Investigation of the cosmological evolution of viscoelastic matter with causal thermodynamics

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The cosmological solutions are investigated with allowance made for the dissipative processes by means of a viscoelastic description that excludes infinite velocities of propagation of signals. The expansions of the thermodynamic functions up to second order in the deviations from the equilibrium quantities are used. The Bianchi type-I model is investigated. It is shown that the effect of matter creation near the initial singularity is preserved, as in the case of simple viscosity investigated by two of the present authors [Sov. Phys. JETP **42**, 205 (1975)], while the effect of isotropization during the contraction disappears. The cosmological singularity necessarily exists at some stage of the evolution, but it may pertain to a new type connected with the accumulation of elastic energy. The stability of the Friedmann solutions in the vicinity of the singular point is investigated. It is shown that for these solutions to be stable, it is necessary to admit the propagation of signals with velocities higher than the velocity of light. Since the existence of such signals is impossible, the Friedmann solutions are unstable in the vicinity of the singularity.

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§1. INTRODUCTION

The dissipative processes occurring in the course of the cosmological evolution is customarily taken into consideration with the aid of a viscous stress tensor. At the same time, it is known that the equations of motion of a viscous fluid are parabolic, and therefore lead to an infinite velocity of propagation of signals (in particular, of explosions). Such a phenomenon contradicts the causality principle. This contradiction arises as a result of the fact that the hydrodynamic theory of a viscous fluid is applicable only under conditions when the derivatives of the velocity of the matter with respect to time and the coordinates are small. This condition is necessarily violated in the vicinity of the singularity, and the description of the dissipative processes with the aid of two viscosity coefficients becomes inapplicable.

The object of the present paper is to eliminate the indicated deficiencies in the investigation of the effect of the dissipative processes on cosmological evolution. This is attained by altering the stress tensor of matter. The relation between strain and stress has the form of a relaxation equation similar to the Maxwell equation in the theory of viscoelasticity. Such a relation implies that matter behaves like a normal viscous fluid if the periods of its motion are long, and like an elastic solid if the periods of the motion are short. The characteristic time is the stress-relaxation time τ. For matter with such a stress tensor (such matter is usually called viscoelastic matter) we can formulate the following causality principle: the velocity of propagation of waves is less than c.

The thermodynamics of viscoelastic matter differs from the previously used[1,2] thermodynamics of a viscous fluid. A consistent relativistic theory of viscoelastic matter is constructed in Ref. 3. Below we give only a summary of the essential results. A systematic exposition of the nonrelativistic theory of viscoelasticity can be found in Astarita and Marrucci's[4] or Blend's[5] book.

The equations of the hydrodynamics of a viscous fluid contain four kinetic coefficients: the dilatational- and shear-viscosity coefficients ξ and η and the dilatational- and shear-stress relaxation times τ_0 and τ_1. In the present paper, as in Ref. 1, it is assumed that the viscosity coefficients are functions of the energy density, which are approximated by power dependences in the regions of small and large values of the argument. The energy-density dependences of τ_0 and τ_1 also turn out under reasonable physical assumptions to be power functions at large and small values of the energy density.

The arrangement of the material in the paper is as follows. In §2 we carry out an analysis of the cosmological evolution, as illustrated by the homogeneous type-I model. The most interesting effects arise at those stages of the cosmological evolution when the viscoelastic stresses become of the same order of magnitude as the energy density, or exceeds it in order of magnitude. One of the effects of this sort is the already well-known "matter-creation" effect.[1] Another interesting effect consists in the fact that isotropic expansion can start from a state in which a significant portion of the initial energy density is elastic energy. On the whole, the behavior of the solutions in the neighborhoods of the singularities differ greatly from the behavior of the corresponding results of Ref. 1, where the type-I model is also investigated. The most important difference consists in the result that isotropic contraction is unstable under certain conditions (following, as shown in §4, from the causality principle).

In §3 we investigate the general solutions of the Einstein equations for viscoelastic matter. We find the

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number of physically different arbitrary functions that enter into the general solution.

In §4 we investigate the evolution of perturbations of the isotropic metric, and find the velocities of propagation of short-wave perturbations. It is shown that, owing to the causality principle, the quasi-isotropic solution cannot be stable, and a general solution that is close to the quasi-isotropic solution cannot exist.

For the understanding of the obtained results, it is important to remember that, in the vicinity of the singularity corresponding to infinite energy density, matter behaves with respect to shear like an elastic body.

The information given below is sufficient for the reading of the paper.

The energy-momentum tensor of viscoelastic matter has the form

$$T_{ik}=\varepsilon u_i u_k+(p+\sigma)\Delta_{ik}+\sigma_{ik}, \tag{1.1}$$

where[1]

$$\Delta_{ik}=g_{ik}+u_i u_k$$

is the tensor of projection onto the space orthogonal to u_i; ε is the energy density; p, the pressure; σ, the dilatational-stress density σ_{ik}, shear-stress tensor, which satisfies the following conditions:

$$\sigma_i{}^i=0,\quad \sigma_k{}^i u^k=0,\quad \sigma_{ik}=\sigma_{ki}. \tag{1.2}$$

The relation between stress and strain has the form[2]

$$\sigma+\tau_0\dot\sigma=(\xi u^n{}_{;n}),\quad \dot\sigma=\sigma_{;i}u^i, \tag{1.3}$$

$$\sigma_{ik}+\tau_1\Delta_{im}\Delta_{kn}\dot\sigma^{mn}=-\eta\Delta_i{}^m\Delta_k{}^n(u_{m;n}+u_{n;j}-{}^2/_3\Delta_{mn}\Delta^{jl}u_{j;l}),\quad \dot\sigma^{ik}=\sigma^{ik}{}_{;j}u^j. \tag{1.4}$$

Here ξ and η are the second- and shear-viscosity coefficients. On the right-hand sides of these equations stand the usual expressions for viscous stresses.[6] The symbols τ_0 and τ_1 denote the stress relaxation times or the Maxwell times.[4]

The thermodynamic functions of viscoelastic matter can be expanded in series in powers of the stress tensor, just as is done in the theory of elasticity.[6] The entropy density S has the form

$$S=S_0-\frac{\sigma_{ik}\sigma^{ik}\tau_1}{2T\eta}-\frac{\sigma^2\tau_0}{T\xi}+C_1\frac{\sigma^3\tau_0^2}{\xi^2 T}+C_2\frac{\sigma\sigma_{ik}\sigma^{ik}\tau_0\tau_1}{\xi\eta T}$$

$$=S_0-\left(\frac{\partial S_0}{\partial\varepsilon}\right)_V\left(\frac{\sigma_{ik}\sigma^{ik}\tau_1}{2\eta}+\frac{\sigma^2\tau_0}{\xi}-C_1\frac{\sigma^3\tau_0}{\xi^2}-C_2\frac{\sigma\sigma_{ik}\sigma^{ik}\tau_0\tau_1}{\xi\eta}\right), \tag{1.5}$$

$$C_1,\ C_2\sim 1.$$

Here S_0 is the equilibrium entropy density. This expression is a generalization of a formula of elasticity theory.[6] As τ_0, $\tau_1\to\infty$, the ratio η/τ_1 goes over into the shear modulus, while ξ/τ_0 goes over into the bulk modulus.

The description of a fluid with the aid of the stress tensor satisfying Eqs. (1.3) and (1.4) is hydrodynamic, and not kinetic, and is therefore valid only when the state of the matter is close to the equilibrium state, i.e., only when those terms in the expansions of the thermodynamic functions which depend on the stress are small.

The first criterion of closeness of the state of the matter to the equilibrium state has the form

$$\frac{\sigma^{ik}\sigma_{ik}}{2T}\frac{\tau_1}{\eta}+\frac{\sigma^2\tau_0}{T\xi}\ll S_0 \tag{1.6}$$

or

$$\frac{\sigma^{ik}\sigma_{ik}}{2}\frac{\tau_1}{\eta}+\frac{\sigma^2\tau_0}{\xi}\ll\varepsilon,$$

which implies that the elastic energy is small compared to the total energy.

The second condition for the state of the matter to be close to the equilibrium state is that the terms of third order in σ and σ_{ik} in the expansion of $S^i_{;i}=(Su^i)_{;i}$ should be small compared to the second-order terms (S^i is the entropy-flux density).

§2. INVESTIGATION OF THE HOMOGENEOUS MODEL

The metric of the homogeneous Bianchi type-I model has the form

$$-ds^2=-dt^2+R_1^2dx_1^2+R_2^2dx_2^2+R_3^2dx_3^2,$$
$$\sqrt{-g}=R_1R_2R_3=R^3.$$

The reference system is synchronous and comoving. From (1.2) we obtain

$$\sigma_i{}^0=\sigma_{0i}=0,\quad \sigma_\alpha{}^\alpha=0.$$

Let us introduce the following notation:

$$H_\alpha=(\ln R_\alpha)^{\cdot}=\frac{\dot R_\alpha}{R_\alpha},\qquad H=\frac{1}{3}\sum_\alpha H_\alpha=(\ln R)^{\cdot}.$$

The Einstein equations can be written in the form

$$\sigma_\beta{}^\alpha=0,\quad \alpha\neq\beta,$$

$$-3\dot H-\sum H_\alpha^2=\frac{1}{2}(\varepsilon+3p+3\sigma), \tag{2.1}$$

$$3H_\alpha H+\dot H_\alpha={}^1/_2(2\sigma^\alpha+\varepsilon-p-\sigma), \tag{2.2}$$

where the σ^α are the diagonal components of the three-dimensional tensor σ_β^α:

$$(\sigma^1,\sigma^2,\sigma^3)=(\sigma_1{}^1,\sigma_2{}^2,\sigma_3{}^3).$$

The hydrodynamic equations $T^i_{k;i}=0$ reduce to the single equation:

$$\dot\varepsilon+3(\varepsilon+p+\sigma)H+\sum_\alpha H_\alpha\sigma^\alpha=0, \tag{2.3}$$

while the equations (1.3) and (1.4) assume the form

$$\sigma+\tau_0\dot\sigma=-3\xi H,$$
$$\sigma^\alpha+\tau_1\dot\sigma^\alpha=-2\eta(H_\alpha-H). \tag{2.4}$$

Adding Eqs. (2.1) and (2.2), we obtain the first integral of these equations:

$$9H^2-\sum_\alpha H_\alpha^2=2\varepsilon. \tag{2.5}$$

Taking into consideration the obvious inequality

$$\sum_\alpha H_\alpha^2\geqslant\frac{1}{3}\left(\sum H_\alpha\right)^2=3H^2,$$

we find from (2.5) that

$$3H^2 \geqslant \varepsilon. \tag{2.6}$$

The equality in (2.6) is possible only in the isotropic case, when $H_\alpha = H$ for all α.

Let us introduce the new variables:

$$X = -H_\alpha \sigma^\alpha, \quad Y = \sum (\sigma^\alpha)^2.$$

Then all the Eqs. (2.1)–(2.4) reduce to five equations:

$$\dot{\varepsilon} = X - 3(w+\sigma)H, \tag{2.7}$$

$$-\dot{H} - 3H^2 = {}^1/_2(w+\sigma) - \varepsilon, \tag{2.8}$$

$$^1/_2\tau_1 \dot{Y} + Y = 2\eta X, \tag{2.9}$$

$$\tau_0\dot{\sigma} + \sigma = -3\xi H, \tag{2.10}$$

$$\dot{X} + \left(3HX + \frac{X}{\tau_1}\right) = -\frac{4\eta}{\tau_1}(\varepsilon - 3H^2) - Y. \tag{2.11}$$

Here $w = \varepsilon + p$ is the enthalpy. We assume the chemical potential $(\partial\varepsilon/\partial n)_{V,T}$ to be equal to zero. In this case all the thermodynamic quantities and the kinetic coefficients can be expressed in terms of ε, and the particle number can vary. We set $w = \varepsilon\gamma$, $1 \leqslant \gamma \leqslant 2$, in the regions of small and large ε values. In this case the equilibrium entropy density S_0 is proportaional to $\varepsilon^{1/\gamma}$, and the second condition for the state of the matter to be close to the equilibrium state can be written in the form

$$\frac{Y}{2\eta} + \frac{\sigma^2}{\xi} \ll 3\left(\frac{\tau_0\sigma^2}{\xi} + \frac{\tau_1 Y}{2\eta}\right)\gamma H - C_1\frac{3\sigma^2(\sigma+3\xi H)\tau_0}{\xi^2} - C_2\frac{Y(\sigma+3\xi H)\tau_1}{\xi\eta} + C_2\frac{2\sigma(2\eta X - Y)\tau_0}{\xi\eta}. \tag{2.12}$$

In the region of intermediate ε values we make only the assumption that the function $w(\varepsilon)$ is a smooth function and that it has no zeros.

Equations (2.7)–(2.11) describe a dynamical system in five-dimensional phase space. We shall be interested in the integral curves in the physical region, which is separated out by the condition $\varepsilon \leqslant 3H^2$. The singular points of the dynamical system (2.7)–(2.11) that correspond to finite values of all the variables coincide with the final singularities of the system of equations for the case, investigated in Ref. 1, of matter in the form of a viscous fluid.

These singular points are the following:

$$\begin{aligned} &1.\ H = \varepsilon = \sigma = X = Y = 0 \text{ (point } O\text{)},\\ &2.\ X = Y = 0,\quad \varepsilon = 3H^2,\quad w(\varepsilon) = \xi(\varepsilon)(3\varepsilon)^{1/2} \text{ (points } N_1 \text{ and } N_2\text{)}. \end{aligned} \tag{2.13}$$

As the calculation shows, the pattern of integral curves in the neighborhoods of these points is the same as the pattern found in Ref. 1. Everything said in that paper about the corresponding singular points, N_1, N_2, and O, are applicable with insignificant changes in our model.

To investigate the behavior of the solutions at infinity, it is necessary to make assumptions about the behavior of ξ, η, τ_0, and τ_1 at large ε. As in Ref. 1, we assume a power dependence of the viscosity coefficients on ε for large values of ε:

$$\xi = \xi_1\left(\frac{\varepsilon}{\varepsilon_1}\right)^{b_2}, \quad \eta = \eta_1\left(\frac{\varepsilon}{\varepsilon_1}\right)^{b_1} \quad \text{for } \varepsilon \to \infty,$$
$$b_1 - b_2 \geqslant {}^1/_2, \quad b_2 \geqslant 0, \quad b_2 < {}^1/_2.$$

Concerning τ_0 and τ_1, we assume that

$$\frac{\tau_0}{\xi} \sim \frac{1}{\varepsilon} \quad \frac{\tau_1}{\eta} \sim \frac{1}{\varepsilon} \quad \frac{\eta}{\tau_1} \approx \delta\varepsilon \tag{2.14}$$

(δ is a positive constant) both for small and large ε. The relation (2.14) is the simplest dimensional relation between ξ and τ_0 and η and τ_1. Moreover, as will be shown in §4, it follows from the physically reasonable assumption that the velocity of propagation of transverse waves in matter has a finite (lower than the velocity of light), nonzero value both in the case of low, and in the case of high, matter densities.

A very important particular case is the Friedmann model with a flat three-dimensional space. Only two variables–σ and ε, or σ and H–remain in this case. The system can be investigated to the end, and the qualitative results will pertain also to the integral curves near the Friedmann surface in the phase space. Under these assumptions, the system (2.7)–(2.11) determines, besides the normal Friedmann evolution from the true singluarity $\varepsilon = \infty$, $H = +\infty$, $R_1 = R_2 = R_3 = 0$, two more families of solutions of a type unknown before. The corresponding integral curves in the (σ, H) plane are shown in Fig. 1. The family AZ_1 has singularities near the $H = 0$ ($\varepsilon = 0$) axis. The curves intersect this axis after a finite time, and the metric coefficients neither vanish, nor become infinite in this case. The criteria (1.6) and (2.12) show that the model is inap-

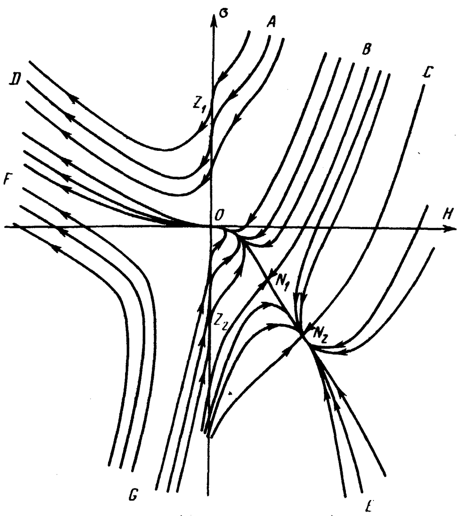

FIG. 1. Integral curves of the flat isotropic model in the (σ, H) plane. Besides the standard solutions EN_2 and OF, which correspond to expansion from the singular state with $\varepsilon = \infty$, $H = \infty$ and contraction from the infinitely rarefied state into the singular state, there are solutions of the type AZ_1 that pass through the $\varepsilon = 0$ state after a finite time, and nowhere have metric singularities. The singular points, N_1, N_2, and O [see (2.13)], of the system of equations are completely anallogous to the corresponding singular points in the case of simple viscosity. The entire pattern corresponds to the $b_2 < \frac{1}{2}$ case. The value $b_2 = \frac{1}{2}$ is not admissible on the basis of the criteria (1.6) and (2.12).

plicable in this region. Since it is difficult to attach a physical meaning to such singularities, they will, apparently, disappear in a more realistic model.

Solutions of the BO type correspond to expansion from the state with $H=+\infty$, $\varepsilon=\infty$, $\sigma \gg \varepsilon$. Undoubtedly, the model is inapplicable in the neighborhood of such a state. But the region of large positive σ, H, and ε should be in any model. This region corresponds to a large accumulation of elastic energy, which forms a significant portion of the total energy. Such an accumulation of elastic energy and its subsequent dissipation are entirely probable, and will, possibly, occur in more realistic models as well. The qualitative behavior of the solutions at high σ and H is as follows. The evolution of the Universe begins at some finite moment of time $t=t_0$. In the $\sigma>0$, $H>0$ region, at the initial moment the metric coefficients are finite and nonzero, and the matter is elastically compressed to such an extent that the dilatational-stress energy density is infinite. A sudden expansion, beginning with infinite velocity, occurs. In the $\sigma<0$, $H<0$ region the behavior of the solutions GF and GZ_2 differs from the behavior of the solutions of the BO family in that H is replaced by $-H$ and R by R^{-1}. At $t=t_0$ the matter is not compressed, but expanded, and it seems unlikely that the elastic energy can, under such conditions, form a significant portion of the total energy, or even attain large values. Such an expansion will lead to explosions.

Although it is difficult to explain why the elastic-energy density should become infinite at $R\neq 0$, the BO-type solutions should, apparently, remain in more realistic models, and should correspond to the real physical situation in which a large portion of the "initial" energy density is made up of elastic compression energy.

To investigate the behavior of all the solutions of the system (2.7)–(2.11) at high H, we carry out a compactification that simultaneously splits off the region where the model is inapplicable (see the Appendix). The Kasner and Friedmann singularities go over into the final singularities of the transformed system of equations. At the same time the points $H=+\infty$ and $H=-\infty$ stick together. The system cannot be linearized in the vicinity of the Kasner singularity. We were unable to find the general solution to the system (2.7)–(2.11) near this singularity, but we found many particular solutions, each of which exists under certain special conditions. We can give the following reasons. As will be shown later (§4), owing to the relativistic-causality conditions, the Friedmann singularity is necessarily unstable. This applies also to the other singular points, except the Kasner singularity. Therefore, only four-parameter families of curves emerge from them. But the complete integral of the system should contain five parameters (or arbitrary contants), since the phase space is five-dimensional. Therefore, a five-parameter family of curves should emanate from the Kasner point, but here, in contrast to Ref. 1, ε not only can vanish, but can also become infinite, at $t=0$. Formally, the criteria (1.6) and (2.12) are therewith violated in the second case, and can be violated in the first: instead of "much greater than" in (1.6) and (2.12), we have equality in order of magnitude. However, it is only for this reason that the solutions cannot be discarded, since it is not difficult to show that, during contraction,

$$\varepsilon\to\infty, \quad Y\sim\varepsilon^2, \quad \tau_1 Y/\eta\sim\varepsilon,$$

i.e., the criterion (1.6) is certainly violated. Thus, besides the solutions describing the "creation" of matter, there can also be solutions with an infinite initial energy density.

The system can be completely integrated in the vicinity of the isotropic singularity (for details, see the Appendix). In terms of the variables H, H_α, σ^α, and ε, the solution has the form

$$H=\frac{2}{3\gamma t}, \quad H_\alpha-H=C_\alpha{}^1 t^{\lambda_3/\gamma-1}+C_\alpha{}^2 t^{\lambda_4/\gamma-1}, \quad \sum_\alpha C_\alpha^{1,2}=0,$$
$$\sigma^\alpha=-\frac{8\delta}{3\gamma^2}\left(\frac{C_\alpha{}^1}{\lambda_3/\gamma-2}t^{\lambda_3/\gamma-2}+\frac{C_\alpha{}^2}{\lambda_4/\gamma-2}t^{\lambda_4/\gamma-2}\right), \tag{2.15}$$

where λ_3 and λ_4 satisfy the equation

$$(\gamma-2-\lambda)(2\gamma-\lambda)+{}^8/_3\delta=0.$$

The criterion (1.6) is always satisfied; the criterion (2.12) can be violated. When $\lambda/\gamma=2$, the corresponding term is replaced by $C_\alpha \ln t$. The dependence $\varepsilon(t)$ is found directly from the first integral, (2.5), of the Eqs. (2.1) and (2.2):

$$\varepsilon=3H^2-\frac{1}{2}\sum_\alpha (H_\alpha-H)^2.$$

The solutions are also valid for $t<0$ ($t^{\lambda/\gamma}$ is replaced by $|t|^{\lambda/\gamma}$). Here $H<0$ $H_\alpha<0$. The dependence $\sigma(t)$ can be found directly from Eq. (2.10) (we assume that $b_2<1/2$):

$$\tau_0\dot\sigma=-(\sigma+3\xi H),$$
$$\dot\sigma\sim\sigma/t, \quad \tau_0\sim t^{2-2b_2}\ll t,$$
$$\sigma+3\xi H\ll\sigma, \quad \sigma\approx-3\xi H.$$

Hence it can be seen that $\sigma\ll\varepsilon$, and therefore σ cannot have an effect on the solution (2.15).

As has already been indicated, relativistic causality requires that λ_3 or λ_4 should be negative. Owing to this, we can elucidate some distinctive features of the behavior of the curves at finite H and ε. The Friedmann singularity of the compactified system of equations will behave like a saddle point in the flat case, i.e., it will attract the curves emanating from the other singular points. Therefore, the curves emanating from the Kasner singularity will, on approaching the Friedmann parabola $3H^2=\varepsilon$, undergo characteristic "bumps." Figure 2 shows the projection of these curves onto the (H,ε) plane. Similar "bumps" are undergone also by the curves that begin at $\varepsilon=0$, $H=0$ (infinite rarefaction) and terminate in a Kasner contraction.

Near the Friedmann point, $\tau_1\dot Y\gg Y$, and matter behaves like an elastic body with respect to shear. The substitution $H\to -H$, $t\to -t$ does not, in the first approximation, alter in this region the pattern of behavior of the trajectories, owing to the insignificance

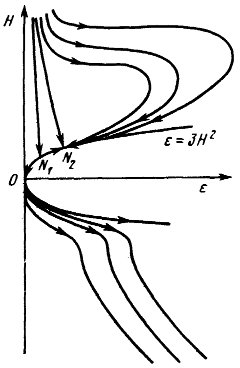

FIG. 2. "Bumps" produced on the integral curves by the instability of the Friedmann singularity. Shown are the projections of the curves onto the (ε, *H*) plane.

of the energy dissipation. For $H<0$, the integral curves in the general case cannot therefore terminate in isotropic contraction, just as when $H>0$ they cannot begin with isotropic expansion (the Friedmann point). The general pattern of behavior of the integral curves is, consequently, as follows. For $H>0$, the integral curves begin in the general case with a Kasner expansion from $\varepsilon=0$, or, possibly, from $\varepsilon=\infty$. They terminate in infinite isotropic expansion at the nodal point N_2 (see Figs. 1 and 2), or at the point O: $\varepsilon=0$, $H=0$.

For $H<0$, the curves begin at the point O and terminate, in the general case, in a Kasner contraction, but now ε should tend to infinity. An integral curve can get caught by a Friedmann singular point (the isotropic singularity) only if the corresponding solution (2.15) in the vicinity of this point does not contain a term with $\lambda<0$.

The instability of the isotropic contraction in the homogeneous model naturally implies the instability of the isotropic solution against small perturbations.

§3. PROPERTIES OF THE GENERAL SOLUTIONS OF THE EINSTEIN EQUATIONS

Two problems are of greatest interest in the investigation of the general solutions of the Einstein equations. The first problem consists in deciding which solution of the Einstein equations can be considered to be the general solution, and the second problem concerns the behavior of the general solution in the vicinity of the singularity.

To determine the number of arbitrary physically different functions that enter into the general solution, it is necessary to set up the Cauchy problem in some definite reference system. The investigation is simplest when the reference system is synchronous.

As the unknown variables, let us take σ_0^α, σ_β^α, $g_{\alpha\beta}$, $\dot g_{\alpha\beta}$, and σ. The quantities u^α can be expressed in terms of them, using relations that follow from (1.2):

$$\sigma_\beta{}^\alpha \frac{u^\beta}{u^0} = -\sigma_0{}^\alpha, \qquad \sigma_\alpha{}^\alpha = \sigma_0{}^\alpha \frac{u_\alpha}{u_0{}^2}.$$

In the process, one of the σ_β^α components also gets expressed in terms of the remaining components. Therefore, for example, σ_1^1 can be eliminated from the unknowns. Using the equation $R_0^0 - \frac{1}{2}R = T_0^0$, we can express the quantity ε in terms of

$$\sigma, \sigma_\beta{}^\alpha, \sigma_0{}^\alpha, g_{\alpha\beta}, \dot g_{\alpha\beta}.$$

From $R_\alpha^0 = T_\alpha^0$, we obtain three other relations between the unknowns, and we can eliminate three $\dot g_{\alpha\beta}$ components. The quantities $\dot\sigma$, $\dot\sigma_\beta^\alpha$, and $\dot\sigma_0^\alpha$ can be expressed in terms of the unknowns from Eqs. (1.3) and (1.4), into which the $\ddot g_{\alpha\beta}$ do not enter. The quantities $\ddot g_{\alpha\beta}$ can be expressed in terms of the unknowns from the equations $R_\beta^\alpha - 1/2R\delta_\beta^\alpha = T_\beta^\alpha$. Thus, we obtain 18 equations of first order with respect to t for 18 unknowns. The initial conditions for the Cauchy problem should be given in the form of 18 functions of the space coordinates, i.e., 18 values of the unknowns at $t=t_0$. The general solution contains 18 arbitrary functions of the x^α, and can be subjected to a transformation containing four arbitrary functions, and not violating the synchronism of the system. The number of physically different functions of the space coordinates in the general solution is 14. This result can also be obtained from a simple, but nonrigorous argument: the free field is specified by four functions; the distribution of matter, also by four (ε and the u^α); the stress tensor, by six functions (five independent components of σ_k^i and one for σ).

Also important for cosmology is the question of the character of the general solution of the gravitational equations, in particular, of its behavior near the cosmological singularity. An analysis of the homogeneous models allows us to conclude that, in the vicinity of the initial singularity, the general solution is oscillatory, and close to the solution constructed in Ref. 7.

The investigation of the behavior of the general solution near the final singularity is more complicated, since the dissipative processes lead to the rapid growth of the energy density, and could eventually lead to the destruction of the Kasner regime and the isotropization of the solution. Indeed, for matter in the form of a hydrodynamic viscous fluid and for viscoelastic matter (in some region of the parameters η and τ_1), we can formally construct an expansion, containing as many arbitrary functions as the general solution, of the solution to the gravitational equations around the quasi-isotropic solution. However, the analysis carried out in §4 shows that signals can propagate with a velocity higher than the velocity of light in a universe described by this solution. This contradicts the casuality principle, and therefore there does not exist for viscoelastic matter a general solution that is close to the quasi-isotropic solution. Nevertheless, the indicated formal expansion is of definite methodological interest, and we shall give without derivation its first order. For simplicity, let us set $\sigma=0$, which does not alter the qualitative character of the results. Let us assume the equation of state in the form $p=\varepsilon/3$, since ε is high. The expansion pertains to the case when the stress tensor satisfies Eqs. (1.3) and (1.4), but we can in the case of matter in the form of a hydrodynamic viscous fluid construct a similar expansion around the quasi-isotropic singularity at the final stage of the evolution.

The reference system is synchronous, and the metric has the form

$$-ds^2=-dt^2+\gamma_{\alpha\beta}dx^\alpha dx^\beta.$$

The expansion of the space metric tensor has the form

$$\gamma_{\alpha\beta}=ta_{\alpha\beta}+t^{\nu_1+1}b^{(1)}_{\alpha\beta}+t^{\nu_2+1}b^{(2)}_{\alpha\beta}+t^{\nu_3+1}b^{(3)}_{\alpha\beta}, \tag{3.1}$$

where the $a_{\alpha\beta}$ and $b^{(i)}_{\alpha\beta}$ are functions of the three-dimensional coordinates, $\nu_1=1$, while ν_2 and ν_3 satisfy the equation

$$(\nu+{}^1/_2)(\nu-2)+{}^3/_2\delta=0, \tag{3.2}$$

δ being the same constant that appears in the relation (2.14). If the expansion (3.1) is constructed around a future singularity (corresponding to the moment of time $t=0$), then we should replace t by $|t|$ in (3.1) and the subsequent formulas, since in this case $t<0$. Imposed on the $b^{(i)\alpha}_\beta$ are the conditions

$$b_\alpha^{(2)\alpha}=b_\alpha^{(3)\alpha}=0,\quad b_\beta^{(1)\alpha}=\frac{P_\beta{}^\alpha-({}^5/_{24}+\delta/8)P\delta_\beta{}^\alpha}{{}^3/_4(\delta-1)}$$

where p^α_β is the curvature tensor, which is constructed from the metric $a_{\alpha\beta}$, just as R^i_k is constructed from g_{ik},

$$P=P_\alpha{}^\alpha,$$

$$u_\alpha=-\sum_i\frac{\nu_i}{2}(b^{(i)\beta}_{\alpha;\beta}-b^{(i)\gamma}_{\gamma;\alpha})t^{\nu_i+1},$$

$$\varepsilon=3/4t^2+P/4t,$$

$$\sigma_\beta{}^\alpha=f_\beta{}^{(1)\alpha}t^{-1}+f_\beta{}^{(2)\alpha}t^{\nu_2-2}+f_\beta{}^{(3)\alpha}t^{\nu_3-1},$$

$$f_\beta^{(1)\alpha}=P_\beta{}^\alpha+{}^3/_4b_\beta^{(1)\alpha}-{}^5/_{24}\delta_\beta{}^\alpha P,$$

$$f_\beta^{(2)\alpha}={}^1/_2\nu_2(\nu_2+{}^1/_2)b_\beta^{(2)\alpha},\quad f_\beta^{(3)\alpha}={}^1/_2\nu_3(\nu_3+{}^1/_2)b_\beta^{(3)\alpha}.$$

The subsequent terms of the expansion of the metric are proportional to some power of t, with coefficients that are time independent, but dependent on the three-dimensional coordinates. Figuring in the exponents are all possible sums of the form $n_1\nu_1+n_2\nu_2+n_3\nu_3+n_4(2b_1-1)+1$, where n_1, n_2, n_3, and n_4 are whole numbers,

$$n_{1,2,3}\geqslant0,\ n_4\geqslant1,\ n_1+n_2+n_3\geqslant1.$$

It follows from the structure of the exact equations that all the terms of the subsequent orders will be expressed in terms of the terms of the preceding orders and, in the final analysis, in terms of the first-order terms.

In order for the constructed expansion to converge and be a solution of the Einstein equations, the condition

$$\nu_2>0,\ \nu_3>0$$

should be fulfilled. As shown in §4, this condition cannot be fulfilled, and the constructed expansion is inadmissible.

The instability of the general solution in the vicinity of the quasi-isotropic singularity implies that the singularity at the final stage of the evolution also has an anisotropic, oscillatory character.

§4. INVESTIGATION OF THE BEHAVIOR OF WEAK PERTURBATIONS

Let us investigate the evolution of weak perturbations of the Friedmann metric. As before, ε is large, and the equation of state is $p=1/3\varepsilon$. The corresponding case of zero viscosity is investigated in Ref. 8. Dilatational viscosity does not introduce any qualitative changes in the vicinity of the isotropic singularity, and therefore let us, as before, set $\sigma=0$. The investigation is simplest in the case when the fundamental metric has a flat three-dimensional space. Moreover, such an investigation is quite sufficient for the determination of the damping law for the perturbations. Indeed, there are among the arbitrary weak perturbations of the flat model those that transform the model into one of constant negative or positive curvature. On account of the linearity of the equations for weak perturbations, they will attenuate in the thus "perturbed" background in the same way as in the flat background. Let us choose the fundamental metric in the form

$$-ds^2=-dt^2+R^2(t)(dx_1{}^2+dx_2{}^2+dx_3{}^2),$$

$$g_{ik}^{(0)}=\mathrm{diag}(-1,R^2,R^2,R^2),\quad \gamma^{(0)}_{\alpha\beta}=R^2\delta_{\alpha\beta}.$$

Let us label the quantities pertaining to the unperturbed solution by the symbol (0). We can assume without any loss of generality that the perturbations do not perturb the synchronism of the reference system. The unperturbed reference system is a comoving frame:

$$u^{i(0)}=(1,0,0,0),\quad u^i=u^{i(0)}+\delta u^i,$$

$$\delta u^\alpha=u^\alpha,\quad u^iu_i=-1,\ \delta(u_iu^i)=2u^{\alpha(0)}\delta u_\alpha$$

$$-2u^{0(0)}\delta u_0=0,\quad \delta u_0=0,$$

$$\delta g_{ik}=h_{ik},\ h_{00}=h_{0\alpha}=0,$$

$$\delta\gamma_{\alpha\beta}=h_{\alpha\beta},\ \gamma^{\alpha\beta}=\delta^{\alpha\beta}/R^2-h^{\alpha\beta},\ h^{\alpha\beta}=h_{\alpha\beta}/R^4.$$

The unperturbed metric is isotropic; therefore,

$$\sigma_k{}^{i(0)}=0,\quad \delta\sigma_k{}^i=\sigma_k{}^i.$$

Here $\sigma^\alpha_0=-\sigma^\alpha_\beta u^\beta/u_0$ and $\sigma^0_0=\sigma^\alpha_0u_\alpha/u_0$ are quantities of higher order in smallness, and we can neglect them. All the operations with the three-dimensional tensors are performed with the aid of the metric $\gamma^{(0)}_{\alpha\beta}$.

The subsequent computations are completely similar to those in Ref. 8. We compute the linearized Einstein equations

$$\delta R_k{}^i=\delta T_k{}^i-{}^1/_2\delta_k{}^i\delta T \tag{4.1}$$

in first order in h^α_β. The equations for σ^α_β have the form

$$\sigma_\beta{}^\alpha+\tau_1\dot\sigma_\beta{}^\alpha=-\eta(\varkappa_\beta{}^\alpha-{}^1/_3\delta_\beta{}^\alpha\varkappa_\gamma{}^\gamma+u^\alpha{}_{;\beta}+u_{\beta;}{}^\alpha-{}^2/_3\delta_\beta{}^\alpha u_\gamma{}^\gamma),\quad \varkappa_{\alpha\beta}=\dot\gamma_{\alpha\beta},\quad \varkappa_\beta{}^\alpha=2\delta_\beta{}^\alpha H+h_\beta{}^\alpha,\quad H=\dot R/R. \tag{4.2}$$

Below we follow the procedure developed by E. Lifshitz.[8] Let us expand the perturbations in terms of $Q=e^{i\mathbf{n}\cdot\mathbf{r}}$, where n is a normal Cartesian vector. The symmetric tensor $h_{\alpha\beta}$ can be expanded into three types of plane waves. To the compression and rarefaction waves correspond the tensors

$$Q_\beta{}^\alpha={}^1/_3\delta_\beta{}^\alpha Q,\ P_\beta{}^\alpha=({}^1/_3\delta_\beta{}^\alpha-n^\alpha n_\beta/n^2)Q.$$

To the transverse waves—the transverse vibrations of the matter—corresponds the tensor

$$S_\beta{}^\alpha=(n_\beta s^\alpha+n^\alpha s_\beta)Q,\ s^\alpha n_\alpha=0.$$

To the gravitational waves corresponds the tensor

$$G_\beta{}^\alpha=g_\beta{}^\alpha Q,\ g_\alpha{}^\alpha=0,\ g_\beta{}^\alpha n^\beta=0.$$

The sought expansion can be written in the form

$$h_\beta^\alpha=\lambda P_\beta^\alpha+\mu Q_\beta^\alpha+\zeta S_\beta^\alpha+\nu G_\beta^\alpha. \tag{4.3}$$

The traceless tensor σ_β^α can be written in the form of the following expansion:

$$\sigma_\beta^\alpha=AP_\beta^\alpha+BS_\beta^\alpha+CG_\beta^\alpha. \tag{4.4}$$

Substituting (4.3) into (4.1), we obtain, assuming that $\varepsilon^{(0)}=3/4t^2$:

$$u_\alpha={}^1/_2 it^2({}^2/_3(\dot\lambda+\dot\mu)n_\alpha-n^2s_\alpha\dot\zeta)Q. \tag{4.5}$$

Substituting (4.3), (4.4), and (4.5) into (4.1), we obtain

$$A+\tau_1\dot A=-\eta\left(\frac{2n^2t^2}{3R^2}(\dot\lambda+\dot\mu)+\dot\lambda\right), \tag{4.6}$$

$$B+\tau_1\dot B=-\eta\left(\frac{n^2t^2}{2R^2}+1\right)\dot\zeta, \tag{4.7}$$

$$C+\tau_1\dot C=-\eta\dot\nu. \tag{4.8}$$

The equations for the coefficients of the expansion (4.3) have the form (in place of H we have substituted $1/2t$)

$$\ddot\mu+\frac{2\dot\mu}{t}+(\lambda+\mu)\frac{2n^2}{3R^2}=0, \tag{4.9}$$

$$\ddot\lambda+\frac{3}{2}\frac{\dot\lambda}{2t}-\frac{n^2}{3R^2}(\lambda+\mu)=2A, \tag{4.10}$$

$$\ddot\zeta+\frac{3}{2t}\dot\zeta=2B, \tag{4.11}$$

$$\ddot\nu+\frac{3}{2t}\dot\nu+\frac{n^2}{R^2}\nu=2C. \tag{4.12}$$

The energy density changes only in the waves of the first type:

$$\delta\varepsilon=\left[\frac{\dot\mu}{2t}+\frac{n^2}{3R^2}(\lambda+\mu)\right]Q. \tag{4.13}$$

The fictitious metric perturbations resulting from a change of reference system are, naturally, the same as those found in Ref. 8:

$$\lambda=C_1+C_2n^2\int\frac{dt}{R^2},\quad \mu=-C_1-C_2n^2\int\frac{dt}{R^2}+3HC_2,$$
$$\zeta=C_3,\quad \nu=0,\quad A=B=C=0,$$
$$\int\frac{dt}{R^2}=\frac{1}{R_0^2}\ln t,\quad R_0^2=\text{const.}$$

The obtained equations (4.6)–(4.12) split into groups of independent equations, which groups can be investigated separatly. Let us investigate the behavior of λ, μ, and A. Let us expand the solution to Eqs. (4.6), (4.9), and (4.10) up to the orders that decrease as $t\to 0$.

The first orders of the expansion are simply a fictitious solution for λ and μ that contains two arbitrary constants C_1 and C_2. Assuming t to be small ($n^2t\ll 1$), we obtain the following orders:

$$A=-\frac{3\delta}{4}\left(C_3\frac{\nu_2t^{\nu_2-2}}{\nu_2-2}+C_4\frac{\nu_3t^{\nu_3-2}}{\nu_3-2}\right), \tag{4.14}$$
$$\lambda=C_3t^{\nu_2}+C_4t^{\nu_3},\quad \mu=C_5;$$

where ν_2 and ν_3 are found from (3.2). If ν_2, $\nu_3>0$, then (4.14) gives the first orders that tend to zero. The complete expansion thus contains five constants, and the obtained solution is the general solution. The first orders can be disposed of through a change of reference system. The appearance of the constant amplitude $\mu=C_5$ is connected with the fact that the perturbation μQ_β^α is the same perturbation that alters the curvature of the space. More precisely, the difference between the metric tensors of a space of constant nonzero, and a space of zero, curvature can be expanded precisely in powers of Q_β^α. The solutions of the remaining equations have exactly the same form:

$$\zeta=C_6t^{\nu_2}+C_7t^{\nu_3},\quad B=-\frac{3\delta}{4}\left(C_6\frac{\nu_2t^{\nu_2-2}}{\nu_2-2}+C_7\frac{\nu_3t^{\nu_3-2}}{\nu_3-2}\right),$$

$\zeta=C_8$ is a fictitious solutions,

$$\begin{aligned}\nu&=C_9t^{\nu_2}+C_{10}t^{\nu_3}+C_{11},\\ C&=-\frac{3\delta}{4}\left(C_9\frac{\nu_2t^{\nu_2-2}}{\nu_2-2}+C_{10}\frac{\nu_3t^{\nu_3-2}}{\nu_3-2}\right).\end{aligned} \tag{4.15}$$

The constant amplitude C_{11} in (4.15) is connected with the fact that in the general case the quasi-isotropic metrics should be related to the fundamental metrics, as was done in §3. The fundamental metric (i.e., the first order of the expansion of the metric tensor) in (3.1) has the form

$$\gamma_{\alpha\beta}=ta_{\alpha\beta}(x_1,x_2,x_3).$$

The nonisotropic part of $a_{\alpha\beta}$ can be expanded in powers of G_β^α.

The perturbations will attenuate if ν_2, $\nu_3>0$. In the opposite case the isotropic solution is unstable against contraction (for large ε and small t). The found laws of evolution of perturbations are valid for both contraction and expansion.

With the aid of Eqs. (4.6)–(4.12), we can find the velocities of propagation of all the types of waves of low amplitude and short wavelength. In the nonrelativistic case the velocity of propagation of the waves increases with frequency,[5] and we can expect this to happen in the relativistic case also. Therefore, we shall retain in the equations only the terms of highest order in n. Such a method yields only the phase of the wave, but this is sufficient for the determination of the wave velocity. Let us go over to the new variable θ:

$$Rd\theta=dt.$$

The solutions of the equations should have the form $A(\theta)e^{ivn\theta}$, where v is the phase velocity. Discarding the terms of lowest order in n ($v\sim 1$), we obtain the solutions

$$\lambda\sim\mu\sim\exp[i(\delta+{}^1/_3)^{1/2}n\theta],\quad v_1=(\delta+{}^1/_3)^{1/2},$$
$$\zeta\sim\exp[i({}^3/_4\delta)^{1/2}n\theta],\quad v_2=({}^3/_4\delta)^{1/2},$$
$$\nu\sim e^{in\theta},\quad v_3=1.$$

The last answer is obvious: the gravitational waves propagate with the velocity of light. The obtained formulas give the velocities of propagation of the waves in a flat isotropic space when the equation of state is $p=\varepsilon/3$. When the equation of state is $p=(\gamma-1)\varepsilon$,

$$v_1=(\gamma-1+{}^4/_3\delta/\gamma)^{1/2},\quad v_2=(\delta/\gamma)^{1/2},\quad v_3=1.$$

For large n the waves are elastic waves, and their velocity is determined by the two parameters:

$$E/\varepsilon=\delta,\quad K/\varepsilon=\gamma-1,$$

where E is the shear modulus and K is the bulk modulus. When $\gamma-1\ll 1$, $\delta\ll 1$, we obtain the standard nonrelativistic formulas.[4,5]

Relativistic causality requires the fulfillment of the conditions

$$v_1<1,\quad v_2<1,\quad \delta<{}^3/_4\gamma(2-\gamma).$$

It follows from (3.2) and (A.8) that

$$v_2v_3<0,\quad \lambda_3\lambda_4<0.$$

This means that relativistic causality precludes the stability of isotropic collapse. The isotropic singularity cannot be the typical initial or final state.

The authors express their gratitude to Professor W. Israel, who drew their attention to this problem.

APPENDIX

Here we outline a qualitative investigation of the system of equations (2.7)–(2.11) in variables in which the system does not have in the region delimited by the conditions (1.6) and (2.12) singular points at infinity, with the exception of the singular point corresponding to infinite rarefaction. The investigation of the singular points is significantly facilitated in such variables.

We consider the case of a linear dependence of the entropy density, $w=\varepsilon+p$, on ε. In place of H, ε, σ, X, and Y, we introduce new variables according to the relations:

$$\rho=\frac{\varepsilon}{(3H)^2},\quad h=\frac{1}{3H},\quad x=\frac{X}{(3H)^3},$$
$$y=\frac{Y}{(3H)^4},\quad \chi=\frac{\sigma}{(3H)^2},$$
$$0\leqslant\rho\leqslant{}^1/_3,\quad y\geqslant 0,$$

and in place of t we introduce a new time variable, τ, according to the law:

$$d\tau/dt=3H.$$

The system of equations (2.7)–(2.11) now assumes the form

$$\dot h={}^3/_2\gamma\rho h+{}^3/_2\chi h-3(\rho-{}^1/_3)h, \qquad (A.1)$$

$$\dot\rho=3\gamma\rho^2+3\rho\chi-6\rho(\rho-{}^1/_3)+x-\gamma\rho-\chi, \qquad (A.2)$$

$$\dot\chi=3\gamma\rho\chi+3\chi^2-6\chi\left(\rho-\frac{1}{3}\right)-\frac{1}{\tau_0}\chi h-\frac{\xi h^2}{\tau_0}, \qquad (A.3)$$

$$\dot x=\frac{9}{2}\gamma x\rho+\frac{9}{2}x\chi-9\left(\rho-\frac{1}{3}\right)x-x-x\frac{h}{\tau_1}-4\eta\frac{h^2}{\tau_1}\left(\rho-\frac{1}{3}\right)-y, \qquad (A.4)$$

$$\dot y=6\gamma y\rho+6y\chi-12y\left(\rho-\frac{1}{3}\right)-\frac{2}{\tau_1}hy+4\frac{h^2}{\tau_1}\eta x. \qquad (A.5)$$

In order to discard the solutions that clearly do not satisfy the criteria (1.6) and (2.12), let us require that $\varkappa\lesssim 1$. Since

$$\chi=\sigma/9H^2\leqslant\sigma/3\varepsilon,$$

we have, for $\varkappa\gg 1$, $\sigma\gg\varepsilon$ and $\tau_0\sigma^2/\xi\gg\varepsilon$, which contradicts the criterion (1.6).

In the physical region, the system (A.1)–(A.5) has, besides the singular points corresponding to the points N_1 and N_2 of the system (2.7)–(2.11), the singular points:

$$h=0,\quad \rho={}^1/_3,\quad \chi=x=y=0, \qquad (A.6)$$

$$h=0,\quad \rho=0,\quad \chi=x=y=0. \qquad (A.7)$$

The singular point (A.7) corresponds to the Kasner solution, since $\varepsilon\ll H^2$ near this point, and it follows from (2.8) that

$$\dot H=-3H^2,\quad H=\frac{1}{3t},\quad \sum_\alpha H_\alpha{}^2\approx 9H^2,$$
$$R_\alpha=\exp\left(\int H_\alpha\,dt\right)=t^{\pi_\alpha},\quad \sum\pi_\alpha=\sum\pi_\alpha{}^2=1.$$

Near a singular point we can discard the terms of higher order in smallness in the system (A.1)–(A.5). We find from (A.1) that in the neighborhood of the singular point (A.7)

$$h=Ce^\tau,\quad t=Ce^\tau.$$

We assume

$$\frac{\xi}{\tau_0}=\delta_V\varepsilon=\delta_V\frac{\rho}{h^2},\quad \frac{\eta}{\tau_1}=\delta\frac{\rho}{h^2}.$$

It is impossible to linearize the system (A.1)–(A.5) in the vicinity of the singular point (A.7), since the condition (2.12) requires that either $h/\tau_0\to\infty$, or $h/\tau_1\to\infty$, depending on which addend in the second-order terms in the expression for $S^i_{;i}$ predominates. If at $t=0$ the quantity $\varepsilon=\infty$, then

$$\tau_1\sim\eta/\varepsilon\gg 1/\sqrt{\varepsilon}\gg t,\quad \tau_1\dot\sigma^\alpha\gg\sigma^\alpha,$$
$$\sigma^\alpha\sim\frac{\eta}{\tau_1}tH,\quad Y\sim\varepsilon^2,\quad \frac{\tau_1 Y}{\eta}\sim\varepsilon.$$

Thus, the criterion (1.6) is violated.

The Friedmann singular point (A.6) is different, in that the system can be linearized in its vicinity if we are not interested in the dependence $\varkappa(\tau)$. We find at once that

$$\dot h=\frac{\gamma}{2}h,\quad h=Ce^{\gamma\tau/2},\quad t=\frac{2}{\gamma}Ce^{\gamma\tau/2},\quad H=\frac{2}{3\gamma t}.$$

The linearized system has the following eigenvalues:

$$1)\ \lambda_1={}^1/_2\gamma,\quad h=Ce^{\gamma\tau/2},\quad \rho-{}^1/_3=\chi=x=y=0;$$

$$2)\ \lambda_2=\frac{3}{2}\gamma-1,\quad h=0,\quad \begin{pmatrix}\rho-{}^1/_3\\ x\\ y\end{pmatrix}=C_2\begin{pmatrix}1\\ -(\gamma/2+1)\\ {}^4/_3\delta\end{pmatrix}\exp(\lambda_2\tau);$$

$$3)\ (\gamma-2-\lambda)(2\gamma-\lambda)+{}^8/_3\delta=0,\quad \lambda_{3,4}={}^3/_2\gamma-1\pm(1-{}^8/_3\delta+{}^1/_4\gamma^2-\gamma)^{1/2},$$

$$h=0,\quad \begin{pmatrix}\rho-{}^1/_3\\ x\\ y\end{pmatrix}=C_{3,4}\begin{pmatrix}-1\\ \gamma-2-\lambda_{3,4}\\ \dfrac{\lambda_{3,4}+2-\gamma}{2\gamma-\lambda_{3,4}}\end{pmatrix}\exp(\lambda_{3,4}\tau). \qquad (A.8)$$

In this case the solution for λ_2 is actually the cross term of the solutions for λ_3 and λ_4. The reason for this is that the original variables H_α and σ^α are proportional to $\varepsilon^{\lambda\tau/2}$. Since the variables entering into the systems (2.7)–(2.11) and (A.1)–(A.5) are quadratic in σ^α and H_α, the squaring of them gives rise to terms of the form $\exp[(\lambda_3+\lambda_4)\tau/2]=\exp(\lambda_2\tau)$.

After ascertaining the behavior of the solutions near the isotropic singularity and making a change of variables to the variables H, ε, σ, H_α, and σ^α, we obtain the solution (2.15). It is shown at the same time that no other solutions exist near the isotropic singularity.

[1)]We use a system of units in which the velocity of light and the gravitational constant are each equal to unity. The metric is written in the form $-ds^2 = g_{ik}dx^i dx^k$, where g_{ik} has the signature $(-+++)$. The Latin indices run from 0 to 3; the Greek indices, from 1 to 3.

[2)]In the present paper we neglect the effect of the thermal fluxes. Such fluxes do not, in fact, arise in the homogeneous models. In the more general cases it must be assumed that we are considering matter with a sufficiently small coefficient of thermal conductivity. Equations that also take account of the effects of thermal conduction can be found in Ref. 3.

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Influence of collisionless particles on the growth of gravitational perturbations in an isotropic universe

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It is shown that the kinetics of the interaction of gravitational perturbations with collisionless particles (neutrinos) in the ultrarelativistic stage of expansion of the universe leads to a behavior of long-wavelength gravitational perturbations which is qualitatively different from that obtained by Lifshitz in 1946 if the energy density of the collisionless particles is more than 5/32 of the total energy density. An important feature of the new long-wavelength asymptotic behaviors is their oscillatory nature. The asymptotic behavior is also found of high-frequency perturbations in an isotropic universe when allowance is made for the influence on the perturbations of the gas of collisionless particles.

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INTRODUCTION

In 1946, Lifshitz[1] solved the problem of the gravitational stability of the isotropic relativistic cosmological model of the universe. He assumed that the matter of a hydrodynamic model with isotropic energy–momentum tensor. The results obtained in Ref. 1 concerning the rate of growth of perturbations were subsequently widely used in studies into the theory of the formation of the large-scale structure of the universe.

In Refs. 2–5 the analogous problem was solved under the assumption that the matter of the universe can be treated in the framework of the model of a collisionless gas, i.e., a model described by a collisionless kinetic equation. This model of the matter is valid in the cases when the characteristic frequency ω of the investigated processes is much higher than the collision frequency ν of the particles of the matter $(\omega \gg \nu)$. The hydrodynamic model of matter used in Ref. 1 is valid if $\omega \ll \nu$.

As is shown in Refs. 3–5, the model of a collisionless gas and the hydrodynamic model of matter lead to very different asymptotic behaviors of perturbations in an isotropic universe. This example suggests that if the universe contains not only matter described by the hydrodynamic model but also a gas of collisionless particles, then this gas could have a significant influence on the rate of growth (or damping) of perturbations.

In a hot universe, a gas of muonic and electronic neutrinos is collisionless.[6] The muonic neutrinos become collisionless $\tau = 0.01$ sec after the start of expansion of the universe, while the electronic neutrinos become collisionless at $\tau = 0.2$ sec (see Ref. 6).

Zel'dovich and Novikov[6] also give relations for the equilibrium energy density of different particles in the universe at a time close to the time of "switching off" of the muonic neutrinos:

$$\varepsilon_\gamma : \varepsilon_{e^\pm} : \varepsilon_{\nu_e,\bar{\nu}_e} : \varepsilon_{\nu_\mu,\bar{\nu}_\mu} : \varepsilon_{\mu^\pm} = 1 : {}^7/_4 : {}^7/_8 : {}^7/_8 : 10^{-4}. \quad (1)$$

These ratios remain valid until the electron–positron pairs are annihilated. It follows from the ratios (1) that the ratio α of the energy density of the collisionless particles to the energy density of the collisional particles at the times $0.01 < \tau < 0.2$ sec is

