Magnetic relaxation in the hydrodynamic region in the superfluid phases of He^3 located in a strong magnetic field

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The magnetic-dynamics equations for superfluid He^3 that describe the relaxation of the magnetization in a strong magnetic field in the hydrodynamic approximation are analyzed. It is shown that the temporal behavior of the relaxation depends on the initial conditions. The laws of variation in time of the longitudinal component of the magnetization are found for two limiting cases. The nonmonotonic relaxation that has been observed in experiment is explained. The possibility of the existence of auto-oscillation regimes in the *B* phase and their effect on the magnetic-relaxation process are discussed.

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1. INTRODUCTION

The process whereby the magnetization in superfluid He^3 relaxes to the equilibrium value is at present not an adequately studied process. The sharp relaxation-time decrease that has been experimentally observed[1,2] to occur in He^3 when it goes over into the superfluid state indicates that a new relaxation mechanism comes into play when this transition occurs. In the one-dimensional case such a mechanism can be the "intrinsic" mechanism proposed by Leggett and Takagi,[3] i.e., relaxation on account of the establishment of spin equilibrium between the Cooper-pair condensate and the Fermi excitations. The theory based on the "intrinsic" mechanism[3] predicts different time dependences for the longitudinal component of the magnetization (i.e., the component parallel to the external magnetic field $\mathbf{H}_0$) for two cases. In the first case the magnetization is parallel to $\mathbf{H}_0$, but its magnitude is not equal to the equilibrium value, and the re-establishment of the magnetiude of the magnetization should occur in time according to a square-root law, which has been observed by Sager, Kleinberg, Warkentin, and Wheatley.[2] In the second case the magnetization has the equilibrium magnitude and relaxes with respect to direction, the relaxation being such that the time dependence of the longitudinal component of the magnetization in the *A* phase should be linear. Such a dependence has been observed by Corruccini and Osheroff,[1] but the observed dependence of the characteristic relaxation time on temperature and the magnetic-field strength does not coincide with the theoretically predicted dependence. The suggestion has been made[1,4] that the indicated discrepancy can be explained by the transport of the magnetization by the superfluid spin currents. This mechanism has, however, not yet been sufficiently studied for us to be sure of the correctness of this explanation. The variation of the temporal relaxation law with the external field and temperature and nonmonotonic relaxation have also been observed.[2] Because the theory has not been adequately developed, these observations cannot, in general, be compared with it.

In view of the foregoing, it is useful to investigate as fully as possible phenomenological equations that do not depend on the details of the relaxation mechanism. The choice of a particular mechanism will then affect only the magnitude and dependence on the external conditions of the phenomenological parameter that enters into the equations. By that means we are able to separate the problem of the temporal behavior of the relaxation from the problem of the elucidation of the specific relaxation mechanism. The equations that we need can be written in the hydrodynamic region, where the frequencies characterizing the motion of the magnetization are low compared to the reciprocal of the time required for the establishment of equilibrium for the given value of magnetization.

In the present paper we shall investigate the phenomenological equations describing magnetization relaxation in the spatially homogeneous case for strong magnetic fields. Those fields are considered to be strong for which the spin-orbit interaction energy is small compared to the Zeeman energy. The simplification arising therewith allows us to investigate the equations thoroughly enough for a comparison with experiment to be possible and, for the important cases, to find the time dependence of the longitudinal component of the magnetization. The dependences thus found are compared with the experimental data of Sager *et al.*,[2] since the experiments in which these data were obtained were performed in that region of temperatures, and magnetic fields, where the conditions for the applicability of the approach used in the present paper are fulfilled.

2. THE EQUATIONS OF MOTION

The equations of motion in the absence of dissipation are obtained with the aid of the Hamiltonian[5]

$$\mathscr{H}=\mathbf{M}^2/2\chi-\mathbf{MH}+U(\mathbf{d}), \tag{1}$$

where $\mathbf{M}$ is the magnetic moment of the amount of helium under consideration, $\mathbf{H}_0$ is the external magnetic field, U is the spin-orbit interaction energy, $\mathbf{d}$ is the vector order parameter in triplet pairing. It is convenient to go over immediately to the dimensionless variables

$$\hat{\mathbf{H}}=\mathbf{H}_0/H_0, \quad \mathbf{S}=-\mathbf{M}/\chi H_0$$

and measure the time in units of the Larmor periods, i.e., make the change of variable $t\to\omega_L t$, where ω_L is

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the Larmor frequency for the field $\mathbf{H}_0$. In terms of these variables, and under the condition that the z axis is antiparallel to $\mathbf{H}_0$, we have

$$\mathscr{H}={}^1/_2\mathbf{S}^2-S_z+U(\mathbf{d}). \tag{1a}$$

To allow for the dissipation in the hydrodynamic approximation, we shall need, besides the Hamiltonian written down above, the dissipation function f. It is determined by the rate of change of $\mathbf{S}$ in the coordinate system rotating with the Larmor frequency. The purely Larmor precession of $\mathbf{S}$ does not lead to the destruction of the equilibrium magnetization distribution between the condensate and the excitations, and, thus, should not, according to Leggett and Takagi's analysis,[3] lead to energy dissipation. We obtain then from symmetry arguments

$$f=\frac{\varkappa}{2}\left(\frac{d\mathbf{S}}{dt}-[\hat{\mathbf{H}}\times\mathbf{S}]\right)^2, \tag{2}$$

where $\varkappa$ is the phenomenological parameter. The expression that has been written down for f coincides, as it should, with the expression obtained for the energy dissipation in the hydrodynamic limit by Leggett and Takagi,[3] who also determine $\varkappa$ in their theory. We, however, decided not to use here the ready expression, but to present the above-stated arguments in order to show that the form of the dissipation function (2) can be obtained under quite general assumptions. For the same reason, we shall not use the explicit expression for $\varkappa$.

The assumption that the magnetic field is strong allows us to use the approach developed earlier for the investigation of the magnetic-dynamics equations for the superfluid phases of He^3 in the absence of dissipation.[6,7] This approach consists in the use of the smallness of the parameter $\Omega^2=(\omega_\parallel/\omega_L)^2$, where $\omega_\parallel$ is the frequency of the longitudinal oscillations. The principal simplification arises as a result of the preservation of the orientation of $\mathbf{S}$ relative to the system of the $\mathbf{d}$ vectors. As a result, one degree of freedom in the system under consideration turns out to be "frozen," and the motion of this system is described by two, and not three, pairs of conjugate variables. As the independent variables, let us choose $S=S_\zeta$, where S_ζ is the component of $\mathbf{S}$ along the direction of the ζ axis of the moving coordinate system connected with the d-vector system. At the initial moment of time, the ζ axis coincides with the z axis of the fixed coordinate system. On account of the above-indicated preservation of the orientation of $\mathbf{S}$ in the moving coordinate system, S coincides in magnitude with $|\mathbf{S}|$. However, these quantities behave differently under time reversal, which is important in the application of the Onsager principle. The variable conjugate to S is $\Phi=\alpha+\gamma$; α, β, and γ are the Euler angles characterizing the rotation of the system of $\mathbf{d}$ vectors from their initial position. The second pair of variables are $P=S_z-S$ and α. In terms of the variables S, Φ, P, and α, the Hamiltonian and the dissipation function have the form

$$\mathscr{H}={}^1/_2(S-1)^2-P+U(P/S,\Phi,\alpha), \tag{3}$$

$$f=\frac{\varkappa}{2}\left\{\frac{2S}{2S+P}\left(\dot{S}^2+\dot{S}\dot{P}-\frac{S}{P}\dot{P}^2\right)-P(2S+P)(\dot{\alpha}+1)^2\right\}. \tag{4}$$

Using the standard procedure,[8] we obtain from these expressions the equations of motion:

$$\frac{dS}{dt}=-\frac{\partial U}{\partial\Phi}, \tag{5}$$

$$\frac{d\Phi}{dt}=S-1+\frac{\partial U}{\partial S}-\frac{\varkappa S}{2S+P}\left(2\frac{\partial U}{\partial\Phi}+\frac{\partial U}{\partial\alpha}\right), \tag{6}$$

$$\frac{dP}{dt}=-\frac{\partial U}{\partial\alpha}-\varkappa P(2S+P)\left(\frac{\partial U}{\partial P}\right)^2, \tag{7}$$

$$\frac{d\alpha}{dt}=-1+\frac{\partial U}{\partial P}+\frac{\varkappa S}{2S+P}\left(\frac{S}{P}\frac{\partial U}{\partial\alpha}-\frac{\partial U}{\partial\Phi}\right). \tag{8}$$

The right-hand sides of these equations contain terms of different orders of magnitude, namely, ~ 1, Ω^2, and Ω^4. To them correspond motions occurring with different time scales. The analysis of the Eqs. (5)–(8) will consist in the separation of the motions occurring with essentially different frequencies. The normal procedure for such a separation was worked out by Bogolyubov and Krylov (see, for example, Ref. 9). It consists in finding the change of variables

$$S=\bar{S}+\varepsilon q_1+\varepsilon^2 q_2+\ldots, \tag{9}$$

$$P=\bar{P}+\varepsilon u_1+\varepsilon^2 u_2+\ldots, \tag{10}$$

$$\Phi=\bar{\Phi}+\varepsilon v_1+\varepsilon^2 v_2+\ldots, \tag{11}$$

$$\alpha=\bar{\alpha}+\varepsilon w_1+\varepsilon^2 w_2+\ldots \tag{12}$$

the coefficients $q_1,q_2,\ldots$, $u_1,u_2,\ldots$, $v_1,v_2,\ldots$, $w_1,w_2,\ldots$ being functions of $\bar{S}$, $\bar{P}$, $\bar{\Phi}$, and $\bar{\alpha}$, that leads to a system of equations for $\bar{S}$, $\bar{P}$, $\bar{\Phi}$, and $\bar{\alpha}$ in the form

$$d\bar{S}/dt=\varepsilon D_1+\varepsilon^2 D_2+\ldots, \tag{13}$$

$$d\bar{P}/dt=\varepsilon A_1+\varepsilon^2 A_2+\ldots, \tag{14}$$

$$d\bar{\Phi}/dt=\bar{S}-1+\varepsilon B_1+\varepsilon^2 B_2+\ldots, \tag{15}$$

$$d\bar{\alpha}/dt=-1+\varepsilon C_1+\varepsilon^2 C_2+\ldots \tag{16}$$

Here ε is a constant of the order of Ω^2 and the functions $D_1,\ldots,A_1,\ldots,B_1,\ldots,C_1,\ldots$ entering into the right-hand sides depend on only the slow variables.

In our case the variables $\bar{S}$ and $\bar{P}$ are the slow variables, since their derivatives are small, $\bar{\alpha}$ is a fast variable ($\bar{\alpha}\sim 1$), and the variable $\bar{\Phi}$ can be either fast or slow, depending on the magnitude of the difference $\bar{S}-1$. We should, accordingly, consider two different cases: 1) $(\bar{S}-1)\sim 1$, the so-called nonresonance case, in which $\bar{\Phi}$ is a fast variable and 2) $(\bar{S}-1)\ll 1$, the "principal resonance" approximation; $\bar{\Phi}$ is a slow variable. The resonance $\bar{S}=1$ is called the principal resonance because there arise in the investigation of the second and subsequnt approximations in Ω^2 still other so-called multiple resonances. In the vicinity of these resonances $\bar{\Phi}$ is a fast variable, but the periods of the $\bar{\Phi}$ and $\bar{\alpha}$ variation are commensurable, and from $\bar{\Phi}$ and $\bar{\alpha}$ we can construct a slow variable. Therefore, the condition $(\bar{S}-1)\sim 1$ is not sufficient for a resonance to be absent, and $\bar{S}$ is required to lie far from all the resonance values that occur in the approximation in question. This will be discussed in greater detail in the investigation of the second approximation in Ω^2.

Below, as an application, we shall consider the A and B phases of He^3 in the open geometry for the case in which the system is in equilibrium at the initial moment of time, i.e., in the A phase the vector $\mathbf{d}$ is perpendicular to $\mathbf{H}_0$, while in the B phase the antisotropy axis for the order parameter—the vector $\mathbf{n}$—is parallel to $\mathbf{H}_0$ (the Leggett configuration). For these cases the potential U has respectively the following form:

$$U_A=-\frac{\Omega_A^2}{8}\left[\left(\frac{P}{S}+2\right)\cos\Phi+\frac{P}{S}\cos(2\alpha-\Phi)\right]^2, \tag{17}$$

$$U_B=\frac{2\Omega_B^2}{15}\left[\frac{1}{2}+\frac{P}{S}+\left(2+\frac{P}{S}\right)\cos\Phi\right]^2. \tag{18}$$

For applications it is useful to represent these potentials in the form of an expansion in double Fourier series:

$$U=\varepsilon\sum_{k,l}h_{kl}(P/S)\,e^{i(k\Phi-l\alpha)}. \tag{19}$$

For the A phase, setting $\varepsilon_A=\Omega_A^2/16$, we obtain the following values for the expansion coefficients:

$$\begin{aligned}
&h_{00}{}^A=-\left[\left(\frac{P}{S}+2\right)^2+\left(\frac{P}{S}\right)^2\right],\\
&h_{20}{}^A=h_{-20}^A=-\frac{1}{2}\left(\frac{P}{S}+2\right)^2,\\
&h_{02}{}^A=h_{0-2}^A=h_{22}{}^A=h_{-2-2}^A=-\frac{P}{S}\left(\frac{P}{S}+2\right),\\
&h_{24}{}^A=h_{-2-4}^A=-\frac{1}{2}\left(\frac{P}{S}\right)^2.
\end{aligned} \tag{20}$$

For the B phase, when $\varepsilon_B=\Omega_B^2/15$, the nonzero coefficients of the expansion have the following values:

$$\begin{aligned}
&h_{00}{}^B=\frac{1}{2}\left(1+2\frac{P}{S}\right)^2+\left(2+\frac{P}{S}\right)^2,\\
&h_{10}{}^B=h_{-10}^B=\left(1+2\frac{P}{S}\right)\left(2+\frac{P}{S}\right),\\
&h_{20}{}^B=h_{-20}^B=\frac{1}{2}\left(2+\frac{P}{S}\right)^2.
\end{aligned} \tag{21}$$

3. SEPARATION OF THE MOTIONS

The system of equations of the first Ω^2-approximation is obtained by averaging the Eqs. (5)–(8) over the first variables (the Van der Pol approximation). In the nonresonance case the equations do not differ from the corresponding equations that neglect dissipation (see Ref. 7, Sec. 5). In the resonance case the equations are as follows:

$$d\bar S/dt=-\partial V/\partial\Phi, \tag{22}$$

$$\frac{d\Phi}{dt}=\bar S-1+\frac{\partial V}{\partial\bar S}-\varkappa\frac{2\bar S}{2\bar S+\bar P}\frac{\partial V}{\partial\Phi}, \tag{23}$$

$$d\bar P/dt=0, \tag{24}$$

$$\frac{d\bar\alpha}{dt}=-1+\frac{\partial V}{\partial\bar P}-\varkappa\frac{\bar S}{2\bar S+\bar P}\frac{\partial V}{\partial\Phi}, \tag{25}$$

where $\bar V(\bar P/\bar S,\bar\Phi)$ is the α-averaged spin-orbit interaction energy (see Ref. 7, Sec. 3). The Eqs. (22) and (23) describe the damping of the oscillations. If the oscillations are weak, then to them correspond the complex frequencies

$$\omega=\omega_\parallel\left(\pm1-\frac{i\varkappa\omega_\parallel}{2+P}\right), \tag{26}$$

where $\omega_\parallel$ is equal to $(\partial^2V/\partial\bar\Phi^2)^{1/2}$ evaluated at the minimum of the potential V as a function of $\bar\Phi$. Thus, the oscillations are attenuated over a period of time $\sim(\varkappa\Omega^2)^{-1}$, after which the system finds itself in the stationary state $\dot{\bar S}=0$, $\dot{\bar\Phi}=0$, and $\dot{\bar P}=0$. As can be seen from Eq. (25), in first order in Ω^2, the dissipation terms have no effect on the precession-frequency shift for the stationary states.

From the equations of the first approximation we can find $\bar S$, $\bar P$, $\bar\Phi$, and $\bar\alpha$. The physical quantities S, P, Φ, and α corresponding to them contain small the corrections $\varepsilon q_1,\ldots,\varepsilon w_1$. There arises in the computation of these corrections an arbitrariness whose elimination requires the imposition of additional conditions on the functions $q_1,\ldots,w_1$, etc. Let us require $\bar S$, $\bar P$, $\bar\Phi$, and $\bar\alpha$ to be equal to the values, averaged over the fast variables, of the corresponding variables in all orders in Ω^2. For the nonresonance case it is convenient to represent the corrections in the form of double Fourier series:

$$q(\bar S,\bar P,\Phi,\bar\alpha)=\sum_{kl}q_{kl}(\bar S,\bar P)\exp\{i(k\Phi-l\bar\alpha)\} \tag{27}$$

etc. The additional condition implies that $q_{00}=u_{00}=v_{00}=w_{00}=0$, and for the remaining components we obtain

$$\begin{aligned}
&q_{kl}=-\frac{kh_{kl}}{k(\bar S-1)+l},\quad u_{kl}=\frac{lh_{kl}}{k(\bar S-1)+l},\\
&v_{kl}=\frac{1}{k(\bar S-1)+l}\left[\frac{\varkappa\bar S}{2\bar S+\bar P}(l-2k)h_{kl}+\frac{ikh_{kl}}{k(\bar S-1)+l}-i\frac{\partial h_{kl}}{\partial\bar S}\right],\\
&w_{kl}=-\frac{1}{k(\bar S-1)+l}\left[\frac{\varkappa\bar S}{2\bar S+\bar P}\left(\frac{\bar S}{\bar P}l+k\right)h_{kl}+i\frac{\partial h_{kl}}{\partial\bar P}\right],
\end{aligned} \tag{28}$$

where the h_{kl} are the coefficients in the expansion (19).

In the resonance case the Fourier series expansion should be performed only in terms of the variable $\bar\alpha$:

$$q(\bar S,\bar P,\Phi,\bar\alpha)=\sum_l q_l(\bar S,\bar P,\Phi)\,e^{-il\bar\alpha}. \tag{29}$$

For $l\neq0$,

$$\begin{aligned}
&q_l=\frac{i}{l}\frac{\partial h_l}{\partial\Phi},\quad u_l=h_l,\\
&v_l=\frac{1}{l}\left[\frac{\varkappa\bar S}{2\bar S+\bar P}\left(2i\frac{\partial h_l}{\partial\Phi}+lh_l\right)-i\frac{\partial h_l}{\partial\bar S}+\frac{1}{l}\frac{\partial h_l}{\partial\Phi}\right],\\
&w_l=-\frac{1}{l}\left[\frac{\varkappa\bar S}{2\bar S+\bar P}\left(\frac{\bar S}{\bar P}lh_l-i\frac{\partial h_l}{\partial\Phi}\right)+i\frac{\partial h_l}{\partial\bar P}\right].
\end{aligned} \tag{30}$$

Here the h_l are the coefficients of the expansion of the potential U in a Fourier series in α:

$$U=\varepsilon\sum_l h_l(P/S,\Phi)\,e^{-il\alpha}. \tag{31}$$

For applications, it is especially useful to know the oscillating corrections in the resonance case. Let us give here the explicit expressions for these corrections in the A phase, assuming that $\bar S$ and $\bar\Phi$ are equal to their steady-state values:

$$\bar S=1+\Omega_A^2(1-x)(3x+1)/8,\qquad \Phi=0.$$

Setting $x=\cos\beta$, we obtain

$$\begin{aligned}
&S=1+{}^1/_{16}\Omega_A^2(1-x)\left[2(3x+1)-2(1+x)\cos2\bar\alpha+{}^1/_2(1-x)\cos4\bar\alpha\right],\\
&P=\bar P+{}^1/_{16}\Omega_A^2(1-x)\left[4(1+x)\cos2\bar\alpha-(1-x)\cos4\bar\alpha\right],\\
&\Phi={}^1/_{16}\Omega_A^2(1-x)\left[(5x+1)\sin2\bar\alpha-{}^5/_8(1-x)\sin4\bar\alpha\right],\\
&\alpha=\bar\alpha+{}^1/_{16}\Omega_A^2\left\{\varkappa\left[2(1+x)\cos2\bar\alpha-\frac{1-x}{2}\cos4\bar\alpha\right]+4x\sin2\bar\alpha\right.\\
&\qquad\left.-\frac{1-x}{2}\sin4\bar\alpha\right\}.
\end{aligned} \tag{32}$$

No oscillating corrections arise in the resonance case in the B phase for the state described by the potential (19). In this case the criterion for the applicability of the hydrodynamic approach is the condition $\omega_{\parallel}\tau \ll 1$, and not $\omega_L \tau \ll 1$, as in the A phase. In order of magnitude, τ is the time between quasiparticle collisions.

In both the resonance and nonresonance cases $\dot{\bar{P}} = 0$ in the first Ω^2-approximation, i. e., the relaxation of the magnetization to the equilibrium value does not occur. To describe the relaxation, we should consider the second Ω^2-approximation. It should, however, be borne in mind that the nondissipative terms in the system (5)–(8) are correct only up to terms of the order of Ω^2, and we do not, generally speaking, have the right to consider the next approximation. In order to get around this difficulty, we shall in what follows be interested in the variation of only those quantities that, in the absence of dissipation, are conserved with the requisite—for us—accuracy on account of the general conservation laws, or on account of physical considerations.

Let us begin the investigation with the nonresonance case. In this case we obtain for $\bar{S}$ and $\bar{P}$ the following closed system of equations

$$\frac{d\bar{S}}{dt} = \varkappa\varepsilon^2 \frac{\bar{S}}{2\bar{S}+\bar{P}} \sum_{kl} h_{kl} h_{-k-l} \frac{k}{k(\bar{S}-1)+l} \left[\frac{\bar{S}}{\bar{P}} l^2 + 2kl - 2k^2 \right], \tag{33}$$

$$\frac{d\bar{P}}{dt} = -\varkappa\varepsilon^2 \frac{\bar{S}}{2\bar{S}+\bar{P}} \sum_{kl} h_{kl} h_{-k-l} \frac{l}{k(\bar{S}-1)+l} \left[\frac{\bar{S}}{\bar{P}} l^2 + 2kl - 2k^2 \right] - \varkappa\bar{P}(2\bar{S}+\bar{P}) \sum_{k,l} \frac{\partial h_{kl}}{\partial P} \frac{\partial h_{-k-l}}{\partial P}. \tag{34}$$

The Eqs. (33) and (34) have singularities at the "resonance" values of $\bar{S}$, i. e., at those value for which the vanishing of one of the denominators $k(\bar{S}-1)+l$ is not accompanied by the vanishing of the corresponding numerator. The nonresonance system (33), (34) is not applicable in the vicinity of these singularities, where $[k(\bar{S}-1)+l] \sim \Omega$. We can, however, consider the behaviro of the solutions to the system (33), (34) at those $\bar{S}$ values at which the resonance denominator satisfies the condition $\Omega \ll k(\bar{S}-1)+l \ll 1$. As the examples considered in the following sections show, the analysis of the behavior of the solutions to the system (33), (34) in the vicinity of the singularities of the system turns out to be important for the understanding of the general character of the solutions.

4. RELAXATION IN THE *A* PHASE

The substitution of the values of h^A_{kl} from (20) into (33) and (34) leads to the following system of equations:

$$\frac{d\bar{S}}{dt} = 4\varkappa \left(\frac{\Omega_A}{4}\right)^4 \left[\frac{1}{1-\bar{S}} \left(\frac{\bar{P}}{\bar{S}}+2\right)^3 + \frac{2\bar{P}}{\bar{S}^2}\left(\frac{\bar{P}}{\bar{S}}+2\right) + \frac{1}{1+\bar{S}}\left(\frac{\bar{P}}{\bar{S}}\right)^3 \right], \tag{35}$$

$$\frac{d\bar{P}}{dt} = -4\varkappa \left(\frac{\Omega_A}{4}\right)^4 \left\{ \frac{\bar{P}}{\bar{S}} \left(\frac{\bar{P}}{\bar{S}}+2\right) \left[\frac{2}{\bar{S}} + 12 + 9\frac{\bar{P}}{\bar{S}}\left(\frac{\bar{P}}{\bar{S}}+2\right)\right] + \frac{2}{1+\bar{S}}\left(\frac{\bar{P}}{\bar{S}}\right)^3 \right\}. \tag{36}$$

The right member of Eq. (36) is positive, and therefor $\bar{P}$ increases monotonically in time. The expression for $d\bar{S}/dt$ has a more complex structure. For $\bar{S} > 1$, the quantity $\dot{\bar{S}} < 0$, and $\bar{S}$ approaches the resonance value $\bar{S} = 1$. For $\bar{S} < 1$, the sign of the derivative is determined by the competion between the first and the two remaining terms in the square brackets in Eq. (35). In the region where the term $\sim 1/(1-\bar{S})$ is the dominant term, $\bar{S}$ is drawn into the resonance region. There is, however, a region where $\dot{\bar{S}}$ and $1-\bar{S}$ have opposite signs. If the initial values of $\bar{S}$ and $\bar{P}$ lie in this region, then $\bar{S}$ initially moves away from the value $\bar{S} = 1$.

For $\bar{P} = -2\bar{S}$, the coefficient attached to $1/(1-\bar{S})$ in Eq. (35) vanishes, and therefore the neighborhood of the point $\bar{S} = 1$, $\bar{P} = -2$ deserves special investigation. Such an investigation is also required for a comparison with the experiments of Sager *et al.*,[2] since the initial state in these experiments was obtained by rotating the magnetization through an angle of 180° from its equilibrium state, i. e., the initial values of $\bar{S}$ and $\bar{P}$ lay in the vicinity of the indicated singular point.

Let us set $\bar{S} = 1 + \sigma$ and $\bar{P} = -2 + \zeta$, with $\sigma, \zeta \ll 1$. Then $(\bar{P}/\bar{S} + 2) = \xi \approx 2\sigma + \zeta$. Retaining in Eqs. (35) and (36) only the dominant—with respect to the small corrections—terms, and changing the time scale by setting $t' = t\varkappa\Omega_A^4/16$, we obtain the following system of equations for ξ and ζ:

$$d\xi/dt' = \xi^3/(\zeta - \xi), \tag{37}$$

$$d\zeta/dt' = 2. \tag{38}$$

A clear idea about the behavior of the solutions to this system is afforded by the phase trajectories, i. e., the $\xi(\zeta)$ curves shown in Fig. 1. The arrows indicate the direction of variation of ξ and ζ with increasing t'.

The physical region $\xi > 0$ is divided into two by the cuvve a), the equation for which is $\zeta = \xi + \xi^3/(2 - 3\xi^2) \approx \xi + \xi^3/2$. The phase trajectories that originate from the region enclosed between the curve a) and the ζ axis remain, as t' increases, in this region, and do not approach the resonance value $\bar{S} = 1$. They, however, get into the domain of influence of the other resonance value $\bar{S} = 0$. For the phase trajectory corresponding to the variation of S along the direction of the magnetic field, $\bar{S} = 0$ is not a resonance value, and in this case the system (35), (36) is integrable, and leads to the law,

$$(\bar{S}_z - 1)^2 = 4 - \varkappa(\Omega_A^2/2)^2 t, \tag{39}$$

obtained earlier by Leggett and Takagi.[3] Analysis of the phase trajectories in the neighborhood of the $\bar{S} = 0$ resonance shows that there are both phase trajectories that enter and phase trajectories that leave the vicinity of this point. Thus, the $\bar{S} = 0$ resonance does not trap the phase trajectories, but is traversed during the motion of the system over a finite period of time. As the

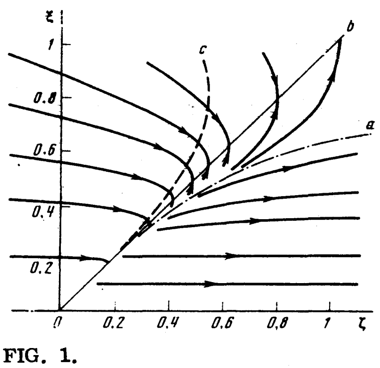

FIG. 1.

phase trajectories recede from the ζ axis, the deviations from the law (39) increase. It should, however, be noted that the neighborhood of the resonance point $\overline{S}=0$ has a "focusing" effect on the phase trajectories. After crossing this neighborhood, the direction of the magnetization approaches that of the z axis. Thus, the estimate for the trajectories passing close to $\overline{S}=0$, but remaining everywhere in the region of applicability of the Eqs. (35) and (36) shows that $(1+\cos\beta)_{-\infty}/(1-\cos\beta)_{+\infty}\approx e^2$. It is to be expected, therefore, that the deviations from the law (39) will be more significant at the initial stage of the relaxation—up to the crossing of the region of $\overline{S}$ values close to zero.

The trajectories originating from above the curve a) terminate on the straight line b) ($\xi=\zeta$), which corresponds to the principal resonance $\overline{S}=1$. The Eqs. (37) and (38) are inapplicable in the neighborhood, $(\xi-\zeta)\sim\Omega\xi^3$, of this straight line. It is easy to see, however, that there are no phase trajectories emanating from this neighborhood of the resonance straight line. This means that, in the case of initial data lying above the curve a), the relaxation will in the end proceed along resonance states.

Figure 1 also shows the curve c): $\zeta=\xi-\xi^3/2$. On this curve $d\overline{S}_z/dt=0$. The phase trajectories emanating from the left of the curve c) intersect the curve c) as they approach the resonance region, $d\overline{S}_z/dt$ changing its sign at the same time. The variable $\overline{S}_z$ will decrease during the motion between the curves c) and b), which apparently explains the experimentally observed[2] nonmonotonic relaxation. The cause of the nonmonotonic character thus consists in the fact that, in the vicinity of the resonance value $\overline{S}=1$, the quantity $\overline{S}$ varies in time much more rapidly than $\overline{P}$, and therefore the rapid decrease of $\overline{S}$ should be componsated by the equally rapid decrease of $\overline{S}_z$. Corroborating this explanation is the fact that the extent of the nonmonotonic section on the relaxation curve varied when the pulse that produced the initial deviation of the magnetization was modulated. The modulation changed the initial $\overline{P}$ value and, with it, the extent of the section of the corresponding—to this value—phase trajectory between the curves c) and b). The Eqs. (35) and (36) allow us to find the equation, in the coordinates $\overline{S}$ and $\overline{P}$, and valid for all $\overline{S}$ and $\overline{P}$ values, of one of the boundaries of the region where the relaxation is nonmonotonic (to wit, the equation of the curve on which $d\overline{S}_z/dt=0$):

$$\frac{1}{\overline{S}-1}\left(\frac{\overline{P}}{\overline{S}}+2\right)^3+\frac{1}{\overline{S}+1}\left(\frac{\overline{P}}{\overline{S}}\right)^3+3\frac{\overline{P}}{\overline{S}}\left(\frac{\overline{P}}{\overline{S}}+2\right)\left[4+3\frac{\overline{P}}{\overline{S}}\left(\frac{\overline{P}}{\overline{S}}+2\right)\right]=0. \tag{40}$$

The other boundary of the nonmonotony region is the straight line $\overline{S}=1$.

In the resonance region the motion of the magnetization is described by three equations for three variables: $\overline{S}$, $\overline{P}$, and $\overline{\Phi}$. Let us note that, according to the results of Sec. 3, the quantities $\overline{S}$ and $\overline{\Phi}$ attain their steady-state values at fixed $\overline{P}$ over a period of time $t\sim(\varkappa\Omega^2)^{-1}$, after which their variation occurs only as a result of the dependence of the steady-state values of $\overline{S}$ and $\overline{\Phi}$ on $\overline{P}$. In order to find how $\overline{P}$ varies in time, let us, following Leggett and Takagi,[3] use the averaged expression for energy dissipation, i.e., let us average over the fast variable α the equality

$$\frac{dE}{dt}=-2f=-\varkappa\left\{\frac{2S}{2S+P}\frac{\partial U}{\partial\Phi}\left(\frac{\partial U}{\partial\Phi}+\frac{\partial U}{\partial\alpha}\right)-\frac{S^2}{P(2S+P)}\left(\frac{\partial U}{\partial\alpha}\right)^2-P(2S+P)\left(\frac{\partial U}{\partial P}\right)^2\right\}. \tag{41}$$

The variation of the energy occurs largely as a result of the variation of the z component of the magnetization. The rate of change of the remaining terms ($S^2/2$ and U) in the energy (1) contains the superfluous power Ω^2. Setting $x=\cos\beta$, and substituting the steady-state values $\overline{S}\approx1$ and $\overline{\Phi}=0$ into (41), we obtain for the relaxation of the angle β the equation:

$$dx/dt=2\varkappa(\Omega_A/4)^4(1-x)(35x^3+55x^2+25x+13). \tag{42}$$

In Ref. 3, the averaging of the expression for the energy dissipation is performed under the assumption that the vector **d** assumes any direction with equal probability during its motion. As a result, the rate of energy dissipation turns out to be a constant. The right member of Eq. (42) is not very different from a constant at large angles of deviation, but it vanishes at $\beta=0$, and at small angles, such that $(1-x)\ll1$, the longitudinal component of the magnetization varies in time according to the exponential law

$$(1-x)\approx(1-x_0)\exp\{-\varkappa\Omega_A^4(t-t_0)\}. \tag{43}$$

The numerical integration of Eq. (42) for all angles leads to the dependence depicted by the solid curve in Fig. 2, which in we also present for comparison the experimental data obtained by Sager *et al.*[2] We took only the data for the open geometry and for the strongest of the magnetic fields used in the experiments.[2] The used data satisfy the conditions for the applicability of the expounded approach. Thus, for an 85-Oe field, in which $\omega_L\tau\approx0.1$, $\Omega_A^2\approx0.003$. The agreement between the theoretical curve and the experimental points is satisfactory. It should, however, be borne in mind that the value of $\varkappa$ is not known *a priori*, and is an adjustable parameter determining the scale along the time axis. A more sensitive verification could be a comparison with experiment of the angle dependence of the relaxation rate, which is given by the right-hand side of Eq. (42). It is, however, difficult to extract dS_z/dt with

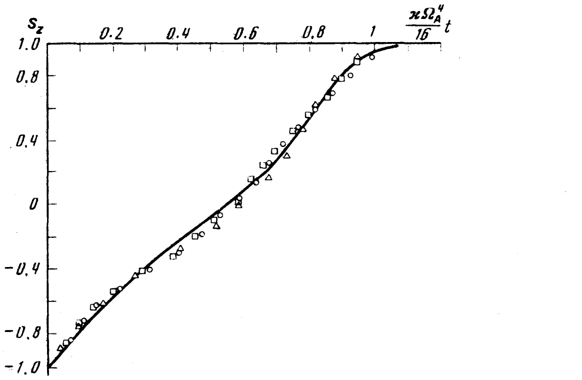

FIG. 2. Result of the integration of Eq. (42)—continuous curve. The points represent the data of Sager *et al.*[2] for magnetic fields, pressures, and $(1-T/T_c)$ values respectively equal to: △) 60 Oe, 21 bar, 0.0028; ○) 85 Oe, 21 bar, 0.0028; □) 85 Oe, 20.7 bar, 0.0053.

reasonable accuracy from the published experiments. It should also be borne in mind in a comparison with experiment that Eq. (42) was derived under the assumption that the system is completely homogeneous. There arise in the presence of an inhomogenity spin currents which can affect the relaxation rate both directly, as a result of magnetization transfer, and indirectly, as a result of a change in the Φ value corresponding to stationary precession.

We thus see that even a small change in the initial values of $\overline{S}$ and $\overline{P}$ can, if these values are close to $\overline{S}=1$, $\overline{P}=-2$, lead to significantly different temporal relaxation laws. In experiments, because of the shift of the precession frequency of the magnetization from the Larmor frequency, the actual initial angle between S and H_0 differs from the nominal angle computed from the pulse duration. This difference is determined by the parameter[6] $\omega_{\parallel}^2/H_0H_1$, where H_1 is the amplitude of the rotating variable field. This parameter depends both on temperatue and on the magnetic field, and it is therefore not surprising that a change in the time dependence of the relaxation was observed in the experiments of Sager *et al.*[2] when these quantities were changed. In order to obtain the predicted dependence for the relaxation, we should control the initial conditions more precisely, which, apparently, can be done more conveniently at points far from the point $\overline{S}=1$, $\overline{P}=-2$.

5. RELAXATION IN THE *B* PHASE

The analysis of the magnetization relaxation in the *B* phase is performed according to the same scheme as the analysis in the *A* phase. For the nonresonance case the equations of motion have, according to (33), (34), and (21), the following form:

$$\frac{d\overline{S}}{dt}=\varkappa\left(\frac{2\Omega_B^2}{15}\right)^2\frac{1}{1-\overline{S}}\left(2+\frac{\overline{P}}{\overline{S}}\right)\left[\left(1+2\frac{\overline{P}}{\overline{S}}\right)^2+\left(2+\frac{\overline{P}}{\overline{S}}\right)^2\right], \tag{44}$$

$$\frac{d\overline{P}}{dt}=-\varkappa\left(\frac{2\Omega_B^2}{15}\right)^2\frac{\overline{P}}{2\overline{S}}\left(2+\frac{\overline{P}}{\overline{S}}\right)\left[18\left(1+\frac{\overline{P}}{\overline{S}}\right)^2+\left(5+4\frac{\overline{P}}{\overline{S}}\right)^2+\left(2+\frac{\overline{P}}{\overline{S}}\right)^2\right]. \tag{45}$$

The main difference between this system and the corresponding system for the *A* phase consists in the fact that here we have only one resonance value, $\overline{S}=1$, as a result of which $\overline{S}$ tends to this value whatever the initial conditions are; $\overline{P}$, as in the *A* phase, increases monotonically. The point $\overline{S}=1, \overline{P}=-2$ is in this case a singular point. Setting in the neighborhood of the singularity, as was done in the analysis of the system (35), (36),

$$\overline{S}=1+\sigma,\quad (\overline{P}/\overline{S}+2)=\xi,\quad t'=\varkappa({}^2/{}_5\Omega_B^2)^2t,$$

we have

$$d\sigma/dt'=-\xi/\sigma, \tag{46}$$

$$d\xi/dt'=(3-2/\sigma)\xi. \tag{47}$$

The equation for the phase trajectories (see Fig. 3) has the form

$$\xi=\xi_0-{}^3/{}_2(\sigma-{}^2/{}_3)^2. \tag{48}$$

The phase trajectory passing through the coordinate origin is a bounding trajectory. The trajectories lying below it tend to the abscissa (i.e., the $\beta=\pi$) axis. As they approach the axis, the dependence of the two variables (σ and ξ) on the time becomes exponential; therefore, the value $\xi=0$ is attained after an infinitely long time. For real systems this means that the relaxation in this case will occur on account of other mechanisms. The trajectories lying above and to the left of the bounding trajectory reach the ξ axis after a finite period of time, after which the motion of the system should be described by the equations for the resonance case. Thus, in the *B* phase the magnetization cannot relax from initial states close to $\overline{S}=1$, $\overline{P}=-2$ along the direction of the magnetic field. The curve on which $d\xi/dt=0$ for the system (46), (47) is $\sigma=1/3$. It lies in a region of fairly large values of σ, and therefore it seems natural that nonmonotonic relaxation was not observed in the *B* phase in the experiments of Sager *et al.*[2]

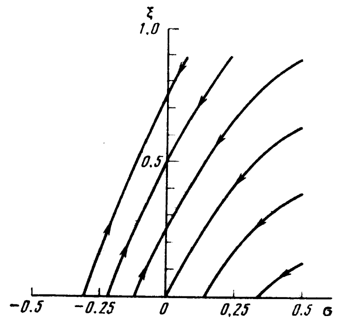

FIG. 3.

For the *B* phase the potential U_B (see (18)) in the Leggett configuration does not depend on α, i.e., the fast variable has already been separated out, and therefore the system of magnetic-dynamics equations for the *B* phase in the resonance case can be obtained from Eqs. (5)–(8) by setting $\partial U/\partial\alpha=0$ in them:

$$dS/dt=-\partial U_B/\partial\Phi, \tag{49}$$

$$\frac{d\Phi}{dt}=S-1+\frac{\partial U_B}{\partial S}-\varkappa\frac{2S}{2S+P}\frac{\partial U_B}{\partial\Phi}, \tag{50}$$

$$\frac{dP}{dt}=-\varkappa P(2S+P)\left(\frac{\partial U_B}{\partial P}\right)^2, \tag{51}$$

$$\frac{d\alpha}{dt}=-1+\frac{\partial U_B}{\partial P}-\varkappa\frac{S}{2S+P}\frac{\partial U_B}{\partial\Phi}. \tag{52}$$

In the absence of longitudinal oscillations, the relaxation of the magnetization is described by Eq. (51), in which the derivative $\partial U_B/\partial P$ should be computed for the S and Φ values corresponding to the steady state. Such a derivative is, as is well known, nonzero in the *B* phase only for $P<-5/4$, i.e., for the angles $\beta>\theta_0=\arccos(-1/4)$; in this case (see Ref. 7, formula (23) for $J=0$)

$$\frac{dP}{dt}=-\varkappa\left(\frac{16}{15}\Omega_B^2\right)^2\times P(P+2)\left(P+\frac{5}{4}\right)^2. \tag{53}$$

Recalling that in the steady state $P=S_z-1$, we find that, in the vicinity of $P=-2$ ($P=-2+\zeta$), the relaxation occurs according to the exponential law:

$$\zeta=\zeta_0\exp[8(t'-t_0')]. \tag{54}$$

Here $t'=\varkappa(\frac{2}{5}\Omega_B^2)^2t$. In the neighborhood of $P=-5/4$, i.e., for $P=-5/4-\psi$, where ψ is small, the relaxation law will be as follows:

$$\psi^{-1}=\psi_0^{-1}+{}^{20}/_3(t'-t_0').$$

When $-5/4\leqslant P\leqslant 0$, for the stable steady states the derivative $\partial U_B/\partial P=0$, i.e., $\dot{P}$ is equal to zero, and the relaxation of the magnetization does not occur.

This result is clearly at variance with experiment. The relaxation observed in experiment occurs over a period of time of the same order as the relaxation time for $P<-5/4$, and therefore it cannot be explained by the effect of the terms of the next order in Ω^2.

To explain the observed relaxation, we can assume that a transition into the auto-oscillating state occurs as P approaches the value $P=-5/4$. In this state Φ and S oscillate, the dissipation of the energy of the oscillations being compensated for as a result of the increase of P and the corresponding decrease of the Zeeman energy. In its turn, the existence of the oscillations [see Ref. 7, formula (17)] leads to a shift of the precession frequency from the Larmor frequency and to the relaxation of P. Such a compensation is impossible for motions whose phase trajectories lie far from the singular trajectories. For the nonsingular trajectories the dissipation is characterized by the time $(\varkappa\Omega^2)^{-1}$, while the inflow of energy is correspondingly characterized by the time $(\varkappa\Omega^4)^{-1}$. The compensation can, thus, occur only for phase trajectories passing close to a separatrix. The possibility in principle of compensation in this case is connected with the fact that as the phase trajectory approaches the separatrix, the frequency of the longitudinal oscillations tends to zero, and S and Φ spend the greater part of the time in the vicinity of the maximum of the potential, i.e., in the region where the rate of change of Φ and, with it, the dissipation of the energy are small. The precession-frequency shift does not, however, vanish in this case.

For a rough estimate of the relaxation rate, we can substitute into the right-hand side of (51) that value of $\partial U_B/\partial P$ which corresponds to the maximum of the potential. Then the relaxation in the case when P is greater than, but not too close to, $-5/4$ will also be described by Eq. (52). In Fig. 4 we compare the right member of Eq. (53) with the rate, computed from the experimental data of Sager *et al.*,[2] of change of S_z. The scale of the curve was adjusted to the points lying in the region $x<-1/4$, since the predictions of the theory in this region are specific. Let us also note that the accuracy with which dS_z/dt can be determined from the data given in Ref. 2 is not high, and, therefore, the comparison carried out in Fig. 4 should not be taken too seriously. The above-presented qualitative arguments constitute an attempt to explain the relaxation observed in the $P>-5/4$ region without giving up the assumption that the system is spatially homogeneous. In the inhomogeneous case, as Webb's experiment[10] and its interpretation[11] show, there exist effective relaxation mechanisms for all angles, relaxation being possible both in the regime of stationary S and Φ and in the autooscillation regime.

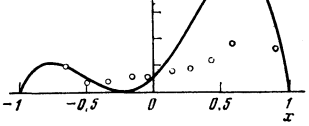

FIG. 4. Plot of the right member of Eq. (53)—continuous curve; the points represent the values of dS_z/dt, computed from the data of Sager *et al.*[2] for $H_0=30$ Oe, a pressure of 19.7 bar, and $(1-T/T_c)=0.0010$.

6. CONCLUSION

The analysis carried out in the present paper of the phenomenological spin-dynamics equations written for the superfluid phases with allowance for the dissipative terms shows that the time dependence of the relaxation of the magnetization is satisfactorily described by these equations. The observed time dependences can be characterized by one phenomenological parameter, $\varkappa$. The existence of the various time dependences is due to the existence of different relaxation regimes—nonresonance and resonance—and the possibility of a transition from one regime into the other. The choice of a regime is determined by the initial experimental conditions. There exist regions of initial data that lead in the end to one or the other regime. In the vicinity of the boundaries of the regions the system is sensitive to small changes in the initial data. In particular, in the A phase the regions corresponding to the various modes of behavior of the magnetization coincide in the neighborhood of the point $S=1$, $P=-2$, which apparently explains the sharp change that is experimentally observed[2] to occur in the character of the relaxation when a relatively small change is made in the external conditions in the case when the initial state is prepared by rotating the magnetization through 180° from the equilibrium state.

In accordance with the terms of the formulated problem, we did not touch upon here the dependences of the phenomenological parameter $\varkappa$ on the external conditions, such as the temperature and the magnetic field; these dependences are determined by the specific relaxation mechanism. The phenomenological theory allows us, however, to interprete experiments in terms of the parameter $\varkappa$. A comparison of the temperature and magnetic-field dependences found experimentally for the parameter $\varkappa$ with the theoretical predictions for the various relaxation mechanisms will allow us to determine which of the mechanisms is dominant under a given set of external conditions.

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Phase transition describable by self-consistent-field theory in an exactly solvable model

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The mechanism of indirect interaction of the order parameter via acoustic oscillations is considered. The special character of the contribution of the static deformation is taken into account. The spectrum of the acoustic oscillations that interact with the order parameter is assumed to be linear. The microscopic description of the thermodynamics with the aid point transformations is reduced to the self-consistent field theory. The principal results of the theory are applied to the model in question. The low-temperature phase transition in $KMnF_3$ is discussed on the basis of these results.

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INTRODUCTION

In the theory of phase transitions, the interaction of the order parameter with the mean field is usually considered only as an approximate description of the initial localized microscopic interaction.[1] At the same time, the interaction with the mean field is actually the result of the special character of the contribution of the homogeneous mode to the indirect exchange.[1)] In the present paper we investigate the case when the total interaction reduces entirely to the interaction of the order parameter with the mean field.

The simplest physical system that has such a behavior is a system of two-level states placed in an elastically isotropic medium and interacting only with the deformation. The effective interaction is made up of the contribution of the static deformation and of the exchange via acoustic oscillations. The linearity of the spectrum of the acoustic oscillations, which is typical of the model, leads to the absence of a short-range dynamic effective interaction. The model can be solved exactly and represents the case when the self-consistent field theory yields a rigorous description of the thermodynamics in the microscopic approach. The model includes a first-order phase transition. In the temperature-pressure plane, the transition line terminates at the critical point. Two possible critical behaviors appear in the model. If the degree of degeneracy of the levels of the two-level system are different van der Waals behavior takes place near the critical point. If the degrees of degeneracy of the levels coincide, the results correspond to the Weiss theory.[5] The low-temperature phase transition recently observed in $KMnF_3$ (Ref. 6) is interpreted as a direct realization of the model.

DESCRIPTION OF MODEL

Assume that a lattice with two-level systems at its site is immersed in a homogeneous elastically isotropic medium. The Hamiltonian of such a composite system can be written in the form

$$H=\int\left(\frac{\lambda}{2}U_{\alpha\alpha}{}^2(\mathbf{r})+\mu U_{\alpha\beta}{}^2(\mathbf{r})\right)dV+\sum_j\left(J_0+gU_{\alpha\alpha}(\mathbf{r}_j)\right)\eta_j, \tag{1}$$

where λ and μ are the elastic moduli, $U_{\alpha\beta}(\mathbf{r})$ is the strain tensor at the point $\mathbf{r}$, summation over repeated indices is implied, $\eta_j=\pm 1$ is a parameter that specifies the occupation of the states of the two-level system with coordinate $\mathbf{r}_j$, and $J_0+gU_{\alpha\alpha}(\mathbf{r}_j)$ is the difference between the level energies of the two-level system and depends on the strain. The summation in (1) is over the lattice sites, and the integration is over the corresponding volume.

Allowance of the deformation contribution to the thermodynamics will be carried out in analogy with Ref. 7. We determine the expansions of the strain tensor and of the parameter η_j in Fourier harmonics by the formulas

$$U_{\alpha\beta}(\mathbf{r})=\bar{U}_{\alpha\beta}+\frac{i}{2V^{1/2}}\sum_{\mathbf{k}}{}'\left(k_\alpha U_\beta{}^{\mathbf{k}}+k_\beta U_\alpha{}^{\mathbf{k}}\right)e^{i\mathbf{k}\mathbf{r}}, \tag{2}$$

$$\eta_j=\frac{1}{N^{1/2}}\sum_{\mathbf{k}}\eta_{\mathbf{k}}\exp(i\mathbf{k}\mathbf{r}_j). \tag{3}$$

Here $\bar{U}_{\alpha\beta}$ is the uniform deformation, U_β^k is the Fourier component of the displacement vector, V is the volume