \quad (18)$$

It follows from this distribution of probabilities that the result of the measurement, expressed in the form of the sequence $\{a_n\}$, leads to the estimate of the spectral component of the force acting on the oscillator in accord with Eq. (13), with the replacement of a_n^{cl} by a_n, while the error of the estimate of the spectral component f_n is described by Eq. (17). For those components for which the error of measurement greatly exceeds the optimal, $\Delta a_n \gg \Delta a_n^{opt}$, we obtain the purely classical formula

$$\Delta f_n = m|\omega^2 - \Omega_n^2|\Delta a_n.$$

For those components for which the optimal measurement error is obtained, $\Delta a_n = \Delta a_n^{opt}$, the error of the estimate of the spectral components of the force reaches its limiting value (15). This limiting value depends on the duration of the measurement τ and decreases without limit with increase of this parameter.

We now consider the case of a long-duration measurement, $\omega\tau \gg 1$, and assume that the force acting on the oscillator has a spectrum in a comparatively narrow frequency band $[\Omega, \Omega + \Delta\Omega]$. In order to find and measure this force, it is convenient to measure all the spectral components of the motion of the oscillator in this band of frequencies and to measure each of them with optimal error (12). If the results of the measurement of the frequency components are expressed by the set of numbers $\{a_k, a_{k+1}, \ldots, a_l\}$, where $k = \Omega/\Omega$, $l = (\Omega + \Delta\Omega)/\Omega$, then, setting the remaining harmonics equal to zero, we can obtain an estimate of the motion of the oscillator from the formula (6):

$$x_0(t) = \sum_{n=k}^{l} a_n \sin \Omega_n t.$$

However, since each harmonic a_n is equal to the classical value (13) only accurate to Δa_n^{opt}, the classical motion of the oscillator $\xi(t)$, which corresponds to the external force given by (4) acting on it, can be different from $x_0(t)$. The difference can be estimated from the mean square deviation

$$\langle \Delta x^2 \rangle = (1/\tau)\int_0^\tau (\xi - x_0)^2\, dt.$$

It is not difficult to see that the maximum deviation is equal to

$$\langle \Delta x^2 \rangle = \sum_{n=k}^{l} (\Delta a_n^{opt})^2 = \frac{\hbar\Delta\Omega}{m|\omega^2 - \Omega^2|}. \quad (19)$$

The amplitude of the quantum noise in the band $\Delta\Omega \gtrsim \tau^{-1}$ is estimated in this way. If the band of frequencies is sufficiently broad (but does not contain the resonance frequency $\Omega = \omega$), then the estimate of the quantum noise is given by the integral

$$\langle \Delta x^2 \rangle = \frac{\hbar}{m}\int_{\Omega_1}^{\Omega_2} \frac{d\Omega}{|\omega^2 - \Omega^2|}. \quad (20)$$

The error in the estimate of the force acting on the oscillator can be found in similar fashion. We get

$$\langle \Delta f^2 \rangle = \sum_{n=k}^{l} (\Delta f_n^{opt})^2 = \hbar m|\omega^2 - \Omega^2|\Delta\Omega \quad (21)$$

for a narrow band of frequencies and

$$\langle \Delta f^2 \rangle = \hbar m \int_{\Omega_1}^{\Omega_2} |\omega^2 - \Omega^2|\, d\Omega \quad (22)$$

for a broad band.

Formulas (19)–(22) are suitable for the estimate of the the accuracy of the optimal measurements in a band of frequencies, i.e., in the case in which we have *a priori* information on the spectrum of the force acting on the oscillator. If there is no such information, then all the spectral components of the motion of the oscillator must be measured in optimal fashion for a reliable determination of a small force or for an accurate measurement of its form. In this case, the error in the determination of the oscillator coordinate is equal to

$$\langle \Delta x^2 \rangle = \sum_{n=1}^{\infty} (\Delta a_n^{opt})^2 = \frac{\hbar}{\tau m\omega^2}\sum_{n=1}^{\infty}\left|1 - \frac{\Omega_n^2}{\omega^2}\right|^{-1}. \quad (23)$$

If the measurement interval is large, $\omega\tau \gg 1$, only terms with numbers $n \lesssim \omega\tau \approx \omega/\Omega$, make a contribution to this sum, and the contribution from each of such terms is approximately equal to unity. This gives the estimate

$$\langle \Delta x^2 \rangle = \hbar/m\omega \quad \text{for} \quad \omega\tau \gg 1, \quad (24)$$

which is identical with the quantum threshold in the case of tracking of the oscillator coordinate (5). In the case of a short-time measurement, $\omega\tau \lesssim 1$, the essential contribution to the sum (23) is made only by the first few terms, which leads to the estimate

$$\langle \Delta x^2 \rangle = \hbar\tau/m \quad \text{for} \quad \omega\tau \lesssim 1, \quad (25)$$

which also agrees with (5). Consequently, in such a regime of measurement, which reliably measures the force of arbitrary shape (i.e., in the absence of *a priori* information), the spectral measurements do not have any advantage over continuous coordinate tracking. This is understandable, for if we know the complete spectrum $\{a_1, a_2, \ldots\}$ we can reconstruct the function $x(t)$ completely, i.e., the measurement of all the spectral components is equivalent to continuous coordinate tracking.

5. CONCLUDING REMARKS

Summing up our results, we can draw the following practical conclusions. For a measurement of the force about which there is no *a priori* information, acting on an oscillator, the spectral measurements do not give any advantage in comparison with continuous coordinate tracking. If the frequency band in which the spectrum of the active force is certain to lie is known beforehand, then the spectral measurements turn out to be better than the continuous tracking of the coordinate (with the exception of measurements at the resonance frequency, see Sec. 3).

Both in the case of continuous coordinate tracking (Sec. 2), and in the case of spectral measurements (Secs. 3,4) we have seen that the optimal measurement regime lies on the boundary of the region in which the measurements are purely classical and are described by non-quantum equations. Since the measurement in classical theory does not lead to a significant perturbation (reduction) of the state of the system, we see that the measurements in the optimal regime (from the viewpoint of quantum errors) should also be nonperturbing or nearly nonperturbing. This illustrates the role which nonperturbing or quantum-nondestructive measurements play in the quantum theory of measurements (see Refs. 2–5, 7–10). The method of Feynman integrals, as we have seen, allows us to consider measurements with account taken of the perturbation of the state to which this measurement leads. In addition to purely theoretical interest, this problem is of practical interest in those cases when the measurement system, even if designed for optimal measurements is subjected to the action of a force for which the measurement process turns out to be not optimal. In this case, to calculate the reaction of the system it is necessary to take into account the perturbation of the state.

Specific estimates of quantum error are given by formulas (19)–(25), in which, naturally, Planck's constant enters. We note that these restrictions manifest themselves in an oscillator even if the energy dissipation in it can be neglected.

Besides these specific conclusions, there is interest in the conclusion that the quantum restrictions on the measurement can be formulated in terms of "measurement quantum noise," which has the spectrum (16). This possibility arises as a consequence of the fact that the measurement of one harmonic has no effect on the accuracy of measurement of the other harmonics. In turn, this independence of the harmonics is a consequence of the fact that the path integral (7) is written as the produce of integrals over the different harmonics. Additionally, the conclusion on the objective character of the quantum measurement noise is confirmed by the fact that the estimates of the quantum noise are identical in continuous tracking of the coordinate (5) and in optimal spectral measurements at all frequencies [Eqs. (24) and (25)].

It would be interesting to analyze more deeply the question of the correctness of the concept of measurement quantum noise, by considering the problem of the quantum measurement restrictions for principally other classes of measurements and for other (different from the oscillator) quantum systems subject to measurement. It can be thought that in the case of optimal measurements, which lie on the boundary of the classical region (see above), the concept of measurement quantum noise is correct. But if the perturbation of the system is substantial, then its character and the character of the measurement noise depend on the type of measurement procedure. One must also consider measurements in the quantum system with account of dissipation, and compare the measurement quantum noise with noise described by the fluctuation-dissipation theorem in the quantum limit. Still another problem is the allowance for "boundary conditions," i.e., the state of the measured system before and after the measurement. In the present paper, the boundary conditions were not taken into account, which is admissible, in any case, for long-duration measurements. However, their account in the general case may turn out to be significant.

The comparison of the results here with the results obtained in quantum radiophysics is another problem, in particular, with calculation of the optimal filtration of quantum signals.[13-15] Such a comparison is made difficult by the essential differences in the setup of the problem and applied methods.[3)]

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[1)] For our purposes, it is convenient to have the expansion in a series and not in a Fourier integral. A transition to the Fourier integral, would transform each spectral line, which corresponds to a frequency Ω_n, into a continuous-spectrum band having a width of the order of τ^{-1}, so that the neighboring bands would run together. In the case of a large measurement time $\tau \gg (\Delta\Omega)^{-1}$ ($\Delta\Omega$ is the band of frequencies of the force acting on the oscillator) the difference between the discrete and continuous spectra becomes unimportant. The transition from one method of description to another corresponds to the replacement of summation over n by integration $(\tau/\pi)\int d\Omega$. Such a transition was used in Eqs. (19)–(22) below.

[2)] It is seen from this formula and from the subsequent calculations that the applied method is correct only in the case in which the frequency Ω_n is not identical with the natural frequency of the oscillator at any $n=1, 2, 3$.

[3)] Recently, the author became acquainted with the work of Gusev and Rudenko [Zh. Eksp. Teor. Fiz. **76**, 1488 (1979), Sov. Phys. JETP **49**, 755 (1979)]. In that paper, methods of filtration and the fluctuation-dissipation theorem are applied for the estimate of quantum errors in the measurement of a force acting on a system of two coupled oscillators.

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Radiative collisions between atoms in a bichromatic field

S. P. Goreslavskiĭ and B. P. Kraĭnov

Moscow Engineering Physics Institute
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We investigate the influence of nonmonochromaticity of radiation on transitions between atomic terms in radiative collisions between atoms. As the simplest analytically solvable example of a nonmonochromatic field we consider the bichromatic field. An analysis is carried out of Landau-Zener-type transitions in such a field at different field amplitudes and at various differences between the harmonic frequencies. New combination Landau-Zener transitions are described. The results are generalized to include the case of arbitrary nonmonochromaticity of the field.

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1. INTRODUCTION

Radiative collisions, i.e., collisions between atoms and molecules, which occur in the presence of an optical field, have recently attracted great interest. In the course of radiative collisions it is possible to have inelastic collisions that are adiabatically forbidden in the absence of an optical field by virtue of the slowness of the collisions. A review of the theory of radiative collisions is contained in a paper by Yakovlenko.[1] The theory of radiative collisions in the presence of a monochromatic perturbing field has been well-developed. At a certain instant of time, the difference between two energy terms of a quasimolecule made up of two colliding atoms becomes equal to the energy of the optical photon and crossing of the terms takes place in the "quasimolecule + field" system. As a result of this crossing, a nonadiabatic transition becomes possible. The mathematical description of the process is similar to the theory of Landau-Zener term crossing (see Ref. 2, Sec. 90).

The task undertaken in the present paper is to investigate the influence of nonmonochromaticity of the radiation on transition between terms in radiative collisions. By way of the simplest example of nonmonochromaticity we consider a bichromatic field, which is a superposition of two harmonic waves with close frequencies ω_1 and ω_2. The matrix element of the bichromatic field between the lower (a) and upper (b) levels of the quasi-molecule then takes the form

$$V_{ab}(t)=2V_{ab}^{(1)}\cos\omega_1 t+2V_{ab}^{(2)}\cos(\omega_2 t+\alpha). \tag{1}$$

The quantity α is the phase difference at the instant of time $t=0$.

We denote the time dependent energies of the terms of the quasimolecule by $E_a(t)$ and $E_b(t)$. Following the usual approach, we expand the term difference near the point of crossing in a series, and confine ourselves to the linear term

$$\omega_{ba}(t)=E_b(t)-E_a(t)=\omega_1+(F_b-F_a)t.$$

For the sake of argument, we reckoned the time from the crossing point due to the first field:

$$E_b(0)-E_a(0)=\omega_1.$$

The derivatives F_a and F_b are small quantities, since they are proportional to the velocities of the colliding atoms u, which are small compared with the atomic velocities. Namely, it is assumed that

$$F_a, F_b\sim\frac{u}{u_a}\frac{\omega_1}{\tau_a}\ll\frac{\omega_1}{\tau_a},$$

where u_a and τ_a are the characteristic atomic velocities and times.

We use henceforth the dimensionless time $\varphi=(F_a-F_b)^{1/2}t$ and the dimensionless frequency difference

$$\Delta\omega=(\omega_1-\omega_2)(F_a-F_b)^{-1/2}.$$

The dimensionless amplitude of the field

$$v_{1,2}=V_{ab}^{(1,2)}(F_a-F_b)^{-1/2}$$

then coincides with the known Landau-Zener parameter.

In Sec. 2 we consider the case of a weak field, and in Sec. 3 the case when the difference between the frequencies of the bichromatic field is large enough, and

