Tunnel transparency of disordered systems

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We investigate the tunneling of a quantum particle through a macroscopically smooth potential barrier in which this particle experiences below-barrier collisions with random scattering centers (e.g., impurity centers in a dielectric layer). The physical situation and the corresponding mathematical technique differ substantially, depending on whether the energy of the tunneling particles falls in the region of the vicinity of the discrete spectrum of the disordered system of impurity centers or far enough from it. In the nonresonant region, a unique "sub-barrier" kinetic equation is derived and makes it possible to find the effective damping, the spectral composition of the tunneling flux, and the average transparency of the plane-parallel barrier. In the resonance region, we study the local-transparency conditions and ascertain the structure and the probability density of resonant-percolation trajectories. Since the local transparency of the barrier turns out to be a strongly fluctuating quantity, we discuss and calculate in addition to the average transparency also other asymptotic quantities that correspond to different possible formulation of the problem. In the first part of the article we discuss in detail and elucidate the quasi-one-dimensional case corresponding to an infinitely large transverse mass of the tunneling particle. In the second part, the same mathematical methods are extended to include a general three-dimensional system.

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INTRODUCTION

The question of tunneling through a macroscopically smooth potential barrier in which the particles undergo sub-barrier collisions with random scattering centers arises in a number of physical problems. This sub-barrier scattering can be either elastic or inelastic. By way of example we can cite the tunneling of an electron through a dielectric plate containing impurity atoms.

It turns out that such a process should be described in certain cases by some analog of a "sub-barrier" kinetic equation, but in other cases it calls for an entirely different procedure. Generally speaking, the problem of tunneling through a disordered medium is not included among the standard problems of quantum theory of random systems. Although the width of the barrier layer is assumed large, all the asymptotic values obtained turn out to be intermediate; on the one hand, this complicates the problem greatly, and on the other it makes for a greater variety of results.

We consider in this paper only elastic collisions with random local centers. We investigate the tunnel transparency $\sigma_L(E)$ of a plane-parallel layer—a rectangular potential barrier of height U_0 and width L, perturbed by a random system of immobile scattering centers (the energy E is reckoned from the barrier U_0). Since the distribution of the scattering centers in the layer is assumed to be statistically homogeneous, the problem retains some essential one-dimensionality features, although the scattering does upset the true one-dimensionality in each concrete configuration of the centers.

First, however, we consider, by way of a very simple model that includes the most significant features typical also of the three-dimensional case, the pure one-dimensional problem. The physical realization of this model corresponds to the case when the effective mass of the tunneling particle is strongly anisotropic, so that the longitudinal mass is much less than the transverse one ($m_\parallel/m_\perp \ll 1$). In this case the three-dimensional system degenerates into a quasi-one-dimensional one consisting of an aggregate of noninteracting filaments on which the scattering centers are located and along which the tunneling takes place. This situation corresponds also to tunneling of elementary excitations through a homopolymer molecule with impurities. Finally, problems having the same mathematical structure can apparently find application also in other problems of optics and radiophysics of waveguides. In the present article, however, we purposely abstain from concrete realizations, which complicate greatly the entire analysis, and confine ourselves to an idealized formulation of the problem.

I. ONE-DIMENSIONAL SYSTEM

1. Formulation of problem and general approach

We express the wave function of the particle on the tunneling section $0<z<L$ in the form $(\hbar^2/2m_\parallel=1)$

$$\frac{d^2\Psi}{dz^2}-\alpha^2\Psi=\sum_{0<z_j<L}\hat{u}_j\Psi,\quad \alpha^2=-E, \tag{1.1}$$

where $\hat{u}_j$ is the operator of the local perturbation produced by the impurity center at the point z_j. In the simplest case we assume the perturbation to be point-like:

$$\hat{u}_j=\beta\delta(z-z_j), \tag{1.1a}$$

a procedure used only to simplify the derivation and corresponding to the requirement that the perturbation be local, $r_0\alpha\ll 1$, where r_0 is the action radius of the local perturbation.

The formulation of the problem of passage through a barrier implies matching the solutions of (1.1) on the boundaries 0 and L with the wave

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$$\Psi_1(z)=e^{ikz}+ae^{-ikz} \text{ if } z<0$$

and the transmitted wave

$$\Psi_2(z)=be^{ik(z-L)} \text{ if } z>L$$

($k^2=U_0+E$), i.e., introduction of the boundary conditions

$$\Psi(0)=1+a, \quad \Psi'(0)=ik(1-a),$$
$$\Psi'(L)/\Psi(L)=ik, \quad \Psi(L)=b, \tag{1.2}$$

where a and b are to be determined. The transmitted flux is then $k|b|^2$, and the transparency is

$$\sigma_L(E, \Gamma)=|b|^2, \quad \Gamma=\Gamma_N=(z_1, z_2, \ldots, z_N). \tag{1.3}$$

We formulate now the problem in somewhat different terms. We consider the equation (1.1) on the entire axis and represent $\Psi(z)$ in the form

$$\Psi(z)=\chi_-(z)+\chi_+(z), \tag{1.4}$$

where $d\chi_\pm/dz=\pm\alpha\chi_\pm$ everywhere except at the points z_j. Thus, Ψ is the sum of the components of a two-component vector χ that satisfies the equation[1)]

$$\frac{d\chi}{dz}+\check{\alpha}\chi=\frac{1}{2\alpha}\check{\beta}\sum_{0<z_j<L}\delta(z-z_j)\chi(z_j),$$
$$\chi=\begin{bmatrix}\chi_1\\ \chi_2\end{bmatrix}, \quad \chi_1=\chi_-(z), \quad \chi_2=\chi_+(z), \tag{1.5}$$
$$\check{\alpha}=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}, \quad \check{\beta}=\beta\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}. \tag{1.6}$$

The boundary conditions that follow from (1.2) and (1.4) take the form

$$\chi_1(0)=-\frac{2ik}{\alpha-ik}+\frac{\alpha+ik}{\alpha-ik}\chi_2(0),$$
$$\chi_1(L)=\frac{\alpha-ik}{\alpha+ik}\chi_2(L). \tag{1.7}$$

Starting with (1.5) and (1.6), we construct a matrix $\check{S}$ (propagator) that "carries" the solution through the region $0<z<L$ from the right to the left boundary of the barrier:

$$\chi(0)=\check{S}\chi(L),$$
$$\check{S}=\check{A}(z_1)\check{T}\check{A}(z_2-z_1)\check{T}\ldots\check{A}(z_N-z_{N-1})\check{T}\check{A}(L-z_N), \tag{1.8}$$
$$0<z_1<z_2<\ldots<z_N<L;$$

$$\check{A}(z)=e^{z\check{\alpha}}=\begin{bmatrix}e^{\alpha z} & 0\\ 0 & e^{-\alpha z}\end{bmatrix}, \quad \check{T}=\check{I}+\frac{1}{2\alpha}\check{\beta}=\begin{bmatrix}1+\beta/2\alpha & \beta/2\alpha\\ -\beta/2\alpha & 1-\beta/2\alpha\end{bmatrix}. \tag{1.9}$$

Here $\check{A}(z)$ is the free propagator that carries the solution from right to left into a region free from the scattering centers, and $\check{T}$ is a matrix that propagates the solution through the scattering center.

The matrix $\check{S}$ is unimodular:

$$D(\check{S})=1,$$

since $\check{T}$ and $\check{A}(z)$ are unimodular. This property is the consequence of the conservation of the flux in the one-dimensional Schrödinger equation and is not connected with assumption (1.1a).

From (1.2)–(1.4), (1.7) and (1.8) we obtain[2)]

$$\sigma_L(E, \Gamma)=1/|(\mathbf{f}, \check{S}\mathbf{g})|^2,$$
$$\mathbf{g}=\frac{1}{2}\begin{bmatrix}1+ik/\alpha\\ 1-ik/\alpha\end{bmatrix}, \quad \mathbf{f}=\frac{1}{2}\begin{bmatrix}1+i\alpha/k\\ 1-i\alpha/k\end{bmatrix}; \quad (\mathbf{f}, \varphi)=f_1\varphi_1^*+f_2\varphi_2^*. \tag{1.10}$$

Averaging $\sigma_L(E, \Gamma)$ over the ensemble of filaments with random configurations Γ, we obtain the average transmission coefficient (transparency)

$$\sigma_L(E)=\langle\sigma_L(E, \Gamma)\rangle=\sum_N\int_{(\Gamma_N)}p(\Gamma_N)\sigma_L(E, \Gamma_N)d\Gamma_N=\sum_N\sigma_N(L, E), \tag{1.11}$$

where $p(\Gamma_N)$ is the probability of realizing the configuration

$$d\Gamma_N=dz_1dz_2\ldots dz_N, \quad 0<z_1<z_2\ldots<z_N<L.$$

Equation (1.10), which expresses $\sigma_L(E, \Gamma)$ in terms of the matrix element on the vectors $\mathbf{f}$ and $\mathbf{g}$, explains well the structure of the solution and permits a number of general conclusions to be drawn. The structure of $\check{S}$ is not connected with any of the boundary conditions at the points 0 and L, while the boundary conditions that describe the flux below the barrier are expressed by the vectors $\mathbf{f}$ and $\mathbf{g}$ in (1.10) and are not connected with $\check{S}$. As will be shown below, all the conclusions concerning the behavior of $\sigma_L(E, \Gamma)$ and $\sigma_L(E)=\langle\sigma_L(E, \Gamma)\rangle$ can be obtained, apart from an insignificant factor ~ 1, from an analysis of $\check{S}$. On the other hand, the propagator $\check{S}(E, \Gamma)$ can by itself serve as convenient characteristic of the spectral properties of the random Hamiltonian:

$$\hat{H}=-\frac{d^2}{dz^2}+\beta\sum_{0<z_j<L}\delta(z-z_j), \tag{1.12}$$

which corresponds to Eq. (1.1) on the entire axis (with the condition $\Psi(\pm\infty)=0$). As can be easily understood, the eigenvalues of $\hat{H}$ at a given configuration Γ are the roots of the equation

$$S_{11}(E, \Gamma)=0. \tag{1.13}$$

In particular, if the configuration Γ reduces to one single scattering center, this yields $T_{11}(E)=0$ or $1+\beta/2\alpha=0$.

Even the noted connection of $\check{S}(E, \Gamma)$ with the spectrum of the operator $\hat{H}$ can easily point to two different situations:

A. Nonresonant tunneling

It is realized whenever the particle energy E is shifted far enough away from the spectrum of the eigenvalues of the random operator $\hat{H}$. This will occur, in particular, at $\beta>0(E=-\alpha^2<0)$. In this energy region $T_{11}\neq 0$, the amplitude of the sub-barrier scattering μ is finite[3)] and moreover, $\mu<1$. The average distance between scattering centers is $\langle x\rangle\sim 1/n$ (n is their linear density). Then, taking the definition of the matrix $\check{S}$ into account, we note that at low concentrations, i.e., $\alpha\langle x\rangle\gg 1$, $c=n/\alpha\ll 1$, the matrix element S_{11} has a probability $(1-c)$ of being exponentially large compared with the remaining ones: $S_{11}/S_{ik}\sim\exp(\alpha\langle x\rangle)\gg 1$. The expression for $\sigma_L(E, \Gamma)$ therefore takes in this case the form

$$\sigma_L(E, \Gamma)\cong\frac{1}{|f_1g_1|^2}\frac{1}{|S_{11}|^2}=\frac{16k^2\alpha^2}{(\alpha^2+k^2)^2}\left|\frac{\chi_1(L)}{\chi_1(0)}\right|^2. \tag{1.14}$$

Since the connection between $\chi_1(L)$ and $\chi_1(0)$ is linear, and σ_L contains only the ratio of these quantities, it follows that, without loss of generality, we can put in (1.14) $\chi_1(0)=1$. Expression (1.14) corresponds then to the solution of Eq. (1.1) [or (1.5)] with boundary conditions

$$\chi_1(0)=1, \quad \chi_2(L)=0. \tag{1.15}$$

These conditions correspond to neglecting the waves reflected from the boundaries 0 and L out of the inner region of the barrier $0<z<L$. However, at off-resonance energies, allowance for these reflections can lead only to the appearance, in the averaged expression for $\sigma_L(E)$, a factor close to unit $(1+0(c))$. Taking (1.15) and (1.14) into account, we have

$$\sigma_L(E)=\frac{16\alpha^2k^2}{(\alpha^2+k^2)^2}\langle|\chi_+(L)|^2\rangle=\frac{16\alpha^2k^2}{(\alpha^2+k^2)^2}\langle|\Psi(L)|^2\rangle. \quad (1.16)$$

Thus, in the off-resonance region the initial problem is practically equivalent to the problem of tunneling of a particle through a layer $0<z<L$, occupied by the scattering centers, in the case when the true boundaries of the homogeneous barrier are moved away from this barrier to an infinite distance (a situation corresponding to condition (1.15)).

Another representation for the quantities S_{11} and σ_L in the off-resonance case follows from the condition $S_{11}\gg S_{22}$. It follows from this condition that $S_{11}\cong \mathrm{Sp}\check{S}=\Lambda_1+\Lambda_2$, where Λ_j are the eigenvalues of $\check{S}$. It follows from the unimodularity of $\check{S}$ that $\Lambda_1\Lambda_2=1$, and the fact that $\check{S}$ is real means that either the Λ_j are real or $\Lambda_{1,2}=e^{\pm i\varphi}$. In the off-resonance case we therefore have $\Lambda_1\equiv\Lambda_{max}$ and

$$\sigma_L(E)\frac{(\alpha^2+k^2)^2}{16\alpha^2k^2}\cong\left\langle\frac{1}{(\mathrm{Sp}\,\check{S})^2}\right\rangle\cong\left\langle\frac{1}{\Lambda^2_{max}}\right\rangle \quad (1.17)$$

(in the resonant case $\Lambda_{1,2}=e^{\pm i\varphi}$, but Eq. (1.17) no longer holds).

B. Resonant tunneling

This case is realized at a given configuration Γ_N whenever the particle energy E lies in the vicinity of the discrete spectrum of the operator $\hat{H}_\Gamma$. In this case, as we shall show, there exists a certain set of resonant configurations $\{\Gamma_N\}_r$, on which

$$\sigma_L(E,\Gamma_N)\sim 1,\quad \Gamma_N\in\{\Gamma_N\}_r.$$

If the total contribution from the resonant configurations with $\sigma\sim 1$ is decisive, then

$$\sigma(E)=\sum_N\sigma_N(E)\sim\sum_N\int_{\{\Gamma_N\}_r}p(\Gamma_N)d\Gamma_N.$$

In the opposite case, we shall seek the probability of resonant configuration and investigate their effect separately.

The off-resonant and resonant tunneling require substantially different methods and will therefore be considered separately.

2. Off-resonant tunneling

As already shown in the preceding section, in the off-resonance region the problem of calculating the transmission coefficient σ_L reduces according to (1.16) to a calculation of $\langle|\Psi(L)|^2\rangle=\langle|\chi_-(L)|^2\rangle$, where $\Psi(z)$ and $\chi_\pm(z)$ are defined by the equations

$$\frac{d^2\Psi}{dz^2}-\alpha^2\Psi=\beta\sum_{0<z_j<L}\delta(z-z_j)\Psi(z_j), \quad (2.1)$$

$$\Psi(z)=\chi_-(z)+\chi_+(z),\quad \chi_-(0)=1,\quad \chi_+(L)=0. \quad (2.1a)$$

The solution of Eq. (2.1) with boundary conditions (1.1a) is[4]

$$\Psi(z)=\exp\{-\alpha z\}-\sum_{0<z_j<L}\varphi_j\exp\{-\alpha|z-z_j|\}, \quad (2.2)$$

$$\chi_-(z)=\exp\{-\alpha z\}-\sum_{0<z_j<z}\varphi_j\exp\{-\alpha|z-z_j|\}, \quad (2.3)$$

$$\chi_+(z)=-\sum_{z<z_j<L}\varphi_j\exp\{-\alpha|z-z_j|\}, \quad (2.3a)$$

$$\varphi_j\equiv\varphi(z_j)=(2\alpha)^{-1}\beta\Psi(z_j). \quad (2.4)$$

Substituting $z=z_j$ in (2.2) and taking (2.4) into account, we have

$$\varphi_j=\mu\exp\{-\alpha z_j\}-\mu\sum_{\substack{0<z_k<L\\k\neq j}}\varphi_k\exp\{-\alpha|z_j-z_k|\},\quad \mu=\frac{\beta}{2\alpha+\beta}, \quad (2.5)$$

where μ is the amplitude of scattering by one center. The system (2.2) and (2.5) is closed and is equivalent to the initial equations (2.1) and (2.1a).

The difficulty of calculating $\langle|\Psi(L)|^2\rangle$ lies in the fact that the macroscopically decreasing function $\Psi(z)$ contains random rapidly increasing terms, against the background of which the separation of the exponentially decreasing particle is a most complicated task. The procedure proposed below and connected with the form in which Eqs. (2.2)–(2.5) are expressed, eliminates these difficulties at least in the off-resonance region, and in addition admits of generalization to the three-dimensional case.

We introduce the shorter notation

$$g(z)=\chi_-(z,\Gamma),\quad \langle|g(z)|_\Gamma{}^2\rangle=G^z \quad (2.6)$$

and proceed to derive an integro-differential equation for G^z. We shall call the points $z_j<z$ "past" and the points $z_k>z$ "future" relative to z. We see then from the definition of $g(z)$, as well as from Eqs. (2.3)–(2.5), that this function depends not only on the past but also on the future, via the coefficients φ_j in (2.5) (for $z_j>z$). This dependence extends over a distance $|z_j-z|\sim 1/\alpha$.

We now write down a relation that follows from (2.3) and (2.5)

$$g(z+t)=\exp\{-\alpha t\}g(z)-\sum_{0<t_j<t}\exp\{-\alpha|t-t_j|\}\varphi_j, \quad (2.7)$$

$$\varphi_j=\mu\exp\{-\alpha t_j\}g(z)-\mu\sum_{\substack{0<t_k<L-z\\k\neq j}}\exp\{-\alpha|t_j-t_k|\}\varphi_k,\quad z_j=z+t_j. \quad (2.8)$$

To calculate the derivative dG^z/dz we use relations (2.7) and (2.8), in which we put $t=\delta z$, an infinitesimally small quantity. In this case there is a probability $n\delta z$ (n is the density of the centers that one term remains in the sum (2.7), and we get

$$g(z+\delta z)=(1-\alpha\delta z)g(z)-\varphi_0|_{0<t_0<\delta z}, \quad (2.9)$$

$$\varphi_j=\mu\exp\{-\alpha t_j\}g(z)-\mu\sum_{\substack{0<t_k<L-z\\k\neq j}}\exp\{-\alpha|t_j-t_k|\}\varphi_k; \quad (2.10)$$

$$\varphi_j\equiv\varphi(z_j),\quad z_j=z+t_j.$$

Squaring (2.9), we average this expression over Γ, and discarding terms $\sim(\delta z)^2$, we get

$$G^{z+\delta z}-G^z=\delta z[-2\alpha G^z-2n\langle g(z)\varphi(z+0)\rangle+n\langle\varphi^2(z+0)\rangle].$$

Hence

$$dG^z/dz=-2\alpha G^z-2n\langle g(z)\varphi(z+0)\rangle+n\langle\varphi^2(z+0)\rangle. \quad (2.11)$$

Since the function $g(z)$ has a discontinuity when it passes through each of the points z_j the form $\langle g(z)\varphi(z$

$+0)\rangle$ means that $g(z)$ is taken on the left of the scattering center located at the point $(z+0)$. When averaging the last two terms of (2.11), we must take into account the connection between $g(z)$ and φ_j, given by the equations in (2.10).

We assume the density to be small $(n/\alpha \ll 1)$ and expand the mean values in (2.11) in powers of n/α. To obtain the terms $\sim (n/\alpha)^{m+1}$ we must take into account in the right-hand side of (2.11) the contribution from configurations that have m additional scattering centers at a distance $|z_j - z| \sim 1/\alpha$ around the center at the point $(z+0)$ (the probability of this situation is $\sim (n/\alpha)^m$). We shall carry out the calculation for $m=0$ and $m=1$.

1. In the case $m=0$ (there are no additional centers near $(z+0)$) we get from (2.10)

$$\begin{gathered}\varphi_0\equiv\varphi(z+0)=\mu g(z),\\ \langle\varphi^2(z+0)\rangle-2\langle g(z)\varphi(z+0)\rangle=\mu^2\langle g^2(z)\rangle-2\mu\langle g^2(z)\rangle,\\ dG^z/dz=-(2\alpha+2n\mu-n\mu^2)G^z.\end{gathered}\tag{2.12}$$

2. $m=1$ (on additional center at the random point $(z+t);\ t\gtrless 0$). Since the additional center can be either on the right or on the left of the point z, and the calculation methods for these cases are different, we write down the sought mean values in the form

$$\langle\varphi^2(z+0)-2g(z)\varphi(z+0)\rangle={}^1/{}_2[\langle\ldots\rangle_{+t}+\langle\ldots\rangle_{-t}]\tag{2.13}$$

and consider the cases $\pm t$ separately.

a) A feature of the case $+t$ is that both centers, $(z+0)$ and $(z+t+0)$, are on the right of z. According to (2.10) we have

$$\begin{gathered}\varphi_0=\mu g(z)-\mu e^{-\alpha t}\varphi_t,\quad \varphi_0\equiv\varphi(z+0);\\ \varphi_t=\mu e^{-\alpha t}g(z)-\mu e^{-\alpha t}\varphi_0,\quad \varphi_t\equiv\varphi(z+t+0).\end{gathered}\tag{2.14}$$

Hence

$$\varphi_0=\mu g(z)-\mu\nu(\alpha t)g(z),\quad \nu(\tau)=[\mu(1-\mu)e^{-2\tau}]/(1-\mu^2e^{-2\tau}).\tag{2.15}$$

Since both centers lie to the right of z, and the correlation between $g(z)$ and the centers on the right (in the future) can be caused only by the close center on the left (in the past), the correlation terms yield the correction of the next (third) order of smallness. Therefore

$$\begin{gathered}\langle 2g(z)\varphi_0-\varphi_0^2\rangle_{+t}=(2\mu-\mu^2)\langle g^2(z)\rangle-[2\mu(1-\mu)\langle\nu\rangle\\ +\mu^2\langle\nu^2\rangle]\langle g^2(z)\rangle,\\ \langle\nu\rangle=n\int_0^\infty w(t)\nu(\alpha t)\,dt,\quad \langle\nu^2\rangle=n\int_0^\infty w(t)\nu^2(\alpha t)\,dt,\end{gathered}$$

where $w(t)$ describes the density correlation

$$\langle n(z)n(z+t)\rangle=n^2w(t),\quad n(z)\equiv n_\Gamma(z)=\sum_{z_j}\delta(z-z_j).$$

b) The case $-t$ is determined by the fact that the point z lies between the centers $(z-t+0)$ and $(z+0)$. Changing the notation, we rewrite formulas (2.7) and (2.8) in the form

$$\begin{gathered}g(z)\,\Big|_{\substack{z-t+0\\ z+0}}=e^{-\alpha t}g(z-t)-e^{-\alpha t}\varphi_{-t},\\ \varphi_0=\mu e^{-\alpha t}g(z-t)-\mu e^{-\alpha t}\varphi_{-t},\quad \varphi_0\equiv\varphi(z+0);\\ \varphi_{-t}=\mu g(z-t)-\mu e^{-\alpha t}\varphi_0,\quad \varphi_{-t}\equiv\varphi(z-t+0).\end{gathered}\tag{2.16}$$

Expressing φ_0, φ_{-t} from (2.16) in terms of $g(z-t)$ and separating in the quantity $2g(z)\varphi_0-\varphi_0^2$ the correlation part due to scattering by the pair of centers, which in fact yields as a result of the averaging the terms $\sim (n/\alpha)^2$, we put

$$\begin{gathered}\left\langle\left\{2g(z)\,\Big|_{\substack{z-t+0\\ z+0}}\varphi_0-\varphi_0^2\right\}\right\rangle=\mu(2-\mu)\langle g^2(z)\rangle+\langle f(\alpha t)e^{-2\alpha t}g^2(z-t)\rangle,\\ f(\tau)=2\mu^2(1-\mu)(2-\mu)\nu(\tau)+\mu^3(2-\mu)\nu^2(\tau).\end{gathered}\tag{2.17}$$

In Eq. (2.17), $g(z-t)$ is now already on the left of the points $(z-t+0)$, $(z+0)$. Therefore, as in the preceding case, accurate to terms of higher order in the density, we should average $g^2(z-t)$ prior to the integration of the entire expression with respect to t. This yields.

$$\langle f(\alpha t)e^{-2\alpha t}g^2(z-t)\rangle=n\int_0^\infty w(t)f(\alpha t)e^{-2\alpha t}G^{z-t}dt.\tag{2.18}$$

According to (2.12), however, accurate to the terms of higher order in the density n/α, we can assume in (2.18) that $G^{z-t}e^{-2\alpha t}=G^z$. Therefore retaining in the derivative dG^z/dz the terms proportional to $(n/\alpha)^2$, we can write

$$\begin{gathered}\frac{dG^z}{dz}=-\left[2\alpha+n\mu(2-\mu)+\frac{n^2\mu^2}{\alpha}S_2\right]G^z,\\ S_2=-(1-\mu)^4\int_0^\infty\frac{w(t)e^{-2\alpha t}\alpha dt}{1-\mu^2e^{-2\alpha t}}-\frac{\mu^2(1-\mu)^4}{2}\int_0^\infty\frac{w(t)e^{-4\alpha t}\alpha dt}{(1-\mu^2e^{-2\alpha t})^2}.\end{gathered}\tag{2.19}$$

If the discreteness of the disposition of the impurity centers in the lattice sites can be neglected $(a\alpha\ll 1)$, and the correlation function is $w(t)=1$, then

$$S_2=\frac{(1-\mu)^4}{4\mu^2}\ln(1-\mu^2)-\frac{1-\mu^2}{4(1+\mu)}\tag{2.20}$$

and consequently

$$\frac{(\alpha^2+k^2)^2}{16\alpha^2k^2}\sigma_L\cong\exp\left\{-\left[2\alpha+n\mu(2-\mu)+\frac{n^2\mu^2}{\alpha}S_2\right]L\right\}.\tag{2.21}$$

Equation (2.21) is valid for distances at which we can neglect the terms $\sim (n/\alpha)^3$.

We conclude this section with a few remarks concerning the meaning of the calculated average transparency σ_L. The averaging of the quantity $g^2(L)_\Gamma$ over the configurations corresponds to averaging over different paths of tunneling in a layer consisting of quasi-one-dimensional filaments or, equivalently, to calculation of the density of the passing flux per unit area of the sub-barrier layer. It is precisely this quantity which is of interest at not too large L, being in fact only an intermediate asymptotic form with respect to L when the number of filaments (i.e. the area of the tunnel layer) increases without limit.

Each of the quantities $g(L)_\Gamma$ is by itself not self-averaging at large L. We shall presently show, however, that in the off-resonance region the specific attenuation coefficient $\ln\sigma_L(\Gamma)/L=2\ln g(L)_\Gamma/L$ is for typical configurations, in the limit as $L\to\infty$ a self-averaging quantity and can be calculated at low densities $n/\alpha\ll 1$.

To this end, noting that

$$\begin{gathered}\ln g(L)/L=\sum_{z=0}^{L}\delta\ln g(z)/L,\\ \delta\ln g(z)=\ln g(z+\delta z)-\ln g(z),\end{gathered}\tag{2.22}$$

we write down the difference between the two close quantities $\ln g(z+\delta z)$ and $\ln g(z)$ in analogy with the procedure used for $g^2(z)$:

$$\ln g(z+\delta z)-\ln g(z)=\ln\left[\frac{g(z)(1-\alpha\delta z)-\varphi_0}{g(z)}\right]_{z<z_0<z+\delta z}$$
$$=-\alpha\delta z+\ln\left[1-\frac{\varphi_0}{g(z)}\right]_{z<z_0<z+\delta z},\quad \varphi_0\equiv\varphi(z+0). \tag{2.23}$$

As verified above, the quantity $\varphi(z+0)/g(z)$ depends only on the local surrounding of the point z by the scattering centers on the segment $(z-t, z+t), t\sim 1/\alpha$. Therefore averaging over z in (2.22) means averaging over the local configuration, and taking the limit as $\delta z\to 0$, we obtain

$$\frac{\ln g(L)_\Gamma}{L}\bigg|_{L\to\infty}=\frac{d\ln g(z)}{dz}=-\alpha+n\ln\left[1-\frac{\varphi_0}{g(z)}\right]. \tag{2.24}$$

In first-order approximation in n/α we have $\varphi_0/g(z)=\mu$, and consequently

$$\frac{\ln\sigma_L(\Gamma)}{L}\bigg|_{L\to\infty}=\frac{\ln g^2(L)_\Gamma}{L}\bigg|_{L\to\infty}=-2\alpha+2n\ln(1-\mu).$$

Accurate to terms of second order in n/α we obtain in analogy with (2.11), (2.15), and (2.18)

$$\frac{\ln\sigma_L(\Gamma)}{L}\bigg|_{L\to\infty}=-2\alpha+2n\ln(1-\mu)+\frac{n^2}{\alpha}C_2(\mu),$$
$$C_2(\mu)=\frac{1}{2}\int_0^\infty\ln\left[\left(1+\frac{\mu\nu(x)}{1-\mu}\right)\left(1-\frac{\mu^2\nu(x)}{(1-\mu)^2}\right)\right]dx. \tag{2.25}$$

As expected, the limit $\ln\sigma_L(\Gamma)/L|_{L\to\infty}$ does not coincide with the previously calculated asymptotic value $\ln G^L/L$. To compare the obtained expressions (2.21) and (2.25) with the transparency of an ordered system we use Eq. (1.17) and express $\check{S}$ in the form

$$S=(\check{s})^N,\quad N=nL,$$
$$\check{s}=\check{A}\left(\frac{L}{2N}\right)\check{T}\check{A}\left(\frac{L}{2N}\right)=\begin{bmatrix} e^{\alpha/2n}(1+\beta/2\alpha) & \beta/2\alpha \\ -\beta/2\alpha & e^{-\alpha/2n}(1-\beta/2\alpha)\end{bmatrix}. \tag{2.26}$$

Hence $\Lambda_{max}=\lambda_1^N$, where λ_1 is the larger eigenvalue of the matrix $\check{s}$. This yields

$$-\ln\sigma_L=\ln\Lambda^2_{max}=2nL\ln[\xi+(\xi^2-1)^{1/2}],$$
$$\xi=\mathrm{ch}\frac{\alpha}{2n}+\frac{\beta}{2\alpha}\mathrm{sh}\frac{\alpha}{2n}. \tag{2.27}$$

At $n/\alpha\ll 1$ we obtain, accurate to terms $\sim e^{-\alpha/2n}$,

$$-\frac{1}{L}\ln\left(\frac{\sigma_L}{\sigma_L|_{\beta=0}}\right)=-2n\ln\left(1+\frac{\beta}{2\alpha}\right)=2n\ln(1-\mu),$$

which coincides with the term of first order in n in (2.25).

3. Resonant tunneling

We proceed now to investigate tunneling of particles having an energy E within the spectrum of the random operator $\hat{H}$. We start from the exact formula (1.10), which we write out in explicit form:

$$\frac{1}{\sigma}=|(\mathbf{f},S\mathbf{g})|^2=\frac{1}{4}\left[\frac{(k^2-\alpha^2)}{\alpha k}\frac{(S_{11}-S_{22})}{2}\right.$$
$$\left.+\frac{(k^2+\alpha^2)}{\alpha k}\frac{(S_{21}-S_{12})}{2}\right]^2+\frac{1}{4}(S_{11}+S_{22})^2. \tag{3.1}$$

Writing down the unimodularity condition $\check{S}(D(\check{S})=1)$ in the form

$$(S_{11}+S_{22})^2+(S_{21}-S_{12})^2-(S_{11}-S_{22})^2-(S_{12}+S_{21})^2=4 \tag{3.2}$$

and introducing the notation

$$\rho^2=(S_{11}-S_{22})^2/4+(S_{21}+S_{12})^2/4,$$
$$\cos\xi=\frac{S_{11}+S_{22}}{2(1+\rho^2)^{1/2}},\quad \sin\xi=\frac{S_{21}-S_{12}}{2(1+\rho^2)^{1/2}}, \tag{3.3}$$
$$\cos\varphi=(S_{12}+S_{21})/2\rho,\quad \sin\varphi=(S_{11}-S_{22})/2\rho,$$

we reduce (31) to the final form

$$\frac{1}{\sigma}=(1+\rho^2)\left\{\cos^2\xi+\frac{(k^2+\alpha^2)^2}{4\alpha^2k^2}\left[\sin\xi+\frac{(k^2-\alpha^2)\rho}{(k^2+\alpha^2)(\rho^2+1)^{1/2}}\sin\varphi\right]^2\right\}. \tag{3.4}$$

Each of the quantities S_{ik} oscillates rapidly as a function of Γ and E, and varies in tremendous intervals $(-e^{\alpha L}, e^{\alpha L})$. The condition for exact resonant tunneling, correspondent to a complete transparency of the barrier $(\sigma_L=1)$, is

$$\rho=0,\quad \sin\xi=0.$$

This corresponds to the three equations

$$S_{11}-S_{22}=0,\quad S_{12}+S_{21}=0,\quad S_{12}-S_{21}=0. \tag{3.5}$$

However, recognizing that the quantity in the curly brackets in (3.4) always remains ~ 1 at $k/\alpha+\alpha/k\sim 1$, and varies in the interval

$$1-\frac{\rho^2}{(\rho^2+1)}\left(\frac{k^2-\alpha^2}{k^2+\alpha^2}\right)^2\leqslant\{\ldots\}\leqslant\frac{1}{4}\left(\frac{k^2+\alpha^2}{\alpha k}+\left|\frac{k^2-\alpha^2}{\alpha k}\right|\left|\frac{\rho}{(\rho^2+1)^{1/2}}\right|\right)^2,$$

we see that we can write with sufficient accuracy[5]

$$\sigma_L(E,\Gamma)\cong[1+\rho^2(E,\Gamma)]^{-1}. \tag{3.6}$$

(The equal sign in (3.6) occurs when $k/\alpha=1$.) Therefore the third condition of "exact resonance" $S_{12}=S_{21}$ does not impose any restrictions on the resonance transparency $(\sigma_L\sim 1)$, and we retain as the "exact resonance" conditions the following two equations that result from $\rho^2(E,\Gamma)=0$:

$$X(E,\Gamma)\equiv S_{11}-S_{22}=0,\quad Y(E,\Gamma)\equiv S_{12}+S_{21}=0. \tag{3.7}$$

At a given energy E, Eqs. (3.7) define a resonant $(N-2)$-dimensional hypersurface in the configuration space Γ_N. On the other hand, the very same equations can be regarded as conditions for the determination of the resonance energy level $E_0(\Gamma_N)$ for the given configuration Γ_N. In a small vicinity of each exact-resonance point $\rho(E_0,\Gamma_N)=0$ resonant tunneling with transparency $\sigma(E,\Gamma_N)\sim 1$, is preserved so long as $\rho(E,\Gamma_N)\sim 1$. The energy width δE of the resonant transparency is therefore determined, for a concrete realization of Γ_N by the obvious relation

$$\delta E\sim\left(\frac{\partial^2\rho^2(E,\Gamma_N)}{\partial E^2}\bigg|_{E_0(\Gamma_N)}\right)^{-1/2}. \tag{3.8}$$

Rewriting the condition $\rho(E,\Gamma_N)\sim 1$ in the form

$$X(E,\Gamma_N)\sim 1,\quad Y(E,\Gamma_N)\sim 1, \tag{3.9}$$

we obtain, at a fixed energy E, the condition for the width of the "resonant layer" around the hypersurfaces (3.7) in the Γ_N space:

$$\delta X\sim 1,\quad \delta Y\sim 1. \tag{3.10}$$

If the resonance is sharp enough, the order of magnitude of the averaged transparency is determined by the relative phase volume of this resonant layer, $(\Delta\Gamma)_r/\Delta T$. We must, however, make here an important remark. We shall consider henceforth the case of low concentration of the scattering centers, $c=n/\alpha\ll 1$. This corresponds to $N\ln N\ll\mathscr{L}=\alpha L$, and only small N are of importance for not too large $\mathscr{L}$, which by itself ensures the required sharpness of the resonance. On the other hand if $N\gg 1$, then in some special "degeneracy" cases the main contribution to the average transparency are made, as we shall show, by configurations with $|\ln\sigma(E,\Gamma_N)|\sim N\ll\mathscr{L}$, which in these cases must be regarded as resonant (see Eq. (3.27) below).

We note finally that it follows from (3.2) and (3.3) that the requirement $\rho\sim 1$ leads automatically to the conditions

$$S_{11}\sim 1,\quad S_{12}\sim 1,\quad S_{21}\sim 1,\quad S_{22}\sim 1. \tag{3.11}$$

The first of these conditions, $S_{11}\sim 1$, explains why the resonant energy levels are close to the eigenvalues of the operator $\hat{H}_\Gamma$. This is due to the giant scale of the oscillations $S_{11}(E,\Gamma)$ near the eigenvalue E^0 defined by the condition $S_{11}(E^0,\Gamma)=0$. The converse, however, does not hold true: proximity of E to the eigenvalue $E^0(\Gamma)$ still does not mean resonant transparency.

One can expect from the foregoing general analysis that resonant configurations, exist near each energy level in the spectrum of the random operator $\hat{H}$ and, conversely, for general-position configurations that satisfy a certain inequality there exist resonant energy values corresponding to $\sigma(\Gamma)\sim 1$. It will be shown below, however, that the energy width of such resonance, or equivalently the probabilities of the resonant configurations for a given energy level, are frequently so small that their relative contribution to the average resonant transparency is of no interest. In the qualitative analysis that follows we shall therefore consider those situations in which the resonance effects are of greatest importance. This can be expected primarily in the vicinity of a level on an individual scattering center.

The general approach referred to above is convenient when $N>2$. Therefore, proceeding to a more concrete analysis, we begin with the cases $N=1$ and $N=2$. Being simpler, they will help understand also a more general situation.

1. Case $N=1$ (one center at the point z_1). Reckoning the energy from the eigenvalue E^0 defined by the condition $S_{11}(E^0,z_1)=0$, and introducing the dimensionless energy $\varepsilon=(E-E^0)/2\alpha_0^2$, $\alpha_0^2=-E^0$, we write down the matrix $\check{T}$ and the propagator $\check{S}$ in the form

$$\check{T}=\begin{bmatrix}\varepsilon & -1\\ 1 & 2\end{bmatrix},\quad \check{S}=\begin{bmatrix}\varepsilon e^{\mathscr{L}} & -e^{2x_1}\\ e^{-2x_1} & 2e^{-\mathscr{L}}\end{bmatrix},$$

$$x_1=\alpha(L/2-z_1),\quad \mathscr{L}=\alpha L.$$

Then, according to (3.6)

$$\sigma(\varepsilon,x_1)\cong(1+\operatorname{sh}^2 2x_1+\gamma_1^2/4)^{-1},\quad \gamma_1=\varepsilon e^{\mathscr{L}}.$$

(The exact equality takes place in this formula at $k/\alpha=1$.)

Recognizing that the probability of finding the center in the interval dx_1 is equal to

$$p_1dx_1=ce^{-c\mathscr{L}}dx_1,\quad c=n/\alpha\ll 1,$$

we obtain in accord with (1.11)

$$\sigma_1(\varepsilon)=\langle\sigma(\varepsilon,x_1)\rangle=\int p_1\sigma(\varepsilon,x_1)dx_1=ce^{-c\mathscr{L}}\Omega_1(\varepsilon),$$

where

$$\Omega_1(\varepsilon)=\int\sigma(\varepsilon,x_1)dx_1=\frac{2}{\varepsilon^2e^{2\mathscr{L}}}\ln(1+\varepsilon^2e^{2\mathscr{L}}).$$

The function $\Omega_1(\varepsilon)$ is concentrated in the vicinity of the point $\varepsilon=0$ and its width is $\delta\varepsilon\sim e^{-\mathscr{L}}$. In those cases when the natural energy width of the beam incident on the barrier is $\Delta\varepsilon\gg\delta\varepsilon\sim e^{-\mathscr{L}}$, the function $\Omega_1(\varepsilon)$ can be replaced by an effective δ function with a normalization coefficient[6])

$$\sigma_1(\varepsilon)\to\sigma_1\delta(\varepsilon),\quad \sigma_1=\int_{-\Delta}^{\Delta}\sigma_1(\varepsilon)d\varepsilon=4\pi ce^{-(1+c)\mathscr{L}}. \tag{3.12}$$

2. The case $N=2$ (two centers at the points $z_1,z_2;z_2>z_1$). In this case the propagator $\check{S}$ is of the form

$$S=\begin{bmatrix}(\varepsilon^2-e^{2y})e^{\mathscr{L}} & -(\varepsilon e^y+2e^{-y})e^{-\xi}\\ (\varepsilon e^y+2e^{-y})e^{\xi} & (4-e^{2y})e^{-\mathscr{L}}\end{bmatrix},$$

where $y=\alpha(z_2-z_1)$, $\xi=\eta_2-\eta_1$, $\eta_1=\alpha z_1$, $\eta_2=\alpha(L-z_1)$. Writing down Eqs. (3.7), which now take the form

$$X(\varepsilon,y)=(\varepsilon^2-e^{-2y})e^{\mathscr{L}}-(4-e^{-2y})e^{-\mathscr{L}}=0,$$
$$Y(\varepsilon,y,\xi)=2(\varepsilon e^y+2e^{-y})\operatorname{sh}\xi=0,$$

we obtain, accurate to the next-order exponentially small terms, the resonant energy ε_0 the resonant configuration ξ_0:

$$\varepsilon_0\approx\pm e^{-y},\quad y\leqslant\mathscr{L}/2;\quad \xi_0=0.$$

From (3.8) we get the energy width of the resonant transparency

$$\delta\varepsilon\sim e^{-\mathscr{L}}/\varepsilon_0\approx e^{-\mathscr{L}+y},$$

and from conditions (3.10) we get the width of the resonant layer in Γ_2 space:

$$\delta y\sim\begin{cases}1, & \varepsilon\leqslant e^{-\mathscr{L}/2}\\ e^{-\mathscr{L}}/\varepsilon^2, & \varepsilon\gg e^{-\mathscr{L}/2}\end{cases};\quad \delta\xi\sim\begin{cases}|\ln(\varepsilon e^{\mathscr{L}/2})|, & e^{-\mathscr{L}}<\varepsilon\ll e^{-\mathscr{L}/2}\\ 1, & \varepsilon\gg e^{-\mathscr{L}/2}\end{cases}.$$

Recognizing further that the probability of finding the centers in the intervals dz_1 and dz_2 is equal to

$$p_2dz_1dz_2=c^2e^{-c\mathscr{L}}dz_1dz_2,\quad z_2>z_1,$$

we obtain

$$\sigma_2(\varepsilon)=c^2e^{-c\mathscr{L}}\Omega_2(\varepsilon), \tag{3.13}$$

where

$$\Omega_2(\varepsilon)\approx\int_{\Delta\Gamma_2(\varepsilon)|_{\rho\sim1}}dy\,d\xi\sim\delta y\delta\xi\sim\begin{cases}\mathscr{L}/2, & \varepsilon<e^{-\mathscr{L}}\\ |\ln\varepsilon e^{\mathscr{L}/2}|, & e^{-\mathscr{L}}<\varepsilon\leqslant e^{-\mathscr{L}/2},\\ 1, & \varepsilon\sim e^{-\mathscr{L}/2}\\ e^{-\mathscr{L}}/\varepsilon^2, & \varepsilon\gg e^{-\mathscr{L}/2}\end{cases}$$

The function $\Omega_2(\varepsilon)$ has a width $\delta\varepsilon\sim e^{-\mathscr{L}/2}$ and under the same conditions as in the preceding case [see (3.12)] can be replaced by an effective δ function. Then

$$\sigma_2(\varepsilon)\sim c^2e^{-\mathscr{L}(1/2+c)}\delta(\varepsilon). \tag{3.14}$$

3. We consider now the general case, when N scattering centers with coordinates $(z_1,z_2,\ldots,z_N)$, $z_1<z_2<\ldots<z_N<L$ are located on a separate filament. To investigate the conditions (3.7) we represent the matrix $\check{S}$ in the form

$$\check{S}=\check{A}(\eta_1)\check{Q}(y)\check{A}(\eta_2),\quad \check{Q}(y)=\check{T}\check{A}(y_1)\check{T}\ldots\check{T}\check{A}(y_{N-1})\check{T};$$
$$\eta_1=\alpha z_1,\quad \eta_2=\alpha(L-z_N),\quad y_k=\alpha(z_{k+1}-z_k),\quad y_k>0; \tag{3.15}$$

$y=(y_1,y_2,\ldots,y_{N-1})$ is an $(N-1)$-dimensional vector. Equations (3.7) then become

$$X=S_{11}-S_{22}=e^{\eta}Q_{11}-e^{-\eta}Q_{22}=0,$$
$$Y=S_{12}+S_{21}=e^{\xi}Q_{12}+e^{-\xi}Q_{21}=0; \tag{3.16}$$
$$\eta=\eta_1+\eta_2=\mathscr{L}-\sum_{k=1}^{N-1}y_k,\quad \xi=\eta_2-\eta_1.$$

Taking into account the definition (3.15) of the matrix $\check{Q}(y)$ and the explicit forms of the matrices $\check{T}$ and $\check{A}$, we note that the left-hand sides of Eqs. (3.16) are polynomials with small powers $\varepsilon \ll 1$. A detailed analysis of such equations is presented in a paper by one of us.[1] It is shown there that for general-position configurations the most probable is localization of the particle wave function in the vicinity of two centers located at the shortest distance. The root of the first equation of (3.16) is then, accurate to the next-order exponentially small terms,

$$\varepsilon=\varepsilon_0=\pm e^{-y_m}, \quad y_m=\min(y_1, y_2, \ldots, y_{N-1}, \eta)$$
$$(X=S_{11}-S_{22}=\varepsilon^N e^{\mathscr{L}}-\varepsilon^{N-2}\exp(\mathscr{L}-2y_m)-\ldots=0) \tag{3.17}$$

and is practically insensitive to changes of any distance $y_i, i \neq m$, at $\exp(y_m - y_i) \ll 1$.

From the second equation of (3.16) we get

$$e^{2\xi}=-Q_{21}(y)/Q_{12}(y), \quad Q_{21}/Q_{12}<0, \quad \xi<\eta. \tag{3.18}$$

Taking next (3.8) into account, we obtain the energy width of the resonant transparency near the energy $\varepsilon = \varepsilon_0$:

$$\delta\varepsilon \sim e^{-\mathscr{L}}/\varepsilon_0^{N-1} \sim \exp\{-\mathscr{L}+(N-1)y_m\}, \tag{3.19}$$

and from the conditions (3.10), (3.11), and (3.16) we get the width of the resonant layer

$$\delta y_m \sim e^{-\mathscr{L}}/\varepsilon_0^N \sim \exp\{-\mathscr{L}+Ny_m\}, \quad \delta\xi \sim 1/(S_{12}-S_{21}) \sim 1. \tag{3.20}$$

The remaining distances y_i and η take on arbitrary values bounded by the inequalities

$$y_j'=y_j-y_m>0, \quad \eta'=\eta-y_m>0, \quad \sum_1^{N-1} y_j'+\eta'=\mathscr{L}-Ny_m. \tag{3.20a}$$

Therefore the resonant phase volume is

$$\Delta\Gamma_N(\varepsilon) \sim e^{-(\mathscr{L}-Ny)}\frac{(\mathscr{L}-Ny)^N}{N!} \sim e^{-N(u-\ln ue)}, \quad u=\frac{\mathscr{L}}{N}-y, \quad \varepsilon=e^{-y}. \tag{3.21}$$

Relations (3.20) and (3.21) correspond to "general-position resonant configurations" for the energies $\varepsilon e^{\mathscr{L}/N} = e^u \gg 1$.

In the opposite limiting case $\varepsilon e^{\mathscr{L}/N} \ll 1$, we construct for the general-position resonant configuration a systematic scheme that makes it easy to determine their statistical weights. We take an arbitrary system of segments $x_k > 0$ satisfying the relation

$$\sum_1^{N+1} x_k = \mathscr{L},$$

and transform the configuration $\Gamma_N\{z\} \equiv \Gamma_N^*\{x\}$ in accordance with them (see Fig. 1):

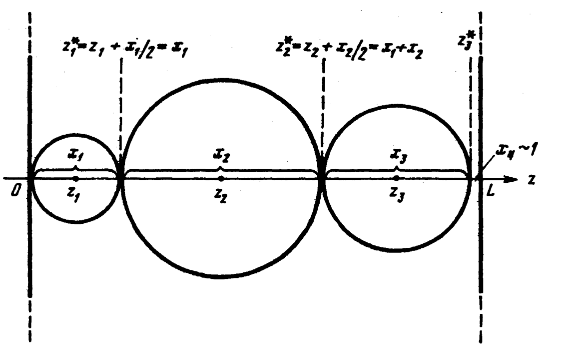

FIG. 1. Example of resonant configuration of general position $\Gamma_N^*\{x\}$ for the case $N=3$ at energies $\varepsilon \lesssim e^{-x_2}$.

$$z_1=x_1/2, \quad y_k=z_{k+1}-z_k=(x_k+x_{k+1})/2 \quad (\alpha z_k \to z_k). \tag{3.22}$$

For this configuration, the matrix $\check{S}$ can be written in the form

$$\check{S}=\check{s}_1\check{s}_2\ldots\check{s}_N\check{A}(x_{N+1}), \quad \check{s}_j=\check{A}\left(\frac{x_j}{2}\right)\check{T}\check{A}\left(\frac{x_j}{2}\right)=\begin{bmatrix}\varepsilon e^{x_j} & -1\\ 1 & 2e^{-x_j}\end{bmatrix}. \tag{3.23}$$

Let $x_k = x_{\max} \equiv y$ be the largest of the x_j. Then at $\varepsilon \lesssim e^{-y}$, for any general-position configuration ($e^{x_j-y} \ll 1/N, e^{-x_j} \ll 1/N, \mathscr{L} \gg N\ln N$), we have

$$\check{S}(\varepsilon)=\check{s}_0^k\check{s}(\gamma)\check{s}_0^{N-k-1}\check{A}(x_{N+1}), \quad \gamma=\varepsilon e^y \leqslant 1;$$
$$\check{s}(\gamma)=\begin{vmatrix}\gamma & -1\\ 1 & 0\end{vmatrix}, \quad \check{s}_0=\check{s}(0)=\begin{vmatrix}0 & -1\\ 1 & 0\end{vmatrix}, \quad \check{s}_0^{2n}=(-1)^n. \tag{3.24}$$

Since $\check{s}_0^k$ corresponds to complete absence of damping, the only additional condition for resonant transparency $\sigma \sim 1$ is $x_{N+1} \sim 1$. The resonance width is then obtained from the condition $\gamma = \varepsilon e^y \sim 1$ and is given by the expression $\delta\varepsilon \sim e^{-y}$. Thus, the resonant configurations for the energy $\varepsilon = e^{-y}, y > \mathscr{L}/N$ are determined by the conditions

$$0<x_j<y=-\ln\varepsilon, \quad \sum_1^N x_j=\mathscr{L}.$$

In particular, at $\varepsilon < e^{-\mathscr{L}}$ all the configurations $\Gamma_N^*\{x\}$ (i.e. $x_j > 0, \sum x_j = \mathscr{L}$) are resonant ($\sigma \sim 1$) and conversely, the remaining configurations are not. It is easily seen that the set $\{\Gamma_N^*\}$ does not account for all the configurations Γ_N. Since the Jacobian of the transition from the variables y to the variables x_j is according to (3.22) $\partial\{y\}/\partial\{x\} = (\frac{1}{2})^N$, and the restrictions on these variables coincide, it follows that the ratio of the total phase volumes is $\Delta\Gamma_N^*/\Delta\Gamma_N = (\frac{1}{2})^N$. Thus, at $\varepsilon < e^{-\mathscr{L}}$ the resonant phase volume is

$$\Delta\Gamma_N(0)=(\mathscr{L}/2)^N/N!=\exp\{N\ln(e\mathscr{L}/2N)\}, \tag{3.25}$$

and at arbitrary $\varepsilon = e^{-y} \ll e^{-\mathscr{L}/N}$ we have

$$\Delta\Gamma_N(\varepsilon) \sim (1/2)^N\varphi_N(y,\mathscr{L}), \quad \varphi_N(y,\mathscr{L})=\int_0^y\ldots\int_0^y\delta\left(\sum_1^N x_j-\mathscr{L}\right)dx_1\ldots dx_N,$$

$$\ln\varphi_{N+1}(y,\mathscr{L})=\begin{cases}N\ln(y-\mathscr{L}/N)e, & y-\mathscr{L}/N \ll \mathscr{L}\ln N/N^2\\ N\ln\mathscr{L}e/N, & y-\mathscr{L}/N \gg \mathscr{L}\ln N/N\end{cases}$$
$$(\mathscr{L} \gg N\ln N \gg 1). \tag{3.26}$$

The seemingly paradoxical fact that damping does not take place along the sequence of points $z_k^* = z_k + x_k/2$ for the general-position configurations Γ^*, but does take place for chains of equidistant resonant centers has a simple physical explanation: the definition of the transparency refers to damping relative to the input amplitude of the wave function, and not to amplitudes at the points of the maxima $z = z_k$ (as in the standard formulation of the problem).

In the energy region $\varepsilon \sim e^{-\mathscr{L}/N}$ the main contribution to the resonant transparency is determined by degenerate configurations close to an ordered arrangement of the centers at equal distances $y_k = x_k = \mathscr{L}/N, \eta_1 = \eta_2 = \mathscr{L}/2N$. In this case, as can be understood from the foregoing, the resonance conditions are determined by the relations

$$\delta y_k \sim \delta x_k \sim \delta\eta_1 \sim \delta\eta_2 \sim 1. \tag{3.27}$$

The conditions (3.27) correspond to an optimal relation between the increasing phase volume $\Delta\Gamma_N$ and the decreasing unbalanced transparency $\sigma(\varepsilon, \Gamma_N)$:

$|\ln\sigma(\varepsilon, \Gamma_N)|\sim\ln\Delta\Gamma_N\sim N.$

Starting from (3.21) and (3.26) and recognizing that the probability $p(\Gamma_N)d\Gamma_N$ of landing in the phase-space element $d\Gamma_N$ is $c^N e^{-c\mathscr{L}}d\Gamma_N(z_1<z_2<\ldots<z_N<L)$, we obtain the magnitude and the form of the maximum of the average resonant transparency in the vicinity of the impurity level:

$$\sigma_N(\varepsilon)=\langle\sigma(\varepsilon, \Gamma_N)\rangle=c^N e^{-c\mathscr{L}}\Omega_N(\varepsilon),\quad \Omega_N\sim\exp\{N\omega_N(u)\},\quad \varepsilon=e^{-\mathscr{L}/N-u};$$

$$\omega_N(u)=\begin{cases}\ln(e\mathscr{L}/2N); & u\gg\mathscr{L}\ln N/N\\ \ln(eu/2); & \ln N\ll u\ll\mathscr{L}\ln N/N^2.\\ \lambda(u),\quad |\lambda|\sim1; & |u|\sim1\\ -|u|+\ln(|u|/e); & u<0,\quad 1\ll|u|\end{cases}\tag{3.28}$$

If the effective width of the function $\sigma_N(\varepsilon)$ is much less in energy $\delta\varepsilon\sim e^{-\mathscr{L}/N}$ than the energy width of the particle or beam of particles incident on the barrier, then $\sigma_N(\varepsilon)$ can be replaced by an effective δ function [see (3.12)]. This yields $\sigma_N(\varepsilon)\to\sigma_N\delta(\varepsilon)$,

$$\sigma_N\sim\exp\{-N\{|\ln c|+\lambda_0\}-\mathscr{L}/N-c\mathscr{L}\},\quad \lambda_0\sim1\tag{3.29}$$

($\lambda_0\ll|\ln c|$, but the accuracy of the employed estimates is insufficient to determine its numerical value).

In the summation over N, the main contribution to $\sigma=\sum\sigma_N$ is made by filaments with extremal transparency. Varying (3.29) with respect to N, we obtain the value $\tilde{N}=(\mathscr{L}/(\lambda_0+|\ln c|))^{1/2}$, that makes the largest contribution to the average transparency. The values of the transparency and of the effective damping coefficients are themselves given by

$$\sigma\sim\exp\{-2[\mathscr{L}(|\ln c|+\lambda_0)]^{1/2}-c\mathscr{L}\},\quad -\mathscr{L}^{-1}\ln\sigma\cong2(|\ln c|/\mathscr{L})^{1/2}-c.\tag{3.30}$$

On the other hand, if the concentration c is defined on each individual filament, then only a single term σ_N corresponds to $N=\langle N\rangle=c\mathscr{L}$ and the normalization factor $e^{-c\mathscr{L}}$ does not enter in (3.28). Then

$$\sigma\sim\exp\left\{-c\mathscr{L}(|\ln c|+\lambda_0)-\frac{1}{c}\right\},\quad -\frac{\ln\sigma}{\mathscr{L}}\cong c|\ln c|+\frac{1}{c\mathscr{L}}.\tag{3.31}$$

We have investigated above the resonant transparency in the vicinity of the single-center local level $\varepsilon=0$. It can analogously be investigated also in the vicinity of discrete local levels[7] made up of several close centers (cluster levels). By regarding the cluster as a single effective scattering center having an internal structure, it is easy to construct a matrix $\check{\theta}$ that propagates the solution through the indicated cluster, in the form of an ordered product of the matrices $\check{T}$ and $\check{A}$ (thus, e.g., for a cluster of two centers we have $\check{\theta}=\check{T}\check{A}\check{T}$). Reckoning next the energy from the corresponding cluster level ε_k^0 defined by the condition $\theta_{11}(\varepsilon_k^0)=0$, we obtain the matrix $\check{\theta}$ and the propagator $\check{S}$ in the region of energies $\varepsilon'=\varepsilon-\varepsilon_k^0\ll1$ close to the cluster level ε_k^0. When "general-position configurations" are considered, it turns out in this case that only the matrix element θ_{11} is small in proportion to the proximity of the energy to the cluster level ($\theta_{11}\sim\varepsilon'$), while all the remaining matrix elements θ_{ik} and T_{ik} do not possess this smallness. In this case the situation is similar to that occurring when there is only one center on a filament in the region of the barrier. The energy width of the resonant transparency is then $\delta\varepsilon'\sim e^{-\mathscr{L}}$. On the other hand the largest contribution to the resonant transparency at these energies is made by degenerate configurations close to an ordered arrangement of identical clusters at equal distances from one another. The corresponding energy width of the resonant transparency turns out to be $\delta\varepsilon'\sim e^{-\mathscr{L}/N_0}$($N_0$ is the number of clusters on the filament).

To conclude this section, we make a few remarks.

1. Since we did not take into account in the investigation of the resonant tunneling the interaction between the tunneling particles, while in the vicinity of the scattering centers their density is high enough, it follows that the condition of applicability of the results imposes a limitation on the density of the particles in the incident beam, namely, it is necessary that the density be small enough to be able to neglect the interaction between the particles within the entire region of the barrier.

2. Since we have used in fact only an integral or matrix form of the Schrödinger equation, a similar analysis is possible also for the finite-difference equations that describe tunneling of elementary excitations. In this case, naturally, the conditions for matching the wave function on the barrier boundary and the law of dispersion of the elementary excitations $\alpha(E)$ may change somewhat, but these circumstances are not essential in the described procedure.

3. The entire investigation of the multiple resonances is based on the initial assumption that the potential barrier is homogeneous and the scattering centers are identical. Violation of these conditions leads readily to an unbalance of the joint resonances. In an unbalanced system, there appears only one resonance (on an individual center or optimal cluster) and therefore $\sigma(E_r)\sim c(E_r)e^{-\mathscr{L}}$ where $c(E_r)$ is the concentration of such clusters for the level E_r.

The presence of inelastic scattering channels also weakens the resonance effects.

II. THREE-DIMENSIONAL SYSTEM

4. Nonresonant tunneling

Proceeding to the three-dimensional case, we start with certain general remarks. Owing to the sub-barrier damping, it is natural to expect that the "quasiclassical tunneling paths," which take into account the multiple scattering, make a substantial contribution if they are close to the shortest ones, i.e., if they lie inside almost right cylinders of radius[8] $\sim1/\alpha$. These paths are independent if the cylinders corresponding to them are separated by distances $\gg1/\alpha$. These cylinders play the role of the individual filaments in the one-dimensional case. This explains why the picture of the tunneling retains in the three-dimensional case many features of the one-dimensional situation.

The equation for the wave function inside a barrier with scattering centers is given by

$$\Delta\Psi(\mathbf{r})-\alpha^2\Psi(\mathbf{r})=\sum_j\hat{u}_j\Psi,\quad \mathbf{r}=(z,\rho),\tag{4.1}$$

where $\hat{u}_j$ is the operator of the local perturbation from a center at the point $\mathbf{r}_j$ with a radius of action $r_0(\alpha r_0 \ll 1)$.

The solution of Eq. (4.1) with a zero right-hand side is given by

$$\Psi_0(z,\rho)=\chi_-(z,\rho)+\chi_+(z,\rho), \tag{4.2}$$

where the functions $\chi_\pm(z,\rho)$ satisfy the equations

$$\partial\chi_\pm(z,\rho)/\partial z=\pm\hat{\alpha}\chi_\pm(z,\rho) \tag{4.2a}$$

and are equal to

$$\chi_-(z,\rho)=e^{-z\hat{\alpha}}\chi_-(0,\rho),\quad \chi_+(z,\rho)=e^{(z-L)\hat{\alpha}}\chi_+(L,\rho);$$

$\hat{\alpha}$ is an operator acting on functions of ρ and is diagonal in the $(z,\varkappa)$ representation (Fourier representation in ρ);

$$[\hat{\alpha}\chi(z,\rho)]_\varkappa=\alpha_\varkappa\chi_\varkappa(z),\quad \alpha_\varkappa^2=\alpha^2+\varkappa^2;$$

$$\chi_\varkappa(z)=\int\chi(z,\rho)e^{i\varkappa\rho}d^2\rho.$$

The general solution of (4.1), on the other hand, will be written using the Green's function of Eq. (4.1) in an infinitely long homogeneous barrier[9]:

$$\Psi(z,\rho)=e^{-z\hat{\alpha}}\chi_-(0,\rho)+e^{(z-L)\hat{\alpha}}\chi_+(L,\rho)-\sum_j\hat{h}\varphi_j, \tag{4.3}$$

where $\hat{h}$ is an integral operator whose kernel is the Green's function

$$h(\mathbf{r}-\mathbf{r}')=\exp\{-\alpha|\mathbf{r}-\mathbf{r}'|\}/4\pi|\mathbf{r}-\mathbf{r}'|, \tag{4.3a}$$

$$\hat{h}\varphi_j=\int h(\mathbf{r}-\mathbf{r}')\varphi_j(\mathbf{r}')d^3r', \tag{4.3b}$$

$$\varphi_j\equiv\varphi_j(\mathbf{r})\equiv\varphi(\mathbf{r},\mathbf{r}_j)=\hat{u}_j\Psi. \tag{4.3c}$$

Using the same reasoning as in the one-dimensional case [see (1.16) and the remark that follows this formula], in the energy region outside the resonant spectrum we shall consider tunneling through the layer $0<z<L$ in homogeneous space, i.e., we omit $\chi_+(L,\rho)$ from (4.3) and assume that $\chi_-(0,\rho)=g^0(\rho)$, where the "initial condition" $g^0(\rho)$ will be connected below with the correlators of the incident particle flux.

We introduce the notation $\chi_-(z,\rho)=g^z(\rho)$. Then, taking (4.2) and (4.3) into account, we write

$$\Psi(z,\rho)=e^{-z\hat{\alpha}}g^0(\rho)-\sum_j\hat{h}\varphi_j, \tag{4.4}$$

$$g^z(\rho)\equiv\chi_-(z,\rho)=e^{-z\hat{\alpha}}g^0(\rho)-\sum_{0<z_j<z}\hat{h}\varphi_j, \tag{4.5}$$

$$\chi_+(z,\rho)=-\sum_{z<z_j<L}\hat{h}\varphi_j. \tag{4.6}$$

Applying the operator $\hat{u}_j$ to the left and right sides of (4.4) and solving the resultant equation with respect to φ_j, we get

$$\varphi_j=\hat{\mu}_je^{-z\hat{\alpha}}g^0-\hat{\mu}_j\sum_{k\neq j}\hat{h}\varphi_k,\quad \hat{\mu}_j=\hat{u}_j(1+\hat{h}\hat{u}_j)^{-1}, \tag{4.7}$$

where $\hat{u}_j$ is the operator of scattering by the center at the point $\mathbf{r}_j$. Equations (4.5) and (4.7) make up a closed system.

By virtue of the locality of the perturbation $\hat{u}_j$, the kernel of the operator $\hat{u}_j$ and the function φ_j are concentrated in regions having the same radius $r_0\ll 1/\alpha$. If, as we shall assume hereafter, the distances between the scattering centers are $|\mathbf{r}_j-\mathbf{r}_k|=r_{jk}\gg r_0$, then we can put in (4.6)

$$(\hat{\mu}_j)_{\mathbf{r},\mathbf{r}'}=\mu\delta(\mathbf{r}-\mathbf{r}_j)\delta(\mathbf{r}'-\mathbf{r}_j),\quad \varphi_k=\tilde{\varphi}_k\delta(\mathbf{r}-\mathbf{r}_k), \tag{4.8}$$

where $\mu=(\hat{\mu}1,1)$ is the sub-barrier scattering amplitude. Its poles in the energy plane correspond to bound states localized in the vicinity of an individual center. Far from these poles, the amplitude μ is small $(\mu\alpha\ll 1)$ (see the Appendix).

When (4.8) is taken into account, Eqs. (4.5) and (4.7) become

$$g^z(\rho)=e^{-z\hat{\alpha}}g^0(\rho)-\sum_{0<z_j<z}h(\mathbf{r}-\mathbf{r}_j)\tilde{\varphi}_j, \tag{4.9}$$

$$\tilde{\varphi}_j=\mu\exp\{-z_j\hat{\alpha}\}g^0(\rho_j)-\mu\sum_{k\neq j}h(\mathbf{r}_j-\mathbf{r}_k)\tilde{\varphi}_k. \tag{4.10}$$

If the density of the scattering centers over the damping length is large $(n/\alpha^3\gg 1)$, then the fluctuation effects are small (in terms of $\alpha^3/n\ll 1$) and the initial equation (4.1) goes over into the smoothed equation

$$\Delta\Psi-(\alpha^2+n\mu)\Psi=0, \tag{4.11}$$

which corresponds to the shift of the renormalized boundary of the continuous spectrum of $\hat{H}$.[1] Equation (4.11) is obtained by local-macroscoping averaging of (4.1), wherein, as seen from a comparison of (4.4) and (4.10),

$$\left\langle\sum\hat{u}_j\Psi\right\rangle=\left\langle\sum\varphi_j\right\rangle\to n\varphi(\mathbf{r})\to n\mu\Psi(\mathbf{r}).$$

In the opposite case of interest to us, that of low density under nonresonant conditions, we use a method similar to that used in Sec. 2 above.

In contrast to the one-dimensional case, in the three-dimensional case the sub-barrier scattering leads to a change of the transverse momentum $\varkappa$ of the tunneling particles, and our task is to investigate both the spectral and the integral transparency of the barrier. To this end it is necessary to obtain at $z=L$ the two-point correlator[10]

$$G^L(\rho)=\langle\Psi(L,\rho)\Psi^*(L,0)\rangle=\langle g^L(\rho)g^{*L}(0)\rangle \tag{4.12}$$

for the given boundary condition

$$G^0(\rho)=\langle g^0(\rho)g^{*0}(0)\rangle,\quad G^0(0)=1.$$

The Fourier component of the correlator $G_\varkappa^L$ determines the spectral transparency, while the integral transparency is

$$\sigma_L\sim\int G_\varkappa^L d^2\varkappa. \tag{4.13}$$

We proceed to derive an equation for $G^z(\rho)$. From (4.6) and (4.7) we have the relations

$$g^{z+t}(\rho)=e^{-t\hat{\alpha}}g^z(\rho)-\sum_{z<z_j<z+t}\hat{h}\varphi_j, \tag{4.14}$$

$$\varphi_j=\hat{\mu}_j\exp\{-t_j\hat{\alpha}\}g^z-\hat{\mu}_j\sum_{\substack{z_k>z\\k\neq j}}\hat{h}\varphi_k,\quad z_j=z+t_j. \tag{4.15}$$

To calculate the derivative $\partial G^z(\rho)/\partial z$, we assume, as in the one-dimensional case, that $t=\delta z$ in (4.14) is an infinitesimally small quantity. In this case there is only one term left in the sum of (4.14), with probability $nS\delta z$ (n is the density of the centers, S is the cross section area of the cylinder in which the wave function of the tunneling particle) is localized, and we get[11]

$$g^{z+\delta z}(\rho)=(1-\delta z\hat{\alpha})g^z(\rho)-\hat{h}\varphi_1|_{z<z_1<z_1+\delta z}. \quad (4.16)$$

Calculating on the basis of (4.16) the difference $G^{z+\delta z}(\rho) - G^z(\rho)$ and taking the limit as $\delta z \to 0$, we obtain in analogy with the one-dimensional case,

$$\partial G^z(\rho)/\partial z=-2\hat{\bar{\alpha}}G^z(\rho)-2n\langle(\hat{h}\varphi_1)_\rho{}^z g^{*z}(0)\rangle+n\langle(\hat{h}\varphi_1)_\rho{}^z(\hat{h}\varphi_1)_0{}^{*z}\rangle. \quad (4.17)$$

Assuming the density to be small, we expand the mean values in (4.17) in powers of $n/\alpha^3 \ll 1$. To obtain the terms $\sim(n/\alpha^3)^{m+1}$ in (4.17) it is necessary to take into account the contribution from the configurations at which, in a sphere $|\mathbf{r}-\mathbf{r}_j|\sim 1/\alpha$ around the point $\mathbf{r}$ at which the scattering center is located, there turn out to be (with a probability $\sim(n/\alpha^3)^m$) m additional scattering centers.

To obtain in (4.17) the terms linear in n, we put $m=0$.

It follows from (4.15) that $\varphi_1(z,\rho)=(\hat{\mu}_1 g^z)_\rho$. Taking into account the locality of the operator μ_1 [see (4.18)], we get

$$-2\langle(\hat{h}\varphi_1)_\rho{}^z g^{*z}(0)\rangle+\langle(\hat{h}\varphi_1)_\rho{}^z(\hat{h}\varphi_1)_0{}^{*z}\rangle=-2\mu\int h(\rho-\rho_1)G^z(\rho_1)d^2\rho_1$$
$$+\mu^2\int h(\rho-\rho_1)h(\rho_1)d^2\rho_1 G^z(0)=\hat{S}_1G^z. \quad (4.18)$$

Equation (4.17) then takes the form

$$\partial G^z(\rho)/\partial z=-2\hat{\alpha}G^z(\rho)+n\hat{S}_1G^z. \quad (4.19)$$

By going over in (4.19) to the $(z,\varkappa)$ representation and taking into account the diagonality of the operator $\hat{\alpha}$ in this representation we obtain for the Fourier components

$$\frac{\partial G_\varkappa{}^z}{\partial z}=-2\left(\alpha_\varkappa+\frac{n\mu}{2\alpha_\varkappa}\right)G_\varkappa{}^z+\frac{n\mu^2}{4\alpha_\varkappa{}^2}\bar{G}^z,$$
$$\bar{G}^z=\int G_\varkappa{}^z d^2\varkappa=G^z(0) \quad (4.20)$$

with the boundary condition

$$G_\varkappa{}^0=\delta(\varkappa). \quad (4.20a)$$

We seek the solution of (4.20) in the form

$$G_\varkappa{}^z=e^{-2\bar{\alpha}z}X_\varkappa{}^z, \quad \bar{\alpha}=\alpha+n\mu/2\alpha. \quad (4.21)$$

The equation for $X_\varkappa^z$ is then

$$\frac{\partial X_\varkappa{}^z}{\partial z}=-q_\varkappa X_\varkappa{}^z+\frac{n\mu^2}{4\alpha_\varkappa{}^2}\bar{X}^z,$$
$$\bar{X}^z=\int X_\varkappa{}^z d^2\varkappa, \quad q_\varkappa=2(\bar{\alpha}_\varkappa-\bar{\alpha}),$$
$$\bar{\alpha}_\varkappa=\alpha_\varkappa+n\mu/2\alpha_\varkappa. \quad (4.22)$$

Taking the Laplace transform in (4.22)

$$X_\varkappa(p)=\int_0^\infty X_\varkappa{}^z e^{-pz}dz \quad (4.23)$$

and taking (4.20a) into account, we obtain with the required accuracy (with account taken of terms $\sim n\mu^2/\alpha$)

$$\bar{X}(p)=\left\{p\left[1-\varepsilon\int_0^\infty\frac{dx}{x^2+(p/2\alpha+1)x+p/2\alpha}\right]\right\}^{-1}$$
$$=\left\{p\left[1-\frac{\varepsilon}{1-p/2\alpha}\ln(p/2\alpha)\right]\right\}^{-1}, \quad \varepsilon=\frac{\pi n\mu^2}{4\alpha}. \quad (4.24)$$

Taking the inverse Laplace transform in (4.24), we obtain

$$\bar{X}^z=\begin{cases}1+\varepsilon 2\alpha z\ln(2\alpha z)\cong 1, & 2\alpha z\ll 1\\ 1+\varepsilon\ln 2\alpha z\cong 1, & 1\ll 2\alpha z\ll e^{1/\varepsilon}.\\ \varepsilon^{-1}\exp(2\alpha z e^{-1/\varepsilon}), & 2\alpha z\gg e^{1/\varepsilon}\end{cases}$$

The explicit solution of (4.22) is

$$X_\varkappa{}^z=\delta(\varkappa)-\varepsilon\int K_\varkappa(\xi)\bar{X}^{z-\xi}d\xi,$$
$$K_\varkappa(\xi)=\frac{\alpha}{\pi\alpha_\varkappa{}^2}e^{-q_\varkappa\xi}. \quad (4.25)$$

Since $\bar{X}^{z-\xi}$ is a slowly varying function compared with a kernel $K_\varkappa(\xi)$, it follows that the principal contribution of the integral (4.25) is determined by small values of ξ. Next, expanding $\bar{X}^{z-\xi}$ in powers of ξ in the vicinity of $\xi=0$ and taking into account only the first term of the expansion, we get

$$X_\varkappa{}^z=\delta(\varkappa)-\varepsilon\bar{X}^z\int_0^z K_\varkappa(\xi)d\xi. \quad (4.26)$$

On the basis of (4.21) and (4.26) we obtain the spectral distribution of the tunneling particles:

$$G_\varkappa^z=e^{-2\bar{\alpha}z}\left[\delta(\varkappa)-\frac{n\mu^2\bar{X}^z}{4\alpha_\varkappa^2 q_\varkappa}(1-\exp\{-q_\varkappa z\})\right]. \quad (4.27)$$

Terms of higher order in the density n do not alter the picture qualitatively. The expressions for them are too unwieldy to be given here.

5. Resonant tunneling

We consider tunneling in three-dimensional systems at energies belonging to the spectrum of the random Hamiltonian $\hat{H}$. As will be shown below, in this case there can appear in the sub-barrier region "resonantly percolating" trajectories, i.e., tunneling paths along which no attenuation takes place. The probabilities of these trajectories and the energy widths of the resonant tunneling on them will determine the transparency conditions.

We consider a solitary trajectory passing through N scattering centers (a criterion of solitarity, i.e., of the absence of a contribution from joint scattering by other centers, will be spelled out below). To this end we write down the boundary conditions (similar to (1.7) in the one-dimensional case) and the relations that follow from (4.3) in the $(z,\varkappa)$ representation:

$$(\chi_+{}^L)_\varkappa=\frac{\alpha_\varkappa+ik_\varkappa}{\alpha_\varkappa-ik_\varkappa}(\chi_-{}^L)_\varkappa, \quad (5.1)$$

$$(\chi_-{}^0)_\varkappa=-\frac{2ik}{\alpha-ik}\delta(\varkappa)+\frac{\alpha_\varkappa+ik_\varkappa}{\alpha_\varkappa-ik_\varkappa}(\chi_+{}^0)_\varkappa, \quad (5.2)$$

$$(\chi_-{}^L)_\varkappa=-\sum_{j=1}^N\frac{\exp[-(L-z_j)\alpha_\varkappa]}{2\alpha_\varkappa}\Phi_j\exp(-i\varkappa\rho_j)+\exp(-L\alpha_\varkappa)(\chi_-{}^0)_\varkappa, \quad (5.3)$$

$$(\chi_+{}^0)_\varkappa=-\sum_{j=1}^N\frac{\exp(-z_j\alpha_\varkappa)}{2\alpha_\varkappa}\Phi_j\exp(-i\varkappa\rho_j)+\exp(-L\alpha_\varkappa)(\chi_+{}^L)_\varkappa, \quad (5.4)$$

$$\Phi_j=-\mu\sum_{j\neq k}h_{jk}\Phi_k+\mu\exp(-z_j\hat{\alpha})\chi_-{}^0|_{\mathbf{r}_j}+\mu\exp[-(L-z_j)\hat{\alpha}]\chi_+{}^L|_{\mathbf{r}_j}. \quad (5.5)$$

Here

$$k_\varkappa{}^2=k^2-\varkappa^2, \quad h_{jk}=h(\mathbf{r}_j-\mathbf{r}_k)=\exp(-\alpha|\mathbf{r}_j-\mathbf{r}_k|)/4\pi|\mathbf{r}_j-\mathbf{r}_k|,$$
$$\mu\exp(-z_j\hat{\alpha})\chi_-{}^0|_{\mathbf{r}_j}=\frac{\mu}{4\pi^2}\int\exp(-z_j\alpha_\varkappa)(\chi_-{}^0)_\varkappa\exp(i\varkappa\rho_j)d^2\varkappa.$$

The system (5.1)–(5.5) is closed and describes completely our problem.

Before we proceed to investigate the general case, we examine the simplest case of a trajectory passing through one scattering center located at the point[12)]

$r_1 = (z_1, \rho_1)\rho_1 = 0$.

1. We consider the case $N=1$. Recognizing that at resonant energies we have $\bar\varphi_1/\alpha_\varkappa \gg (\chi_-^0)_\varkappa, (\chi_+^L)_\varkappa$, we write down on the basis of (5.3)–(5.5)

$$(\chi_-^L)_\varkappa = -\frac{\exp[-(L-z_1)\alpha_\varkappa]}{2\alpha_\varkappa}\tilde\varphi_1, \tag{5.3$'$}$$

$$(\chi_+^0)_\varkappa = -\frac{\exp(-z_1\alpha_\varkappa)}{2\alpha_\varkappa}\tilde\varphi_1, \tag{5.4$'$}$$

$$\varphi_1 = \mu e^{-z_1\hat\alpha}\chi_-^0|_{\rho_1=0} + \mu e^{-(L-z_1)\hat\alpha}\chi_+^L|_{\rho_1=0}. \tag{5.5$'$}$$

Substituting (5.3$'$) in (5.1), we get

$$e^{-(L-z_1)\hat\alpha}\chi_+^L|_{\rho_1=0} = \frac{1}{4\pi^2}\int \frac{e^{-2(L-z_1)\alpha_\varkappa}}{2\alpha_\varkappa}\frac{(\alpha_\varkappa+ik_\varkappa)}{(\alpha_\varkappa-ik_\varkappa)}d^2\varkappa\tilde\varphi_1$$
$$= -\frac{(\alpha+ik)}{(\alpha-ik)}\frac{e^{-2(L-z_1)\alpha}}{8\pi(L-z_1)}\tilde\varphi_1. \tag{5.6}$$

From (5.4$'$) and (5.2) we obtain similarly

$$e^{-z_1\hat\alpha}\chi_-^0|_{\rho_1=0} = -\frac{2ik}{(\alpha-ik)}\frac{e^{-z_1\alpha}}{4\pi^2} - \frac{(\alpha+ik)}{(\alpha-ik)}\frac{e^{-2z_1\alpha}}{8\pi z_1}\tilde\varphi_1. \tag{5.7}$$

Substituting now (5.6), (5.7) in (5.5$'$) and putting $\xi = (z_1 - L/2)$, we get

$$\varphi_1 = -\mu\frac{ik}{2\pi^2(\alpha-ik)}\exp\left\{-\alpha\left(\frac{L}{2}+\xi\right)\right\}\left\{1+\frac{\mu}{2\pi L}\frac{(\alpha+ik)}{(\alpha-ik)}e^{-\alpha L}\operatorname{ch}2\alpha\xi\right\}^{-1}. \tag{5.8}$$

As $\mu\to\infty$ we have

$$|\varphi_1|^2 = \frac{k^2L^2e^{\alpha(L-2\xi)}}{(\alpha^2+k^2)\pi^2\operatorname{ch}^2 2\alpha\xi}.$$

Taking the inverse Fourier transform in (5.3$'$), we get

$$\chi_-^L(\rho) = -\frac{\varphi_1}{4\pi^2}\int\frac{\exp(-\eta\alpha_\varkappa)}{2\alpha_\varkappa}e^{i\varkappa\rho}d^2\varkappa$$
$$= -\frac{\varphi_1}{4\pi}\frac{\exp[-\alpha(\eta^2+\rho^2)^{1/2}]}{(\eta^2+\rho^2)^{1/2}}, \quad \eta = (L-z_1) = \frac{L}{2}-\xi.$$

Recognizing, finally, that

$$\Psi_\varkappa(L) = (\chi_+^L)_\varkappa + (\chi_-^L)_\varkappa = \frac{2\alpha_\varkappa}{\alpha_\varkappa - ik_\varkappa}(\chi_-^L)_\varkappa,$$

we get

$$\Psi(L,\rho) = -\frac{2\alpha}{(\alpha-ik)}\frac{\varphi_1}{4\pi}\frac{\exp[-\alpha(\eta^2+\rho^2)^{1/2}]}{(\eta^2+\rho^2)^{1/2}}. \tag{5.9}$$

The density of the particles that pass at the point (L,ρ) is

$$|\Psi(L,\rho)|^2 = \frac{\alpha^2}{4\pi^2(\alpha^2+k^2)}|\varphi_1|^2\frac{\exp[-2\alpha(\eta^2+\rho^2)^{1/2}]}{(\eta^2+\rho^2)}. \tag{5.10}$$

At $\mu\to\infty$, $\xi \ll L/2$ we have

$$|\Psi(L,\rho)|^2 = \frac{\alpha^2k^2}{\pi^4(\alpha^2+k^2)^2\operatorname{ch}^2 2\alpha\xi}\exp\left(-\frac{2\alpha\rho^2}{L}\right). \tag{5.10a}$$

Expression (5.10) determines the local transparency of the barrier $\sigma_L(\rho) \cong |\Psi(L,\rho)|^2$ (i.e., the ratio of the flux density at the exit to the density of the incident flux). At exact resonance $\mu=\infty$, $\xi=0$ the local transparency at the center of the spot at the point $\rho=0$ is

$$\sigma_L(\rho=0) = \alpha^2k^2/\pi^4(\alpha^2+k^2)^2, \tag{5.11}$$

and the area of the spot is $\pi\rho^2 \sim L/\alpha$. From Eqs. (5.10a) and (5.10) it is evident that the resonant passage is preserved at $|\xi| \lesssim 1$, $\mu\alpha \gtrsim \alpha L e^{\alpha L}$. Recognizing that near the pole $\mu\alpha \sim 1/\varepsilon$ ($\varepsilon = (E-E_0)/2\alpha_0^2$, $E_0 = -\alpha_0^2$ is the energy corresponding to the pole of the scattering amplitude), we obtain the energy width of the resonant transparency $\delta\varepsilon \sim e^{-\alpha L}$.

2. We consider now the general case of the "resonance-percolation" trajectory passing through N scattering centers located at the points $(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N)$, with $\rho_1 = 0$. As is already clear from the analysis of the one-dimensional systems, the largest energy width of the resonant transparency (at a fixed number N of scattering centers along the trajectory and at a fixed path length L') should have trajectories in which all the distances between neighboring scattering centers are equal. We start with a study of these trajectories and investigate next the admissible deviations that preserve their "percolation" properties. We put

$$2\mathbf{y}_n = \mathbf{r}_{n+1} - \mathbf{r}_n, \quad 2y = |\mathbf{r}_{n+1} - \mathbf{r}_n| = L'/N,$$
$$z_1 = (L - z_N) = L'/2N = y; \quad (\widehat{\mathbf{y}_n, \mathbf{z}}) = \theta_n,$$

$\theta_n < 1$ (see Fig. 2). Recognizing that in the vicinity of the resonance $\mu\alpha \gg 1$, and consequently $\bar\varphi_j/\alpha_\varkappa \gg (\chi_\pm^0)_\varkappa$, $(\chi_\pm^L)_\varkappa$, we simplify the general expressions (5.3)–(5.5), retaining in them only the principal terms. Then

$$(\chi_-^L)_\varkappa = -\frac{\exp(-y\alpha_\varkappa)}{2\alpha_\varkappa}\tilde\varphi_N\exp(-i\varkappa\rho_N), \tag{5.3$''$}$$

$$(\chi_+^0)_\varkappa = -\frac{\exp(-y\alpha_\varkappa)}{2\alpha_\varkappa}\tilde\varphi_1, \tag{5.4$''$}$$

$$\varphi_j = -\mu h(\varphi_{j+1}+\varphi_{j-1}), \quad 2\leqslant j\leqslant N-1, \tag{5.5$''$a}$$

$$\varphi_1 = \frac{\mu}{4\pi^2}\int\exp(-y\alpha_\varkappa)(\chi_-^0)_\varkappa d^2\varkappa - \mu h\varphi_2, \tag{5.5$''$b}$$

$$\varphi_N = \frac{\mu}{4\pi^2}\int\exp(-y\alpha_\varkappa)(\chi_+^L)_\varkappa\exp(i\varkappa\rho_N)d^2\varkappa - \mu h\varphi_{N-1}, \tag{5.5$''$c}$$

where $h = e^{-2\alpha y}/8\pi y$.

We consider now Eq. (5.5$''$a), which we rewrite in the form

$$\varphi_{j+1} - 2x\varphi_j + \varphi_{j-1} = 0, \quad x = 1/2\mu h, \quad 2\leqslant j\leqslant N-1. \tag{5.12}$$

Its solution is

$$\varphi_n = c_1\lambda_1^n + c_2\lambda_2^n, \tag{5.13}$$

where $\lambda_{1,2}$ are determined by the characteristic equation

$$\lambda^2 - 2x\lambda + 1 = 0. \tag{5.14}$$

The condition for the appearance of an energy band (i.e. for the absence of damping along the trajectory), is $|\lambda_{1,2}| = 1$ and, as seen from (5.14), is equivalent to the requirement

$$x\leqslant 1, \quad \mu\alpha \geqslant \alpha/2h = 4\pi\alpha y e^{2\alpha y}. \tag{5.15}$$

Substituting (5.3$''$) in (5.1) and (5.4$''$) in (5.2) and taking (5.5$''$b) and (5.5$''$c) into account, we get

$$\varphi_N = -\frac{\mu h}{1+(\alpha-ik)\mu h/(\alpha+ik)}\varphi_{N-1},$$

$$\varphi_1\left[\frac{1+(\alpha+ik)\mu h/(\alpha-ik)}{\mu h}\right] + \varphi_2 = -\frac{2ik}{(\alpha+ik)}\frac{e^{-\alpha y}}{4\pi^2 h}.$$

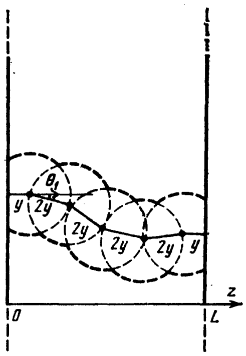

FIG. 2. One of the simplest "resonance-percolation" trajectories. The length of the broken line is L', $2y = L'/N$. The criterion that determines the solitarity of this trajectory is that there be no additional scattering center inside the region bounded by the thick dashed lines.

Hence

$$|\Phi_1|\sim|\Phi_2|\sim\ldots|\Phi_N|\sim 2kye^{\alpha y}/\pi(\alpha^2+k^2)^{1/2}.$$

Then, taking (5.3″) into account, we get

$$(\chi_-^{L})_\varkappa\sim-\frac{\exp(-y\alpha_\varkappa-i\varkappa\rho_N)}{2\alpha_\varkappa}\frac{2kye^{\alpha y}}{\pi(\alpha^2+k^2)^{1/2}}$$

and consequently

$$\Psi(L,\rho)\sim\frac{2\alpha}{\alpha-ik}\chi_-^{L}(\rho)\sim\frac{2\alpha}{\alpha-ik}\Phi_N\frac{1}{4\pi^2}\int\frac{\exp[-y\alpha_\varkappa+i\varkappa(\rho-\rho_N)]}{2\alpha_\varkappa}d^2\varkappa$$

$$\sim\frac{2\alpha}{\alpha-ik}\Phi_N\frac{1}{4\pi}\frac{1}{(y^2+|\rho-\rho_N|^2)^{1/2}}\exp[-\alpha(y^2+|\rho-\rho_N|^2)^{1/2}],$$

$$|\Psi(L,\rho)|^2\sim\frac{\alpha^2k^2}{\pi^4(\alpha^2+k^2)^2}\exp\left\{-\frac{\alpha|\rho-\rho_N|^2}{y}\right\}. \tag{5.16}$$

Thus, the integral flux Q through one "percolation" trajectory with "step" $2y$ is

$$Q=k\int|\Psi(L,\rho)|^2d^2\rho\sim\alpha yk^3/\pi^3(k^2+\alpha^2)^2.$$

Recognizing now that $\varepsilon\sim1/\mu\alpha<(\alpha y)^{-1}e^{-2\alpha y}$, $y=L'/2N$, we obtain the energy width of the resonant transparency $\delta\varepsilon\sim e^{-\alpha L'/N}$. The maximum width $\delta\varepsilon$ at a fixed number of steps N corresponds to $L'=L$, i.e., to the shortest percolation path. To estimate the average transparency it is necessary to find the probabilities of the percolation trajectories, determine their optimal structures, and find their integral contribution. These trajectories, naturally become more tortuous with increasing center concentration $c=n/\alpha^3$. In the present paper we consider only the simplest case of low concentration, when the main contribution to the transparency is made by trajectories close to the shortest ones (this is certainly the case, for example, at $c\mathscr{L}^2\ll1$, $\mathscr{L}=\alpha L\gg1$).

We characterize the degree of tortuousness of the trajectory by the average aperture angle θ of the vectors $\mathbf{y}_j(\langle\cos\theta_j\rangle=1-\theta^2/2;\langle(\cos\theta_j-\langle\cos\theta_j\rangle)^2\rangle=\theta^4/12;$ $\theta^2\ll1)$. Fixing the dimensionless step of the trajectory $u=2y\alpha=\alpha L'/N$, we introduce the quantity

$$z_N=u\sum\cos\theta_j,\quad\langle z_N\rangle=Nu\left(1-\frac{\theta^2}{2}\right),\quad\left\langle\left(\frac{z-\langle z_N\rangle}{\langle z_N\rangle}\right)^2\right\rangle=\frac{\theta^4}{12N}.$$

At $N\gg1$ we get from the obvious condition $z_N=\langle z_N\rangle=\mathscr{L}-u$

$$N=N_\theta=\frac{\mathscr{L}}{u}\left(1+\frac{\theta^2}{2}\right)=N_0\left(1+\frac{\theta^2}{2}\right),\quad N_0=\frac{\mathscr{L}}{u},\quad\mathscr{L}=\alpha L.$$

The possible scattering-center fluctuations that leave the trajectory "resonantly percolating" are equal to

$$\alpha\delta(z_{j+1}-z_j)\sim1,\quad\alpha\delta\rho_j\sim u\theta,$$
$$\alpha^3\delta V_j\sim\alpha^3\delta(z_{j+1}-z_j)(\delta\rho_j)^2\sim u^2\theta^2.$$

The condition that the trajectory be solitary consists in the absence of other centers (other than those making up this trajectory) in a cylinder of radius u around it. Then the probability of realizing a resonance-percolation trajectory per unit plate area is

$$W_{u,\theta}\sim(cu^2\theta^2)^{N_\theta-1}\exp\{-cN_\theta\pi u^3\},\quad c=n/\alpha^3,$$

and the contribution of such trajectories to the average trajectory is (with logarithmic accuracy)

$$\sigma_{u,\theta}(\varepsilon)\to\sigma_{u,\theta}\delta(\varepsilon),\quad\sigma_{u,\theta}\sim W_{u,\theta}e^{-u},$$

$$\ln\sigma_{u,\theta}=\frac{\mathscr{L}}{u}\left(1+\frac{\theta^2}{2}\right)\{\ln cu^2\theta^2-c\pi u^3\}-u.$$

With the sum degree of accuracy, the total σ is determined by the extremal value of $\sigma_{\tilde u,\tilde\theta}$ with respect to θ and u (of course, in that region of the values of c and $\mathscr{L}$, in which the initial assumptions $\tilde u\gg1$, $\tilde\theta^2\ll1$ are satisfied). Thus, for example, in the region $\mathscr{L}^{3/2}c|\ln c|\ll1$ (and all the more $c\mathscr{L}^2\ll1$) we have $\mathscr{L}\gg1$, and the term $c\pi u^3$ in the curly brackets can be neglected; we then obtain asymptotically

$$\tilde u=(\mathscr{L}|\ln c\mathscr{L}|)^{1/2},\quad\tilde N=(\mathscr{L}/|\ln c\mathscr{L}|)^{1/2},$$
$$\theta^2\cong2/|\ln c\mathscr{L}|,\quad-\ln\sigma=2(\mathscr{L}|\ln c\mathscr{L}|)^{1/2}.$$

We can similarly investigate also the resonant transparency in the vicinity of the discrete local levels made up of several close centers (cluster levels), for which it is necessary to introduce the effective amplitude μ_k of scattering by the cluster; this amplitude will be large the closer the energy to the considered cluster level. All the remaining scattering amplitudes, however, will be small. It is clear that near the cluster levels the highest energy width of the resonant transparency is ensured by configurations with identical clusters separated by equal distances, along the shortest "resonance-percolation" trajectory.

Just as in the one-dimensional case, the resonant transparency is a strongly fluctuating quantity, and the determination of various intermediate asymptotic forms with the aid of the estimates presented above is governed by the concrete formulation of the problem.

Investigation of higher concentrations, particularly the approach to the critical values c_{cr} at which infinite percolation paths arise, calls for special methods and is beyond the scope of the present article.

APPENDIX

In the expression containing a damped plane and scattered spherical wave

$$\Psi(\mathbf{r})=e^{-\alpha z}+\mu e^{-\alpha r}/4\pi r$$

(r_0 is the effective radius of the local perturbation $\hat u$), the amplitude μ of the sub-barrier scattering takes, according to (4.7) and (4.8), the form

$$\mu=(\hat\mu1,1),\quad\hat\mu=\hat u(1+\hat h\hat u)^{-1}. \tag{A.1}$$

Inasmuch as in the three-dimensional cases a δ-function perturbation in the form $\hat u=\beta\delta(\mathbf{r})$ (i.e., with a kernel $u(\mathbf{r},\mathbf{r}')=\beta\delta(\mathbf{r})\delta(\mathbf{r}')$ leads to divergences we shall obtain the required estimates by using as the simplest local perturbation of radius r_0 and consider an operator $\hat u$ with a kernel

$$\hat u(\mathbf{r},\mathbf{r}')=\tau f(\mathbf{r})f(\mathbf{r}'), \tag{A.2}$$

where $f(\mathbf{r})$ is a function with a sharp maximum and differs from zero inside a sphere of radius $r_0(\alpha r_0\ll1)$. In this case, according to (A.1) and (A.2), we have for the kernel $\mu(\mathbf{r}_1,\mathbf{r}_2)$ of the operator μ

$$\mu(\mathbf{r}_1,\mathbf{r}_2)=\tau f_1f_2\Big/\left(1+\tau\iint f_1f_2h_{12}dV_1\,dV_2\right), \tag{A.3}$$

$$h_{12}=h(r_{12})=e^{-\alpha r_{12}}/4\pi r_{12},\quad f_1=f(\mathbf{r}_1),\quad f_2=f(\mathbf{r}_2).$$

The value of the dimensionless constant τ is given by the normalization $f(\mathbf{r})$, which we choose in the form

$$\frac{1}{4\pi}\iint f_1f_2\frac{dV_1\,dV_2}{r_{12}}=1. \tag{A.4}$$

By virtue of (A.3) and (A.4), recognizing that $\alpha r_0 \ll 1$, we obtain a simple expression for the scattering amplitude μ:

$$\mu = \iint \mu(\mathbf{r}_1, \mathbf{r}_2) dV_1 dV_2 = \frac{4\pi\tau\tilde{r}}{1+\tau-\tau\alpha\tilde{r}}, \quad \hat{\mu} \to \mu\delta(\mathbf{r})\delta(\mathbf{r}'),$$
$$\tilde{r} = \iint f_1 f_2 dV_1 dV_2 \Big/ \iint f_1 f_2 \frac{dV_1 dV_2}{r_{12}} \sim r_0. \qquad (A.5)$$

A bound state exists on the center at $\tau < -1$. The corresponding eigenvalue α_0 is determined by the pole μ, i.e., by the condition $1+\tau = \tau\alpha_0\tilde{r}$ (at $|1+\tau| \ll 1$). Near this value we have

$$\mu\alpha = \frac{4\pi\alpha}{\alpha_0-\alpha} = \frac{8\pi\alpha_0^2}{E_0-E}, \quad \left|\frac{E-E_0}{E_0}\right| \ll 1. \qquad (A.6)$$

On the contrary, at $|(1+\tau)/\tau| \gg \alpha r_0$ we have

$$\mu = \tau\tilde{r}/(1+\tau). \qquad (A.7)$$

1)We use below a hacek ∨ ($\check{\alpha}, \check{\beta}, S\ldots$) to denote a 2×2 transformation matrix on the components of the vector χ, and a circumflex ∧ ($\hat{H}\ldots$) to denote operators acting on functions of the spatial coordinates.

2)The transparency $\sigma_L(E, \Gamma)$ depends on three arguments, L, Γ, and E, but in order not to clutter up the notation we shall write out explicitly, depending on the context, only those arguments which are of interest to us here.

3)It is easy to verify that the amplitude of forward sub-barrier scattering is $\mu = (1 - T_{11}/T_{11}$ [see also Eq. (2.5)].

4)The solution of the equation

$$d^2\Psi/dz^2 - \alpha^2\Psi = -\delta(z-z_0), \quad 0<z_0<L$$

with boundary conditions $\chi_-(0) = 0$, $\chi_+(L) = 0$ coincides with the the Green's function on an infinite axis

$$\Psi = (2\alpha)^{-1} \exp\{-\alpha|z-z_0|\},$$

and therefore the general solution of (2.1) takes the form

$$\Psi(z) = \chi_-(0)\exp\{-\alpha z\} + \chi_+(L)\exp\{-\alpha(L-z)\} - \sum_{0<z_j<L} \varphi_j \exp\{-\alpha|z-z_j|\}.$$

5)The estimate (3.6) pertains to the case $q^2 = (k^2+\alpha^2)^2/\alpha^2k^2 \sim 1$. For $q \gg 1$ the averaging of σ gives rise to an additional factor $\sim 1/q$.

6)Expression (3.12) should be regarded as symbolic, since the transparency $\sigma_1(\mathcal{E}) \leq 1$ at any energy $\mathcal{E}$; however, it is precisecisely the coefficient σ_1 which determines the ratio of the integral input and output fluxes.

7)The cluster levels may be discrete because the positions of the scattering centers in the crystal-lattice sites are discrete.

8)More accurately, the cylinders can be slightly bent, provided that the angle θ between their generators and the z axis lies within the limits $\theta^2 \lesssim 1/\alpha L$.

9)The solution of the equation $\Delta\Psi(\mathbf{r}) - \alpha^2\Psi(\mathbf{r}) = -\delta(\mathbf{r}-\mathbf{r}')$, $0 < z' < L$ with the homogeneous boundary conditions $\chi_-(0, \boldsymbol{\rho}) = 0$, $\chi_+(L, \boldsymbol{\rho}) = 0$ coincides with Green's function in an infinitely long homogeneous barrier (4.3a) $\Psi(\mathbf{r}-\mathbf{r}') = h(\mathbf{r}-\mathbf{r}')$. We note that the kernal of the operator $\hat{h}$ in the $(z, \varkappa)$ representation is is diagonal in $\varkappa$, with

$$h(z-z', \varkappa) = \exp\{-\alpha_\varkappa|z-z'|\}/2\alpha_\varkappa.$$

10)By virtue of the macroscopic homogeneity, the propagator $\langle\Psi(z, \rho')\Psi^*(z, \boldsymbol{\rho}'')\rangle$ depends only on $\boldsymbol{\rho} = \boldsymbol{\rho}' - \boldsymbol{\rho}''$.

11)It is assumed here that the density of the scattering centers is low enough, so the $nSr_0 \ll 1$, where r_0 is the perturbation action radius.

12)The case of one center at the midpoint of the barrier was considered in Ref. 2 by another method.

[1]I. M. Lifshtiz, Usp. Fiz. Nauk **83**, 617 (1964) [Sov. Phys. Usp. **7**, 549 (1965)].

[2]H. Knauer, J. Richter, and P. Seidel, Phys. Stat. Sol. (a), **44**, 303 (1977).

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