Some stationary generalizations of plane-wave solutions of general relativity theory

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The problem of superposition of plane waves is considered in a general metric of the form $g_{\mu\nu}=g_{\mu\nu}(x^1)$, where x^1 is a spacelike coordinate. It is shown that in this case the interacting-wave propagation trajectories are isotropic geodesics without convergence, but with rotation and shift.

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1. A number of workers[1-5] pointed out one of the possible ways of generalizing the plane-wave solutions of the gravitational equations to include the case when space-time contains two waves propagating in different directions. It consists, in brief, in the following. The metric of the plane wave

$$-ds^2=dx^2+dy^2-2dpdq-H(x,y,q)\,dq^2 \tag{1}$$

can be reduced by a coordinate transformation to the form

$$ds^2=g_{11}dx^2+2g_{12}dxdy+g_{22}dy^2+2g_{34}dpdq, \tag{2}$$

where $g_{\mu\nu}=g_{\mu\nu}(q)$ depends only on the isotropic coordinate q (q-wave). It is assumed that the interacting waves should correspond to a metric similar to (2), but now dependent on both isotropic coordinates p and q:

$$ds^2=g_{\mu\nu}dx^\mu dx^\nu,\quad g_{\mu\nu}=g_{\mu\nu}(p,q). \tag{3}$$

It then becomes possible to break space-time up into four regions: a planar region, a region containing the p wave, a region containing the q wave, the the region of interaction with the metric (3). The Lichnerowicz matching conditions are satisfied on the boundary.

By using the Newman-Penrose spin-wave coefficient method to solve the problem[1)] it is possible to describe both the kinematics of the process of the interaction (the geometric characteristics of the propagation trajectories) and the dynamics (the change of the structure of the tetrad components of the Weyl tensor and of the energy-momentum tensor).

In the case of solutions of the form (3), mutual focusing of the waves takes place. The congruences of the isotropic geodesics, along which the waves propagate in the interaction region, have convergence, and the field quantities characterizing the intensities of the wave fields become infinite on some spacelike hypersurface, which is called caustic in analogy with geometrical optics.

Solutions of the described type, corresponding to different interacting wave fields, can be found in a number of papers.[1-5,7] In the present communication we indicate another possibility of generalizing the solutions for free plane waves. By wave of example we present an exact solution that describes the gravitational interaction of two fluxes of high-frequency radiation.

2. We present the initial metrics. The metric (1) at $H=B^2(q)(x^2+y^2)/2$ is a solution of the gravitation equation with right-hand side in the form

$$T_{\mu\nu}=Pl_\mu l_\nu,\quad l_\mu l^\mu=0 \tag{4}$$

and corresponds to space-time filled with a flux of electromagnetic or neutrino radiation (plane wave). The vector l^μ determines the wave propagation direction, while the quantity $B^2(q)$ characterizes the effective density of the radiation energy. At constant B^2 this metric is reduced by the coordinate transformation

$$r^2=x^2+y^2,\quad re^{i\varphi}=x+iy,\quad z=2^{-1/2}(p-q),\quad t=2^{-1/2}(p+q) \tag{5}$$

to a form where the metric tensor depends only on one spacelike variable $x^1=r$:

$$-ds^2=dr^2+r^2d\varphi^2+(1-{}^1/{}_4B^2r^2)\,dz^2-{}^1/{}_2B^2r^2dzdt-(1+{}^1/{}_4B^2r^2)\,dt^2. \tag{6}$$

If we choose the Newman-Penrose tetrad field in the form

$$\begin{aligned} l^\mu&=(0,0,2^{-1/2},2^{-1/2}),\\ n^\mu&=(0,\,-2^{-1/2}B,\,-2^{-1/2},\,2^{-1/2}),\\ m^\mu&=(2^{-1/2},\,i\cdot2^{-1/2},\,-iBr\cdot2^{-3/2},\,-iBr\cdot2^{-3/2}), \end{aligned} \tag{7}$$

then the only nonzero quantities are

$$\mu=2\gamma=iB\cdot2^{-1/2},\quad \alpha=-\beta=-1/2^{3/2}r,\quad \Phi_{22}={}^1/{}_2B^2. \tag{8}$$

The introduced coordinates have the meaning of ordinary cylindrical coordinates. The optical scalar $\omega=\operatorname{Im}\mu=2^{-1/2}B$ is the rate of rotation of the congruence of the isotropic geodesics with a tangential vector n^μ. The motion of test particles along this congruence constitutes (in the three-dimensional sense) motion along helical lines ($n^1=n^r=0$) around the z axis ($n^2=n^\varphi=-2^{-1/2}B$) in a direction opposite to the radiation flux ($n^3=n^z=-2^{-1/2}$).

There exists one other coordinate system in which $g=g\,(x^1)$ for the metric in question. In this system

$$-ds^2=d\tilde{x}^2+d\tilde{y}^2-2^{3/2}B\tilde{x}d\tilde{y}dq-2d\tilde{p}dq, \tag{9}$$

where the tilde denotes coordinates subjected to transformation. The explicit form of the transformation can be found in the paper of Dinariev and Sibgatullin.[8]
If we choose

$$\begin{aligned} &l^\mu=(0,0,1,0),\quad n^\mu=(0,0,0,1),\\ &m^\mu=(2^{-1/2},\,i\cdot2^{-1/2},\,-iB\tilde{x},\,0), \end{aligned} \tag{10}$$

then, just as in (8)

$$\mu=2\gamma=2^{-1/2}iB,\quad \Phi_{22}=B^2/2, \tag{11}$$

but $\alpha=\beta=0$.

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For a transverse gravitational wave in the form[2)]

$$-ds^2=dx^2+dy^2-2dpdq+2A^2[(x^2-y^2)\cos 2^{1/2}Ap+2xy\sin 2^{1/2}Ap]dp^2$$

we can obtain relations similar to (9)–(11):

$$\begin{gathered}-ds^2=d\tilde{x}^2+d\tilde{y}^2+2^{1/2}A\tilde{x}d\tilde{y}dp-2dpd\tilde{q}+4A^2\tilde{x}^2dp^2,\\ l^\mu=(0,-2^{1/2}A\tilde{x},1,-A^2\tilde{x}^2),\\ n^\mu=(0,0,0,1),\quad m^\mu=(2^{-1/2},i\cdot 2^{-1/2},0,iA\tilde{x}),\\ \rho=-\sigma=\varepsilon=-iA\cdot 2^{-1/2},\quad \Psi_0=-2A^2,\quad A=\mathrm{const.}\end{gathered} \tag{12}$$

3. The solutions (6)–(12) can be generalized by a common metric of the type

$$g_{\mu\nu}=g_{\mu\nu}(x^1)=\begin{bmatrix} g_{11} & 0 & 0 & 0\\ 0 & g_{22} & g_{23} & g_{24}\\ 0 & g_{32} & g_{33} & g_{34}\\ 0 & g_{42} & g_{43} & g_{44}\end{bmatrix}. \tag{13}$$

We choose the tetrad field in the form

$$\begin{gathered}l^\mu=(0,l^2,l^3,l^4),\quad n^\mu=(0,n^2,n^3,n^4),\quad m^\mu=(m^1,m^2,m^3,m^4),\\ m^1=\bar{m}^1,\quad m^2=-\bar{m}^2,\quad m^3=-\bar{m}^3,\quad m^4=-\bar{m}^4.\end{gathered} \tag{14}$$

For the metric (13) only the Christoffel symbols Γ^1_{ik} and Γ^i_{1k} $(i,\ k\neq 1)$ differ from zero. Using this circumstance, we can easily show that

$$\begin{gathered}\tau=\bar{\tau},\quad \pi=\bar{\pi},\quad \alpha=\bar{\alpha},\quad \beta=\bar{\beta},\quad k=\bar{k},\quad \nu=\bar{\nu},\\ \rho=-\bar{\rho},\quad \mu=-\bar{\mu},\quad \sigma=-\bar{\sigma},\quad \lambda=-\bar{\lambda},\quad \varepsilon=-\bar{\varepsilon},\quad \gamma=-\bar{\gamma}.\end{gathered} \tag{15}$$

Substitution of (14) and (15) in the commutator yields additional limitations on the spin coefficients:

$$\tau+\pi=0,\quad \rho=2\varepsilon+\sigma,\quad \mu=2\gamma+\lambda \tag{16}$$

and equations for the components of the tetrad vectors:

$$\begin{gathered}\delta l^\mu=(\alpha+\beta+\tau)l^\mu+kn^\mu+2\sigma m^\mu,\\ \delta n^\mu=(\tau-\alpha-\beta)n^\mu-\nu l^\mu+2\lambda m^\mu,\quad \delta m^\mu=(\alpha-\beta)m^\mu-\mu l^\mu-\rho n^\mu,\\ \delta=m^1d/dx^1,\quad \mu\neq 1.\end{gathered} \tag{17}$$

We put $k=\nu=0$. Then the system of Newman-Penrose equations takes the form

$$\begin{gathered}\rho^2-\sigma^2+\Phi_{00}=4\varepsilon^2+4\varepsilon\sigma+\Phi_{00}=0,\\ \Psi_0=2(\sigma-\rho)\sigma=-4\varepsilon\sigma,\\ \Psi_1=(\sigma-\rho)\tau-\Phi_{01}=-2\varepsilon\tau-\Phi_{01},\quad \Psi_2=\tau^2-\Phi_{11},\\ \delta\varepsilon=(\alpha+\beta+3\tau)\varepsilon+(\beta-\alpha+\tau)\sigma+\Phi_{01},\\ \delta(\alpha-\beta)=(\alpha-\beta)^2-\tau^2+\rho\mu-\sigma\lambda+2\gamma\rho+2\varepsilon\mu+2\Phi_{11},\\ \delta\tau=(2\tau+\alpha-\beta)\tau-4\varepsilon\gamma-2\varepsilon\lambda-2\gamma\sigma-\Phi_{11},\\ \delta\tau=(\tau+\beta-\alpha)\tau-2\varepsilon\lambda-2\gamma\sigma+\Phi_{02},\\ \mu^2-\lambda^2+\Phi_{22}=4\gamma^2+4\gamma\lambda+\Phi_{22}=0,\\ \Psi_4=2(\lambda-\mu)\lambda=-4\gamma\lambda,\quad \Psi_3=(\mu-\lambda)\tau+\Phi_{12}=2\gamma\tau+\Phi_{12},\\ \delta\gamma=(3\tau-\alpha-\beta)\gamma+(\tau-\alpha+\beta)\lambda+\Phi_{12}.\end{gathered} \tag{18}$$

4. We present the simplest, most lucid, and in a certain sense most general solution of this system of equations, which describes the interaction between two radiation fluxes in the high-frequency limit, when we can put $T_{\mu\nu}\sim Pl_\mu l_\nu+Qn_\mu n_\nu$. The solution takes the form

$$\begin{gathered}\Phi_{00}=\frac{A^2}{2(1-{}^3/_4ABr^2)^2},\quad \Phi_{22}=\frac{B^2}{2(1-{}^3/_4ABr^2)^2},\\ A=\bar{A}=\mathrm{const},\quad B=\bar{B}=\mathrm{const},\quad \nu=\lambda=\Psi_0=0,\quad k=\sigma=\Psi_4=0,\\ \mu=2\gamma=\frac{iB}{2^{1/2}(1-{}^3/_4ABr^2)},\quad \rho=2\varepsilon=\frac{iA}{2^{1/2}(1-{}^3/_4ABr^2)},\\ \Psi_1=-\frac{iA^2Br}{4(1-{}^3/_4ABr^2)^2},\quad \Psi_3=\frac{iAB^2r}{4(1-{}^3/_4ABr^2)^2},\\ \tau=ABr/2^{1/2}(1-{}^3/_4ABr^2),\end{gathered}$$

$$\begin{gathered}\Psi_2=A^2B^2r^2/8(1-{}^3/_4ABr^2)^2,\quad \alpha=-\beta=-\tau-1/2^{1/2}r,\\ l^\mu=(0,-A/2^{1/2}f,\ 1/2^{1/2}f,\ 1/2^{1/2}f),\\ n^\mu=(0,-B/2^{1/2}f,\ -1/2^{1/2}f,\ 1/2^{1/2}f),\\ m^\mu=\left(\frac{1}{2^{1/2}},\ \frac{i(1+{}^1/_4ABr^2)}{2^{1/2}fr},\ \frac{i(A-B)r}{2^{3/2}f},\ -\frac{i(A+B)r}{2^{3/2}f}\right),\\ f=(1-{}^3/_4ABr^2)^{1/2},\quad g_{11}=-1,\quad g_{22}=\frac{1}{f^4}(ABr^2-1)r^2,\\ g_{33}=-\frac{1}{f^4}\left[1-\frac{ABr^2}{2}-\frac{(A+B)^2r^2}{4}+\frac{A^2B^2r^4}{16}-\frac{AB(A+B)^2r^4}{8}\right]\\ g_{44}=\frac{1}{f^4}\left[1-\frac{ABr^2}{2}+\frac{(A-B)^2r^2}{4}+\frac{A^2B^2r^4}{16}-\frac{AB(A-B)^2r^4}{8}\right],\\ g_{23}=\frac{1}{8f^4}AB(A-B)r^4,\quad g_{24}=\frac{1}{8f^4}AB(A+B)r^4,\\ g_{34}=\frac{1}{4f^4}(A^2-B^2)\left(1-\frac{ABr^2}{2}\right)r^2,\quad \|g\|=-r^2.\end{gathered} \tag{19}$$

It is easily seen that as $A\to 0$ or $B\to 0$ these relations go over into Eqs. (6)–(8).

Since $\mathrm{Re}\rho=\mathrm{Re}\mu=0$, there is no focusing. The radiation-propagation trajectories are isotropic geodesics with rotation, and the rotation rate is larger the higher the energy density of the opposing radiation flux. At $AB<0$, which corresponds to rotation of the rays in opposite directions, the field quantities are finite everywhere. The maximum of the density is reached at $r=0$. On the other hand if $AB>0$, then on going through the point $r=r_0=(4/3AB)^{1/2}$ the direction of rotation of the flux trajectories is reversed. At the point r_0 itself, the field quantities have a singularity, which indicates an infinite attraction between layers rotating in opposite directions.

The interaction leads also to the appearance of longitudinal (Ψ_1 and Ψ_3) and Coulomb (Ψ_2) components of the free gravitational field.

5. Under other assumptions with respect to Ψ_0, Ψ_4, and Ψ_m we can obtain similar solutions of the system (18), which describe the superposition of the gravitational, neutrino, and electromagnetic waves. In these cases, however, the limiting metrics should be chosen to be (9) and (12).

We dwell in conclusion on certain properties of the solution for interacting neutrino and gravitational waves. A neutrino field with a current vector parallel to l^μ is described by a component ψ that satisfies the equation

$$D\psi=(\rho-\varepsilon)\psi,\quad \delta\psi=(\tau-\beta)\psi. \tag{20}$$

The tetrad components of the energy-momentum tensor are of the form

$$\begin{gathered}\Phi_{02}=-i\sigma\psi\bar{\psi},\quad \Phi_{11}=-i\rho\psi\bar{\psi},\\ \Phi_{12}={}^1/_2 i(\alpha-\beta-\tau)\psi\bar{\psi},\quad \Phi_{22}=-i\mu\psi\bar{\psi}.\end{gathered} \tag{21}$$

We see therefore that upon interaction with the gravitational wave the neutrino field loses its isotropic character. It follows from (18) and (21) that $\mathrm{Im}\mu=\psi\bar{\psi}>0$, i.e., polarized neutrino radiation admits of rotation in only a definite direction.

The gravitational wave corresponds to the conditions $\Psi_0\neq 0$ and $\Phi_{00}=0$. In this case

$$\rho=-\sigma=\varepsilon\neq 0. \tag{22}$$

Thus, a transverse gravitational wave causes, besides rotation, a shift of the rays of the opposing wave. It follows also from (22) that the energy-momentum tensor of the neutrino field satisfies the energy-dominance condition[10]

$$\mathrm{Im}\,\rho \geqslant {}^1/_2 |\sigma| \tag{23}$$

only if $\mathrm{Im}\rho > 0$.

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[1)]A description of the method and several of its applications can be found in the paper by Frolov.[6]

[2)]The physical interpretation of the tetrad components of the Weyl tensor Ψ_i is the subject of a paper by Szekeres.[9]

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One-soliton cosmological waves

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Exact solutions of the gravitational equations which describe the evolution of gravitational solitons against the background of Friedmann cosmological models with the equation of state $\epsilon = p$ are derived and examined. The corresponding vacuum solutions are given.

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§1. INTRODUCTION

The method of inverse solution of the scattering problem has been used by Zakharov and the writer[1] to describe a procedure for integrating the gravitational equations for the case of a metric tensor depending on only two variables. The metric we used was written in the form[1)]

$$-ds^2 = f(-dt^2 + dz^2) + g_{ab}dx^a dx^b, \tag{1.1}$$

where the functions f and g_{ab} depend on the coordinates t and z. Our notation for the coordinates is $(x^0, x^1, x^2, x^3) = (t, x, y, z)$. The first Latin letters a and b always run through the values 1 and 2 and refer to the coordinates x and y. The Latin indices i and k, which occur later, refer to four-dimensional space and take the values 0, 1, 2, 3.

In the previous paper[1] we considered the Einstein equations corresponding to the interval (1.1) only in empty space. The application of a similar method to the integration of these equations in a space filled with matter is as yet an unsettled question. Meanwhile the solutions belonging to the class of metrics (1.1) include such fundamental exact solutions as the Friedmann cosmological models, for which the presence of matter is essential. It would certainly be interesting to construct new exact cosmological solutions describing the evolution of finite disturbances such as gravitational solitons, appearing against the background of a Friedmann space. For the reason we have noted, this cannot at present be done in general form.

There is, however, one special case in which the method already described[1] can still be applied even in a space with matter. This is the case of an ideal fluid with the "superrigid" equation of state $\varepsilon = p$, proposed by Zeld'ovich.[2] The specific form of this equation of state will not play any decisive part in our work, since we shall deal with soliton perturbations of the gravitational field itself, not of the matter, which remains unperturbed in our solutions. From this point of view the matter serves only for the provision and maintainance of the Friedmann background solution, and it can be hoped that the qualitative picture of the behavior of gravitational solitons on this background will remain approximately the same for other equations of state. Besides this, exact solutions of the Einstein equations, analogous (in the sense that the behavior of the metric coefficients g_{ab} remains the same in them) to those obtained here for a space with matter, exist also in vacuum. The way they are found in the general case is described in Sec. 2, and the actual construction is given in Sec. 4.

In this paper we shall consider one-soliton solutions on the background of Friedmann models of all three types. Let us point out their main qualitative peculiarities. These solutions are inhomogeneous cosmological models, in which the distribution of the gravitational field at the inital time shows a clearly expressed max-

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