The early stages of evolution of spatially nonhomogeneous models of the Universe

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We analyze the problem of evolution of spatially nonhomogeneous models of the Universe over times when the characteristic scale L of inhomogeneity is considerably larger than the size of the event horizon ct. In the approximation $ct/L \ll 1$ we have constructed a mathematical model embedded into a Friedmann space with isolated spatial inhomogeneity, generating rotating motions of matter with parameters corresponding to the vortex theory of galaxy formation. We show that the existence of pure vortex motions during the quasi-Friedmann stage leads for $t \to 0$ to the appearance of potential motions caused by a redistribution of matter into the inhomogeneities. This is due to the different character of the time variation of matter energy density at different points of the inhomogeneity.

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1. INTRODUCTION

The presently observable large-scale isotropy and homogeneity of the properties of the Universe unfortunately does not allow one to conclude to what degree the Universe satisfied or did not satisfy these properties during the early stages of its evolution. The reason for this is the fact that the Friedmann solution, which describes the evolution of a homogeneous and isotropic Universe, for $t \to 0$ becomes unstable with respect to small perturbations of the metric, the energy density and the velocities of the motions of matter.[1] The answer to the question: how fast does the Universe "forget" a putative initial lack of homogeneity and isotropy is indelibly related to the problem of initial conditions which in cosmology. However, since the latter problem has remained unresolved to this day, there is only one way to analyze the early stages of evolution of the Universe: to reconstruct these from the structure of the Universe observable at the present epoch.

Turning to the vortex theory of galaxy formation,[2-6] we have to note that it requires the construction of an essentially non-Friedmann cosmological model for the early stages, including spatial inhomogeneities of the vortex type. These inhomogeneities must be responsible for the occurrence of intense vortex motions, which for $t \gtrsim t_{eq}$ (t_{eq} is the instant of time when the energy densities of radiation and matter become equal: $t_{eq} \approx 3.7 \times 10^{11}(\Omega h^2)s$ can, according to Refs. 2–6, lead to the observable picture of metagalactic structure. The incompatibility of an early Friedmann Universe with large-scale ($L \gg ct$) spatial inhomogeneities of vortex type follows already from the instability ($t \to 0$) of the Friedmann solution with respect to small vector perturbations of the metric above the horizon.[7]

The following question arises naturally: how will such a spatially nonhomogeneous model evolve in time during the early stages of its evolution? An answer to this question will shed light on the physical picture during the early nonhomogeneous Universe, and in particular, will clarify to what extent the putative difficulty of the vortex theory, related to the chemical composition of primordial matter, is plausible.

At present there exist two alternative approaches to the problem of constructing a spatially inhomogeneous (in the differential sense) model of the Universe, which for $t > t_{eq}$ should exhibit vortex motions of the matter. The first approach (cf.[8,9]) requires identity of the geometric properties and of the anisotropies of deformations of space at each of its points, i.e., a "group" homogeneity of the space under consideration. The velocity field of the particles in flat three-dimensional space can be described in a simplified manner in the following way. In an arbitrarily chosen plane $z = \text{const}$ there exists a uniform flow of particles with some orientation with respect to the x and y axes. As z varies the direction of the velocity vector varies periodically, its magnitude remaining constant. Rotational motions of matter, such as the motion of the particles around a closed trajectory, occur in this model only if one takes into account the interaction between infinitely close layers of moving particles, i.e., if viscosity is taken into consideration.

A second approach[10,11] does not require taking into account dissipative processes in order to obtain rotational motions. The latter can appear directly in the process of time evolution of the Universe as a consequence of a primordial spatial nonhomogeneity. However, in this case one must give up the requirement of homogeneity of space not only in the differential sense, but also in the group sense, which complicates considerably the analysis of such models in distinction from the first approach described above.

The presence of even initially small irregularities of the vortex type (as well as of other types) necessarily leads to a local anisotropy of space.[12] The development of the irregularities in the reverse-time evolution of the Universe is accompanied by a growth of the anisotropy of the deformation of space[12] in the localization region of a given anisotropy. In turn, the growth for $t \to 0$ of the deformation anisotropy related to vortex motion, derived earlier by Zel'dovich and Novikov,[13] leads in time to an essential change in the character of the temporal evolution of a spatially nonhomogeneous Universe. As was shown by Tomita,[10] in the process

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of the reverse-time evolution such a system gradually deviates from the Friedmann solution, and from a certain time t_F on goes over into Kasner asymptotic behavior, i.e., becomes essentially anisotropic. The values of t_F at each point of a spatially nonhomogeneous model will be determined by quantities which describe the degree of anisotropy of the deformation of space at the given point.

It is necessary, however, to note that the deformation anisotropy of an arbitrarily chosen volume element of matter is determined, in the most general case, not only by the local anisotropy of deformation introduced by spatial inhomogeneities. Moreover the background space into which vortex inhomogeneities are embedded may have its own anisotropies. Then t_F at a given point of space will be determined by the total value of the deformation anisotropy with regard to the background space (a constant at each point) and by the vortex inhomogeneities. In this case, it is naturally impossible to relate uniquely the vortex parameters to parameters which determine the deformation anisotropy at each point of the spatially nonhomogeneous model. And thus there arises the necessity of appealing to additional hypotheses which allow one to define the geometry of the background space.

The mathematical formulation of the problem, when the spatial irregularities are embedded into a background space which itself satisfies the requirements of group homogeneity is given in §2.

In §3 we write down the solutions of the Einstein equations obtained similarly to Ref. 10, and describing the evolution of a spatially nonhomogeneous model in the approximations $L \gg ct$ and $u_\alpha \ll 1$. Section 4 is devoted to an analysis of the characteristic stages of evolution of the model in the case when the deformation anisotropy at all points of space is determined only by vortex space inhomogeneities. The vortex parameters are determined in agreement with the vortex theory of galaxy formation.[2-5] In §5 special examples of the listed results are considered.

2. AN INHOMOGENEOUS MODEL

We consider an inhomogeneous space for which the metric in a synchronous reference system has the representation

$$g_{ik}=\begin{cases} -c^2 & \text{for } i=k=0 \\ 0 & \text{for } i=0,\ k\neq 0;\ i\neq 0,\ k=0. \\ \gamma_{(ab)}e_\alpha^{(a)}e_\beta^{(b)} & \text{for } i=\alpha,\ k=\beta. \end{cases} \tag{2.1}$$

Assume that the coordinate functions $e_\alpha^{(a)}(x^\nu)$ which represent the set of three frame vectors labeled by a depict the structure of a background space into which spatial inhomogeneities may be embedded. Since the background space may exhibit its own anisotropy, in the absence of spatial irregularities the frame functions $e_\alpha^{(a)}$ will determine to which type according to the Bianchi classification the space belongs. This, of course, is true only in the case when the background space itself satisfies at least the requirement of group homogeneity.

Since nonhomogeneous spaces, in the most general case, do not allow coordinate transformations transforming the space into itself, the metric tensor which determines the geometry of a given space, must essentially be a function of coordinates which does not admit the invariance of independent differential forms of the type

$$e_\alpha^{(a)}\,dx^\alpha \tag{2.2}$$

(Greek and Latin indices run over 1, 2, 3; summation is understood over repeated indices) with respect to any three-parameter group of motions.[1)] Therefore in a synchronous reference system, where the spatial length element dl^2 can be separated from the time interval

$$ds^2=-c^2dt^2+dl^2, \tag{2.3}$$

the matrix $\gamma_{(ab)}$ which enters the expression of dl^2 when expressed in terms of the independent differential forms (2.2)

$$dl^2=\gamma_{(ab)}e_\alpha^{(a)}e_\beta^{(b)}\,dx^\alpha\,dx^\beta,\quad \gamma_{\alpha\beta}=\gamma_{(ab)}e_\alpha^{(a)}e_\beta^{(b)}, \tag{2.4}$$

must be a function not only of the time, but also of the coordinates, i.e.,

$$\gamma_{(ab)}=\gamma_{(ab)}(x^\nu, t).$$

Thus, the presence of spatial irregularities in the model under discussion, are reflected only in the coordinate-dependence of the matrix $\gamma_{(ab)}$.

The coordinate functions $e_\alpha^{(a)}(x^\mu)$ form a set of vectors reciprocal to the coefficients $e_{(a)}^\alpha(x^\mu)$ of the 1-forms (2.2):

$$e_{(a)}^\alpha e_\alpha^{(b)}=\delta_a{}^b,\quad e_{(a)}^\alpha e_\beta^{(a)}=\delta_\beta{}^\alpha. \tag{2.5}$$

The operations of raising and lowering of Greek indices are defined in terms of $\gamma_{\alpha\beta}$ and $\gamma^{\alpha\beta}(\gamma_{\alpha\varepsilon}\gamma^{\varepsilon\beta}=\delta_\alpha^\beta)$ and for Latin indices in terms of the matrices $\gamma_{(ab)}$ and $\gamma^{(ab)}$ $(\gamma_{(ac)}\gamma^{(cb)}=\delta_a^b)$.

The projections of the Einstein equations, describing the evolution of a universe with inhomogeneous 3-space, onto the corresponding frame vectors in the synchronous coordinate system considered here can be written in the form

$$\tfrac{1}{2}\dot{\varkappa}_{(a)}^{(a)}+\tfrac{1}{4}\varkappa_{(a)}^{(b)}\varkappa_{(b)}^{(a)}=\varkappa(p+\varepsilon)u_0u^0+\varkappa p, \tag{2.6}$$

$$\tfrac{1}{2}(\varkappa_{(b)|(a)}^{(b)}-\varkappa_{(a)|(b)}^{(b)})=\varkappa(p+\varepsilon)u_{(a)}u^0, \tag{2.7}$$

$$\tfrac{1}{2}(-\gamma)^{-1/2}((-\gamma)^{1/2}\varkappa_{(a)}^{(b)})^{\cdot}+P_{(a)}^{(b)}=\varkappa(p+\varepsilon)u_{(a)}u^{(b)}+\varkappa\delta_a{}^b(p-\tfrac{1}{2}T), \tag{2.8}$$

where $\varepsilon, p, (u_0, u_{(a)})$ are respectively the energy density, the pressure of matter and the components of the 4-velocity; T is the contraction of the energy-momentum tensor; $\varkappa_{(a)}^{(b)}=\gamma^{(bc)}\dot{\gamma}_{(ac)}$ (here and in the sequel the dot denotes differentiation with respect to time ct);

$$P_{(a)}^{(b)}=P_\alpha{}^\beta e_{(a)}^\alpha e_\beta{}^{(b)},$$

and, in its turn, P_α^β is the Ricci tensor of 3-space with the metric $dl^2=\gamma_{\alpha\beta}dx^\alpha dx^\beta$.

Here $P_{(a)}^{(b)}$ has the form:

$$P_{(a)}^{(b)}=\gamma^{(bg)}\{\Gamma_{[ag],(p)}^{[p]}-\Gamma_{[ap],(g)}^{[p]}+\Gamma_{[ag]}^{[d]}\Gamma_{[dp]}^{[p]}-\Gamma_{[ap]}^{[d]}\Gamma_{[gd]}^{[p]}\}. \tag{2.9}$$

Moreover, in (2.7) we have introduced a notation which allows one to write the equation in a compact form and reminiscent in its structure of the covariant derivative of a mixed second-tank tensor:

$$\varkappa^{(b)}_{(a)|(c)}=\varkappa^{(b)}_{(a),(c)}+\Gamma^{[b]}_{[fc]}\varkappa^{(f)}_{(a)}-\Gamma^{[f]}_{[ac]}\varkappa^{(b)}_{(f)}. \tag{2.10}$$

In their turn, the functions $\Gamma^{[c]}_{[ab]}$ have the form

$$\Gamma^{[c]}_{[ab]}=\Gamma^{(c)}_{(ab)}-a^{c}_{\cdot ab}, \tag{2.11}$$

where

$$\Gamma^{(c)}_{(ab)}={}^1/_2\gamma^{(cl)}(\gamma_{(al),(b)}+\gamma_{(bl),(a)}-\gamma_{(ab),(l)}), \tag{2.12}$$

$$a^{c}_{\cdot ab}={}^1/_2(C^{c}_{\cdot ab}+\gamma^{(cl)}\gamma_{(af)}C^{f}_{\cdot bl}-\gamma^{(cl)}\gamma_{(bf)}C^{f}_{\cdot la}). \tag{2.13}$$

Finally, $C^{c}_{\cdot ab}$, which in the inhomogeneous models plays the role of the group structure constants, have the same form as for the homogeneous models, and for the same reason we have conserved the same notation, as well as for the functions $a^{c}_{\cdot ab}$:

$$C^{c}_{\cdot ab}=e^{\nu}_{(a)}e^{(c)}_{\nu,(b)}-e^{\nu}_{(b)}e^{(c)}_{\nu,(a)}. \tag{2.14}$$

In Eqs. (2.9), (2.10), (2.12), (2.14) and everywhere below, the symbol (f) after the comma denotes the differentiation in the direction (f), related to ordinary differentiation with respect to the coordinate x^{α} by[2)]

$$\varphi_{,(f)}=\varphi_{,\alpha}e^{\alpha}_{(f)}. \tag{2.15}$$

The equations (2.6)–(2.8) represent a rather complicated system of partial differential equations (PDE), which can be analyzed only if one considers more concrete physical problems. Below we shall investigate a spatially nonhomogeneous model of the Universe for $t<t_{eq}$, including inhomogeneities of the vortex type, which at times close to t_{eq} correspond in their parameters to the vortex motions adopted as initial conditions in the vortex theory of galaxy formation.[2-5]

3. LARGE-SCALE SPATIAL INHOMOGENEITIES

Whereas for the vortex theory of metagalactic structure[2-5] an essential requirement is that the sizes of the fundamental energy containing vortex, L, and of the event horizon, ct, should be close to each other at the instant t_{eq} when the matter energy density and radiation energy density become equal, for $t<t_{eq}$ the characteristic scale of the inhomogeneities which produce a vortex of the required scale will exceed the size of the horizon. Thus, one of the approximations simplifying the analysis of the equations (2.6)–(2.8) for $t<t_{eq}$ is an expansion in terms of the small parameter $ct/L\ll 1$. In this approximation one may neglect in the left-hand side of Eq. (2.8) the terms of $P^{(b)}_{(a)}$ of order L^{-2} vis-a-vis the other terms which are of order $(ct)^{-2}$. However, this is not always admissible over the whole interval of times $0\leqslant t\leqslant t_{eq}$, since it is possible that, as will be shown below, after the onset of the Kasner stage and further for $t\to 0$, the terms in $P^{(b)}_{(a)}$ along directions with negative Kasner exponent become important. Although this situation has its specificity, it is similar to that which can be encountered in the investigation of homogeneous models, when we are dealing with the change of Kasner regimes (cf., e.g., Ref. 15).

As a second assumption, also simplifying the investigation of the system (2.6)–(2.8) and yielding a more intuitive picture of the evolution of a spatially nonhomogeneous Universe during the stage $t<t_{eq}$, we assume:

$$u_{(a)}u^{(b)}\ll 1,\quad u_0u^0\sim -1,$$

i.e., we shall assume that the velocities of the motions of matter at a time near t_{eq} are small. It should be noted, however, that like the first assumption, this second assumption has a limited range of applicability for $t\to 0$, since during the Kasner stage the separate components of the velocity may increase taking on relativistic values (cf. infra).

Making use of these approximations and choosing as equation of state for matter during this stage of evolution of the Universe the equation of state of ultrarelativistic matter $p=\varepsilon/3$, we rewrite the system of (2.6)–(2.8) in the following simpler and more convenient form for the further analysis:

$$^1/_2\dot{\varkappa}^{(a)}_{(a)}+{}^1/_4\varkappa^{(b)}_{(a)}\varkappa^{(a)}_{(b)}=-\varkappa\varepsilon, \tag{3.1}$$

$$^1/_2(\varkappa^{(b)}_{(b)|(a)}-\varkappa^{(b)}_{(a)|(b)})={}^4/_3\varkappa\varepsilon u_{(a)}, \tag{3.2}$$

$$(-\gamma)^{-1/4}[(-\gamma)^{1/4}\varkappa^{(c)}_{(c)}]^{\cdot}+{}^1/_4\varkappa^{(b)}_{(a)}\varkappa^{(a)}_{(b)}=0, \tag{3.3}$$

$$(-\gamma)^{-1/2}[(-\gamma)^{1/2}(\varkappa^{(b)}_{(a)}-{}^1/_3\delta_a{}^b\varkappa^{(c)}_{(c)})]^{\cdot}=0. \tag{3.4}$$

It should be noted that the equations (3.3) and (3.4) do not involve the characteristics of matter: the energy density and the velocity of matter. Consequently they are a system of equations describing the evolution of the geometric model under consideration with time.

Integration of Eq. (3.4) yields

$$\varkappa^{(b)}_{(a)}=\delta_a{}^b\frac{\dot{X}}{X}+A^{(b)}_{(a)}X^{-3/2}, \tag{3.5}$$

where $X\equiv(-\gamma)^{1/3}$, $A^{(b)}_{(a)}$ are arbitrary functions of the coordinates, slowly varying over scales ct and satisfying the condition $A^{(a)}_{(a)}=0$. In the linear approximation the coordinate functions

$$A_{\alpha}{}^{\beta}=A^{(b)}_{(a)}e_{\alpha}{}^{(a)}e^{\beta}_{(b)}$$

correspond to vortex perturbations, gravitational waves and to the nondiagonal part of the potential perturbations in the Lifshitz expansion[1] in terms of eigenfunctions of the linearized nonregular part of the metric tensor.

The relation (3.5) determines in fact a deformation of the elements of the medium under consideration. The diagonal part characterizes the change of the specific volume of the medium under deformation, and the nondiagonal part, containing only the terms with $A^{(b)}_{(a)}$, describes the anisotropic deformations of different elements of space. Since the $A^{(b)}_{(a)}$ are determined only by giving a Cauchy hypersurface and are, in general, functions of the coordinates, one should expect that at different points of the space under consideration the anisotropy deformation will be different. The regions of space with identically vanishing values of $A^{(b)}_{(a)}$ will evolve without experiencing any deformation anisotropy, i.e., according to the Friedmann law.

Further, substituting (3.5) into (3.3) and integrating the equation so obtained, it is easy to obtain a relation which determines the time-dependence of X:

$$2R_0c(t-t_0)=[X(X+\xi)]^{1/2}-\xi\ln[(X^{1/2}+(X+\xi)^{1/2})/\xi^{1/2}]; \tag{3.6}$$

here

$$\xi = \frac{1}{24R_0^2} A^{(a)}_{(b)} A^{(b)}_{(a)},$$

$R_0 = \mathrm{const}(\sim 10^{25}\,\mathrm{cm})$, t_0 is an arbitrary slowly varying function of the coordinates, which according to[10] determines a retardation at each point of the irregularities when the system comes out of the cosmological singularity (in the linear approximation t_0/t corresponds to that part of the function μQ in the Lifshitz expansion[1] for potential perturbations, which was removed by infinitesimal coordinate transformations).[3)] We shall assume below that the spatially nonhomogeneous system under consideration comes out of the singular state simultaneously at each point of space, i.e., that t_0 does not depend on the coordinates. Then, without loss of generality one can set $t_0 = 0$ everywhere.

Integration of the system of equations (3.5) by means of the method proposed in Ref. 10 allows one to represent the matrix $\gamma_{(ab)}$ in the following form:

$$\gamma_{(ab)} = a^2_{(c)} \delta_{cd} M^{cd}_{ab}, \quad \delta_{cd} = \begin{cases} 1, & c=d \\ 0, & c \neq d' \end{cases} \tag{3.7}$$

where

$$a^2_{(c)} = X[((X+\xi)^{1/2} - \xi^{1/2})/((X+\xi)^{1/2} + \xi^{1/2})]^{3p_c - 1}. \tag{3.8}$$

In the general case the matrix M^{cd}_{ab} in (3.7) defines a local reorientation of the frame vectors $e^{(a)}_\alpha$ in the field of spatial inhomogeneities. However, in order to facilitate the remaining analysis of the solutions we shall assume that the background space does not exhibit its own anisotropy and the frame vectors $e^{(a)}_\alpha$ introduced before now reflect the presence of local space inhomogeneities. The latter fact will in general contradict the definition of $e^{(a)}_\alpha$ given before, as characteristics of the background space. However, such a redefinition of the $e^{(a)}_\alpha$ allows one to use all the relations introduced above without any changes of notation, and the matrix M^{cd}_{ab} reduces in this case to the form

$$M_{ab}{}^{cd} = {}^1/_2 (\delta_a{}^c \delta_b{}^d + \delta_a{}^d \delta_b{}^c). \tag{3.9}$$

Then the matrix $\gamma_{(ab)}$ takes the form

$$\gamma_{(ab)} = a^2_{(a)} \delta_{(ab)}. \tag{3.10}$$

The coordinate function p_a in (3.8) satisfies the conditions

$$\sum_a p_a = \sum_a p_a{}^2 = 1$$

and is determined by the relation

$$p_a = {}^1/_3 (1 + r_a/(4R_0{}^2 \xi)^{1/2}),$$

where r_a are the eigenvalues of the matrix $A^{(b)}_{(a)}$.

The equalities (3.6), (3.8), (3.10) describe completely (within the framework of the assumptions made above) the time evolution of the geometry of the space under discussion within the causally connected region. One should however note to highest order in the parameter $ct/L \ll 1$, within this causal region $A^{(b)}_{(a)}$ does not depend on the coordinates, i.e., for a given element of space of size ct the matrix $A^{(b)}_{(a)}$ is an array of constants which determine the deformation anisotropy of the selected element of the medium. Since the model under consideration is essentially nonhomogeneous, in distinction from the homogeneous models, one cannot say anything on the fate of other arbitrarily chosen volumes of matter in terms of the solution at one given point in space. The character of evolution of the former will be determined by their appropriate array of constants $A^{(b)}_{i\,(a)}$ (the index i labels the chosen volume element). In principle a situation is possible when the deformation anisotropy of space is the same at each of its points, i.e., the matrix $A^{(b)}_{(a)}$ is strictly coordinate-independent. In this case (adding the condition that the geometric properties be identical at all points of three-space) the space under discussion can be considered as homogeneous. A field of vortex velocities in such a space can be defined by means of a special choice of the frame vectors.[8,9] An investigation of the character of the evolution of such models is significantly simplified compared to the general case since the Einstein equations reduce to a system of ordinary differential equations, and the solution at an arbitrarily chosen point allows one to draw conclusions about the evolution of the model as a whole. However, such an approach to the problem of the possible appearance of intense vortex motions in the Universe, in spite of its simplicity and attractiveness, does not exclude other hypotheses, but on the contrary, by its results makes them necessary.

In spite of the fact that in the approach considered here the functions $A^{(b)}_{(a)}$ vary little over scales ct, the values of the space derivatives of these functions at an arbitrary point inside the region ct are not necessarily close to zero. This allows one, using Eqs. (3.1), (3.2) (3.5), (3.8), (3.10), to relate quantities which characterize the deformation anistropy of space with the velocity of matter relative to the chosen synchronous frame:

$$\varkappa\varepsilon = 3R_0{}^2/X^2, \tag{3.11}$$

$$u_{(a)} = -\frac{X^{1/2}}{8R_0{}^2} [8R_0 (X+\xi)^{1/2}_{,(a)} + f_{(a)}], \tag{3.12}$$

where

$$f_{(a)} = A^{(b)}_{(a),(b)} + C^f_{\cdot ab} A^{(b)}_{(f)} - C^b_{\cdot fb} A^{(f)}_{(a)}. \tag{3.13}$$

The components of the physical velocity, defined in the approximation $u^{(a)} u_{(b)} \ll 1, u_0 u^0 \approx -1$ by the relation $v_{(a)} = [u_{(a)} u^{(a)}]^{1/2}$ (no sum over a), can be written with the aid of (3.8), (3.12) in the form

$$v_{(a)} = \frac{1}{8R_0{}^2} \left[\frac{(X+\xi)^{1/2} - \xi^{1/2}}{(X+\xi)^{1/2} + \xi^{1/2}} \right]^{1/2(1-3p_a)} [8R_0 (X+\xi)^{1/2}_{,(a)} + f_{(a)}]. \tag{3.14}$$

The relations (3.6), (3.8) (3.10), (3.11), and (3.12) describe, within the domain of applicability of the approximations circumscribed above, and within the span of a separate causally connected region, the time evolution of the geometry of space, the energy densities of matter and of the vortex velocity field, i.e., these solutions allow one to obtain a general idea on the evolution of matter in a given space element, where the geometry varies with time in a known manner. It should be noted that the solutions given here have a Kasner asymptotic behavior for $t \to 0$, and a Friedmann behavior for $t \to \infty$:

$$X=\begin{cases}2R_0ct, & t\to\infty\\ 4\xi[{}^3/_8\xi^{-1}R_0ct]^{2/3}, & t\to 0\end{cases},\qquad(3.15)$$

$$v_{(a)}=\begin{cases}f_{(a)}/8R_0^2, & t\to\infty\\ {}^3/_{16}R_0^2\xi^{-2}[{}^3/_8\xi^{-1}R_0ct]^{-4/3}, & t\to 0\end{cases},\qquad(3.16)$$

$$\varkappa\varepsilon=\begin{cases}3/4(ct)^2 & t\to\infty\\ {}^3/_{16}R_0^2\xi^{-2}[{}^3/_8\xi^{-1}R_0ct]^{-4/3}, & t\to 0\end{cases},\qquad(3.17)$$

$$a_{(a)}^2=\begin{cases}2R_0ct, & t\to\infty\\ 4\xi[{}^3/_8\xi^{-1}R_0ct]^{2p_a}, & t\to 0\end{cases}.\qquad(3.18)$$

Thus, the spatially nonhomogeneous model, being essentially anisotropic on the scale of the irregularities during the early stages, isotropizes itself with the passage of time and gradually goes over into the Friedmann solution. The instant of isotropization of space at a given point will be determined by its degree of deformation at the given point. And since the system under consideration is essentially inhomogeneous, i.e., the degree of the deformation anisotropy of space varies from point to point, the isotropization of space will occur at different times in different regions of space.

4. THE CHARACTERISTIC STAGES OF EVOLUTION

For a more pictorial concept on the evolution of a spatially nonhomogeneous system in the stage $t<t_{eq}$ we estimate the characteristic times when the functional time-dependence of all its characteristics undergoes a significant change, namely: t_F the instant of isotropization of the system, t_r the time when, on account of a kinematic increase of the velocity of matter motion relative to the synchronous reference system they become relativistic, t_c, the instant of time from which on spatial curvature begins to play an essential role in the dynamics of the system under consideration, i.e., when it becomes necessary to take into account $P_{(a)}^{(b)}$ in the solutions of Eqs. (2.8).

In fact the time t_F can be considered as the instant when the asymptotic behaviors written out above go over into each other. Before t_F we are dealing with a vacuum solution, the presence of matter does not influence in fact the functional time dependence of the metric, and for t larger than t_F the influence of matter on the behavior of the system is substantial, leading to an isotropization of the model. We therefore define t_F, e.g., as the point of intersection of the curves $\varepsilon(t)$ given by (3.17), corresponding to the two asymptotic regimes. It is easy to see that to this point corresponds a value t_F equal to

$$t_F=9\xi/8cR_0.\qquad(4.1)$$

Making use of the latter relation, the exact solution (3.6) and (3.11) which determines the parametric dependence of the energy density of matter on time as one goes from the Kanser regime to the Friedmann regime can be replaced, according to Ref. 16, by the more intuitive asymptotic expression

$$\varkappa\varepsilon=\frac{3}{4}\frac{1}{c^2t^{4/3}(t+t_F)^{2/3}}.\qquad(4.2)$$

One can also represent in a similar asymptotic form the solutions which describe the evolution of the velocity field (3.14) and the time change of the scale factor:

$$v_{(a)}=\frac{2^{1/2}}{4R_0^2}\left(\frac{t}{t+{}^{64}/_{27}t_F}\right)^{1/3(1-3p_a)}\left[\left(\frac{R_0}{c}\right)^{1/2}\frac{\xi_{,(a)}}{(t+{}^{16}/_9t_F)^{1/2}}+f_{(a)}\right],\qquad(4.3)$$

$$a_{(c)}^2=2R_0ct^{2/3}(t+t_F)^{4/3}[t/(t+{}^{64}/_{27}t_F)]^{2/3(3p_c-1)}.\qquad(4.4)$$

It can be seen from the relation (4.1) that t_F is proportional to the quantity ξ which characterizes the deformation of space at the given point. Since in the case under consideration at $t\to 0$ the deformation anisotropy is generated by the field of vortex velocities at an instant of time close to that when the energy densities of matter and radiation become equal, it is natural to assume that the magnitude of the deformation anisotropy of space at an arbitrary point due to an individual vortex will be determined by the value of the velocity of matter in that vortex at $t\sim t_{eq}$. Since in any individual vortex the velocity vanishes at the center of the vortex[4] and outside the vortex, it is also natural to assume that there will be no deformation anisotropy of space in these regions, and that it will attain its maximal value in the intermediate region of the inhomogeneity.

Taking into account these considerations, one can imagine the evolution of the spatial inhomogeneities in the following manner: At the first instant of time, when the vortex just disappears beyond the horizon, the whole system evolves according to Friedmann's law. Further, as $t\to 0$ there appears a stage when the system gradually goes over from the Friedmann behavior to the Kasner behavior, first in the regions with maximal deformation anisotropy, then towards the center and the periphery of the vortices, as the influence of the deformation anisotropy spreads. The other regions continue to evolve according to a law close to Friedmann's. With time the deformation of space becomes substantial for all the anisotropy regions. Thus with time, as $t\to 0$, the whole system goes over into a Kasnerian evolution regime.

As was noted above, initially the small velocities of motion of matter after the system enters the Kasner stage, begin to grow along different directions for $t\to 0$ according to a law determined by the second relation (3.16), and starting from some time t_r, they become relativistic. This circumstance does not influence the character of the time evolution of the geometry of space, since the Kasner solution is a vacuum solution, i.e., the influence of matter is negligible. However the form of the functional dependence on time of all physical characteristics of the system is radically changed, in particular that of the energy density and the velocity of matter motion.[17]

Since, according to the second equation (3.16), the component of velocity along the direction with the Kasner exponent p_3 ($p_3\geq p_2\geq p_1$) grows faster than the others for $t\to 0$, and consequently attains its relativistic value earlier than the others, it makes sense to define t_r by the condition $t=t_r$ when $u_{(3)}u^{(3)}$ becomes comparable to one, as was done by Tomita.[11] It is easy to see that in this case t_r will be

$$t_r=\frac{8}{3}\frac{\xi}{R_0c}\left|\frac{\Phi_{(3)}}{2R_0^2}\right|^{3/(3p_3-1)}\qquad(4.5)$$

where

$$\Phi_{(a)}=R_0\xi_{,(a)}\xi^{-1/_2+1/_4}f_{(a)}.$$

Making use of the Lifshitz-Khalatnikov relations[17] which are valid during the vacuum stage and have been obtained from the equations of relativistic hydrodynamics:

$$u_{(a)}(\varkappa\varepsilon)^{1/4}=C_{(a)},\quad (-\gamma)^{1/2}u_0(\varkappa\varepsilon)^{3/4}=C \tag{4.6}$$

(where in general $C_{(a)}$ and C are arbitrary functions of the coordinates), and knowing the explicit form of the solution for $t>t_r$, (3.13)–(3.16), it is easy to obtain expressions for ε and $u_{(a)}$ which reflect their functional time-dependence in the ultrarelativistic stage.

It is first necessary to relate the functions $C_{(a)}$ and C with the quantities which have been introduced earlier. This is easily done if one substitutes into (4.6) the relations (3.13)–(3.16) valid for $t>t_r$:

$$C=(3R_0^2)^{3/4},\quad C_{(a)}=-(3R_0^2)^{1/4}\Phi_{(a)}/2R_0^2. \tag{4.7}$$

Then, since in the ultrarelativistic stage ($t<t_r$)

$$u_0=u_{(3)}a_{(3)}^{-1},$$

one can obtain from (4.6) by means of relations (3.18) and (4.7) the expressions for ε and $u_{(a)}$, valid for $t>t_r$:

$$\varkappa\varepsilon=\frac{3}{4}\left(\frac{t_F}{t_r}\right)^{4/3}\left(\frac{t}{t_F}\right)^{2(p_3-1)}(ct_F)^{-2}, \tag{4.8}$$

$$u_{(a)}=-2\xi^{1/2}\left|\frac{\Phi_{(a)}}{\Phi_{(3)}}\right|\left(\frac{t}{t_r}\right)^{1/2(1-p_3)},\quad u^{(b)}=\gamma^{(ab)}u_{(a)}. \tag{4.9}$$

Since the velocities of motion of matter relative to the chosen synchronous reference system become relativistic for $t<t_r$, the proper time τ in a comoving reference system will differ from the time t. The relation between these two times is given by

$$d\tau=dt(1-v^2)^{1/2}, \tag{4.10}$$

where the components of the physical velocity $v_{(a)}$ are related to the components of the four-velocity by means of the relation

$$v_{(a)}^2=u_{(a)}u^{(a)}/(u^0)^2$$

(no summation with respect to a). Then, making use of (4.9), (4.10) and the last relation we obtain (cf. Refs. 17, 11)

$$\tau=\frac{2t_r}{3p_3+1}\left(\frac{t}{t_r}\right)^{1/2(3p_3+1)}. \tag{4.11}$$

Substituting this into (4.8) we obtain for the energy density

$$\varkappa\varepsilon=\frac{3}{4}\left(\frac{t_F}{t_r}\right)^{4/3}\left[\frac{3p_3+1}{2}\frac{\tau}{t_r}\right]^{4(p_3-1)/(3p_3+1)}(ct_F)^{-2}. \tag{4.12}$$

It should be noted that similar to the way in which the model gradually, within the scale of a single vortex, goes over from a Friedmann regime to a Kasner regime of evolution, also gradually, starting from regions with initially maximal velocities towards the edges and the center the velocities of matter relative to the reference system (2.1) will attain relativistic values. Moreover, if matter moved with relativistic velocities already for t close to t_{eq} then the compatibility condition for the system (2.6)–(2.8) in the approximation $L\gg ct$ implies that in fact immediately after the appearance of the vortex above the horizon, a Kasner evolution regime starts in those regions where the deformation of space is maximal.

Thus, if one moves backward in time from the instant when the energy densities of radiation and matter are equal, then in the more general case, when the velocities of matter at the initial time are not too large, one will observe the following picture. The spatially nonhomogeneous system gradually forsakes the Friedmann solution and starting from time t_F (different for each causally connected region) goes over into the Kasner regime. In spite of the fact that starting from $t=t_r$ the velocities of motion of matter attain relativistic values, the solution remains Kasnerian up to $t\sim t_c$, since the influence of matter on the metric is negligibly small. Starting from this time it is necessary to take into account in Eqs. (2.8) the curvature of space.

The latter circumstance is related to the fact that for $t\to 0$ the terms which enter into $P_{(a)}^{(b)}$ which have the characteristic structure

$$P_{(a)}^{(b)}\sim{}^1/_2(a_{(1)}/a_{(2)}a_{(3)})^2l^{-2}, \tag{4.13}$$

will grow along directions with Kasner exponent p_1 faster than

$$((-\gamma)^{1/2}\varkappa_{(a)}^{(b)})^{\cdot}/2(-\gamma)^{1/2}\sim(ct)^{-2},$$

and at the instant corresponding to t_c both expressions become equal in order of magnitude, whence we obtain for t_c

$$t_c=\left(\frac{4}{3}\right)^3\left[\frac{9}{8}\frac{R_0l}{(2R_0ct_F)^{1/2}}\right]^{p_1/2}t_F \tag{4.14}$$

The quantity l which enters the expressions (4.13), (4.14) represents the coordinate distance over which there is a noticeable change of the geometric properties of space. But since in the most general case the latter are determined by two factors: the geometric properties of space itself, without taking into account inhomogeneities, and the geometric properties of the inhomogeneities embedded into the background space, then one should take for in (4.13) $l=\min\{\tilde{L},\tilde{\mathscr{L}}\}$. The proper dimensions L and $\mathscr{L}$ corresponding to the characteristic sizes of spatial inhomogeneities and of noticeable changes of the curvature of the homogeneous part of space will be determined by the product of a scale factor with the appropriate coordinate distances $\tilde{L}$ and $\tilde{\mathscr{L}}$. The analysis of the evolution is considerably simpler for $t\lesssim t_c$, when $L\gg\mathscr{L}$. In this case the Universe evolves according to laws which are proper to homogeneous models, which is understandable, since the two inequalities $L\gg ct$, $L\gg\mathscr{L}$ eliminate in fact the inhomogeneities from the equations which determine the geometric properties of space. The character of evolution of such a system for times $t<t_c$ will be determined by the Bianchi type to which the model belongs.

For $L\ll\mathscr{L}$ the earliest stages of evolution of the World can be investigated only in the case when one succeeds in excluding $P_{(a)}^{(b)}$ from the equations (2.13) by means of some "model considerations."

5. A UNIVERSE WITH A VORTEX

As a particular application of the results obtained above one may consider the case when a single vortex is embedded into a background space which in itself (i.e., in the absence of the irregularity) satisfies the requirements of homogeneity and isotropy. One can represent this mathematically in the following fashion. We place the coordinate system at the center of the spatial inhomogeneity and prescribe on a spacelike Cauchy hypersurface the matrix A_α^β, which enters the relations (3.5), (3.13) and into the definition of ξ, in the form

$$A_\alpha^\beta = A_0\varphi(\rho, y)\begin{Bmatrix} \sin 2\gamma & 0 & -\cos 2\gamma \\ 0 & 0 & 0 \\ -\cos 2\gamma & 0 & -\sin 2\gamma \end{Bmatrix}, \tag{5.1}$$

where

$$\rho = \frac{\pi}{2L}(x^2+z^2)^{1/2}, \quad \gamma = \mathrm{arctg}\frac{z}{x}$$

(x, y, z are the Cartesian coordinates). Such a representation of the matrix A_α^β corresponds to prescribing a vortex inhomogeneity with rotation in the (x, z) plane.

It is necessary to note that the form of the functions A_α^β was chosen out of the consideration that at the stage when the deformation anisotropy of space is small, i.e., when it can be considered as a small perturbation of the Friedmann solution ($t > t_F$), the velocity field inside the inhomogeneity corresponds to a vortex motion of matter. It is easy to see that in this case, for $t > t_F$ the components of the vortex velocity can be represented in terms of the previously introduced quantities in the form:

$$\begin{aligned} v_1 &= -\frac{\pi}{16L}\frac{A_0}{R_0^2}\left(\varphi_{,\rho} + \frac{2\varphi}{\rho}\right)\sin\gamma, \\ v_3 &= \frac{\pi}{16L}\frac{A_0}{R_0^2}\left(\varphi_{,\rho} + \frac{2\varphi}{\rho}\right)\cos\gamma, \end{aligned} \tag{5.2}$$

from where one can see that the vortex velocities are proportional not only to the derivative of the function φ which determines the degree of spatial deformation as a function of the increasing distance from the center of the inhomogeneity but also to the value of the function itself. Consequently, in those regions where the vortex velocities vanish identically the anisotropy of deformation of the space must necessarily vanish.

The explicit form of the function φ can be chosen, e.g., from considerations of localization of the inhomogeneity in some region of space. This condition can be satisfied, in particular, by the following expression

$$\varphi = \mu\rho^2 \exp\left[-\rho^n - (\pi y/2L)^m\right] \tag{5.3}$$

(here μ is a normalization constant).

The inhomogeneity amplitude A_0 which enters into the relations (5.1) and (5.2) can be estimated starting from the condition that at the instant when the energy densities of radiation and matter are equal, t_{eq}, the spatial inhomogeneity under consideration must lead to vortex motions with parameters corresponding to the vortex theory of galaxy formation,[2-5] whence, by analogy to Tomita's reasoning,[10] one can obtain for A_0 the following estimate

$$A_0 = \frac{20}{\mu}\frac{\alpha_{opt}}{\alpha}(R_0^3 c t_{eq})^{1/2} w^2, \tag{5.4}$$

where $w \equiv v/c$, and α is the vortex spreading parameter (cf.[4]).[5)]

In terms of the eigenvectors

$$e_\alpha^{(a)} = \frac{1}{2^{1/2}}\begin{Bmatrix} \cos\gamma+\sin\gamma & 0 & \cos\gamma-\sin\gamma \\ 0 & \sqrt{2} & 0 \\ -\cos\gamma+\sin\gamma & 0 & \cos\gamma+\sin\gamma \end{Bmatrix} \tag{5.5}$$

the matrix (5.1) can be transformed to diagonal form, with the eigenvalues along the diagonal:

$$r_1 = -A_0\varphi, \quad r_2 = 0, \quad r_3 = A_0\varphi.$$

Then, considering that in the case under discussion

$$\xi = A_0^2\varphi^2/12R_0^2,$$

one obtains for the Kasner exponents which determine the dynamics of the early stages of evolution of the model in the vortex localization region the following values:

$$p_1 = (1-3^{1/2})/3, \quad p_2 = {}^1/_3, \quad p_3 = (1+3^{1/2})/3, \tag{5.6}$$

i.e., within the framework of the model under consideration the Kasner exponents are coordinate-independent. Consequently at different points of the spatial inhomogeneity the character of the temporal evolution will differ only by the time t_F when each volume element goes into the Kasner stage. This is related to the fact that t_F is a coordinate function:

$$t_F \approx 75\pi^{-2}\varphi^2(\alpha_{opt}/\alpha)^2 w^4 t_{eq}. \tag{5.7}$$

The same can be said about the instant of time when the velocities of motion of matter relative to the chosen synchronous reference system become relativistic:

$$t_r \approx \frac{800}{\pi^2}\left(\frac{\alpha_{opt}}{\alpha}\right)^2\left(\frac{w}{3^{1/2}}\right)^{4+3^{1/2}}\varphi^2\left|\left(1-\frac{3^{1/2}}{4}\right)\varphi_{,\rho} - \frac{3^{1/2}}{2\rho}\varphi\right|^{3^{1/2}} t_{eq}. \tag{5.8}$$

It is necessary to note that in leading order of the parameter $ct/L \ll 1$ the chosen model does not require taking into account the spatial curvature in solving Eqs. (3.1)–(3.4) up to $t = 0$. This is easily verified by substituting (5.5) and (3.18) with the Kasner exponents (5.6) into the expression (2.6) for $P_{(a)}^{(b)}$. However, before one can assert that this solution is asymptotically correctly describing the evolution of the model up to $t = 0$, it is necessary to prove rigorously that if one takes into account higher orders in $ct/L \ll 1$ in $P_{(a)}^{(b)}$ there do not appear terms which grow for $t \to 0$ faster than t^{-2}.

6. CONCLUSION

The spatially inhomogeneous model described in the last section with a single vortex embedded in a homogeneous and isotropic background space, can be considered as a special case of a more general problem, when the background space is more or less uniformly filled with a large number of similar inhomogeneities. Without indulging in the mathematical analysis of the possible peculiarities of such models one may assert that they preserve all the properties enumerated above, i.e., in the process of the reverse evolution in time, in the localization region of any inhomogeneity, the system will gradually deviate from the isotropic solution, changing from a Friedmann character of evolution to a Kasner behavior. In the same manner gradually, starting from regions with maximal value of the velocity and encompassing larger and larger regions of inhomogeneity, the system will go over into an ultrarelativistic stage.

It should also be noted that the initial presence of only vortex motions of matter leads with time ($t\to 0$) to the appearance of intense potential motions. These potential motions will be responsible for the redistribution of matter through space, related to differences in the evolution of the system in different regions. The appearance of potential motions (for $t\to 0$) from pure vortex motions is not the result of generation of the former by the latter, as would happen in hydrodynamics. This is a specifically relativistic effect, related to a rearrangement of the character of evolution of the model as a result of the development of the deformation anisotropy of space introduced by the vortices. For instance, in distinction from Friedmann's law of time variation of the matter energy density $\varepsilon\sim t^{-2}$ in those regions where the solution has a Kasner asymptotic behavior we obtain $\varepsilon\sim t^{-4/3}$.

Thus, one of the consequences of the presence of spatial inhomogeneities in the early Universe one can consider a change of its temperature history, which, as should have been expected, will be different in different regions of the inhomogeneous space.

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1) We have in mind inhomogeneity of three-space.

2) I do not give here a detailed derivation of the equations (2.6)–(2.8) with the relations (2.9)–(2.15), in view of its length. I only note that it is analogous to the derivation of the same equations in the case of homogeneous models.[14] Here and everywhere below we have in mind homogeneity in the group-theoretic sense.

3) A detailed comparison of the linear approximation of the solutions with the analysis of Lifshitz[1] can be found in Tomita's paper.[10]

4) It is assumed here that there is no overall displacement of the vortex relative to the synchronous reference system.

5) The introduction into (5.4) of the parameter α presupposes, of course, the presence in the system of a large number of inhomogenities of the type under consideration, which lead to the appearance of intensive vortex motions which for $t>t_{eq}$ participate in the process of hydrodynamic mixing. Here this parameter is introduced only for the purpose of a quantitative comparison of the parameters in which we are interested.

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