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Longitudinal waves and two-stream instability in a relativistic plasma

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We investigate the problem of longitudinal waves in a relativistic plasma, with phase velocity lower than the velocity of light. We show that such waves exist at arbitrary particle momentum distribution function that falls off rapidly enough at large momenta. Earlier papers dealing with these questions are analyzed in connection with the problem of pulsar radiation, and the causes of the unclear and confused views advanced in some of these papers are discussed. Relativistic-plasma instabilities due to Cerenkov buildup of longitudinal waves by beams of high-energy and low-energy particles are investigated. The quasilinear relaxation of a beam of high-energy particles in a relativistic plasma is considered and the main regularities of this relaxation are explained.

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1. INTRODUCTION

Laboratory experiments have stimulated the development of a theory of collective processes induced by a beam of relativistic electrons in a nonrelativistic plasma.[1] This branch of plasma theory is used also in the interpretation of astrophysical phenomena (see, e.g., Chap. 3 of Ref. 2). At the same time, interest attaches in a number of astrophysical problems to the interaction of a beam of relativistic particles with a relativistic plasma. An example is the problem of radioemission from pulsars (see Ref. 1 and the literature cited therein).

Since two-stream instability is due to Cerenkov interaction of the particles with the waves, $\omega = kv$ (ω, k are the frequency and wave number, and v is the particle velocity), the question of two-stream instability in a relativistic plasma is frequently associated with the question of longitudinal (potential) waves of such a plasma with a phase velocity lower than that of light, $v_{\rm ph} \equiv \omega/k < c$. Longitudinal waves in a relativistic plasma were considered initially by Silin[4] and by him with Rukhadze[5] for the case of an isotropic Boltzmann distribution of the particles in energy. Only waves with $\omega/k > c$ were observed in the cited studies. Silin and Rukhadze have therefore made a general statement that waves with $\omega/k < c$ cannot exist in a relativistic plasma, and consequently Cerenkov interaction of longitudinal waves of particles is impossible. The problem of longitudinal waves of particles is impossible. The problem of longitudinal waves in a relativistic plasma was later investigated by Tsytovich.[6] He has considered a plasma having the same momentum distribution function as Silin and Rukhadze,[4,5] but, on the contrary, found waves with $\omega/k < c$ and calculated their damping decrement due to the Cerenkov interaction with particles. This result contradicts the aforementioned conclusion of Silin and Rukhadze, but Tsytovich made no note of this contradiction and did not explain its causes.

Interest in longitudinal waves in the relativistic plasma has increased after the discovery of pulsars. The question of longitudinal waves with $\omega/k < c$ was considered anew by Tsytovich and Kaplan[7] (without reference to the earlier paper[6]!), but no longer for an isotropic Boltzmann distribution function, as in Refs. 4–6, but for one-dimensional power-law distributions. Tsytovich and Kaplan[6,7] have stated that longitudinal waves with $\omega/k < c$ exist and that Cerenkov generation

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of such waves is possible. This result, however, was critized by Suvorov and Chugunov,[8] who reaffirmed the previously noted conclusion of Silin and Rukhadze[4,5] that there are no waves with $\omega/k < c$, and reached the conclusion that Cerenkov generation of such waves is impossible, i.e., that two-stream instability is impossible in a relativistic plasma. A compromise point of view is advanced by Ginzburg and Zheleznyakov,[3] who noted that the realization of two-stream instability in a relativistic plasma depends on the concrete form of the plasma particle momentum distribution function. That this is a moot question is stated also in Sec. 5 of Zheleznyakov's book,[9] where it is noted, in addition, that if the phase velocity of the plasma waves v_{ph} becomes smaller than the velocity of light c, then strong Landau damping is to be expected on particles with velocities $v \approx c$, and it is doubted whether solutions of the dispersion equation $\varepsilon(\omega, k) = 0$ exist at all in this region for an equilibrium plasma (ε is the longitudinal permittivity of the plasma).

As to applications of the theory of instabilities of a relativistic plasma to the pulsar emission problem, the situation seems here quite ambivalent, with two groups of authors adhering to two alternate points of view. One group[3,8,9] denies outright or by implication the possibility of development of two-beam instability under pulsar conditions, and take the keystone to be other types of instability. The other group,[10-15] on the contrary, base their analysis on the concept of two-stream instability.

It is clear from the foregoing that the question of longitudinal waves and two-stream instability in a relativistic plasma still remains unclear. We must therefore first establish which of the preceding results are correct and which are in error. Such an analysis is given in Sec. 2 of the present paper. In agreement with Tsytovich and Kaplan[6,7] we shall show, first, that plasma (longitudinal) waves with phase velocity lower than that of light exist at an arbitrary particle momentum distribution,[1] second, that these waves, even though $v_{ph} < c$ are self-damping, and third, that they can be excited by fast-particle beams. In addition, we shall show that two-stream instability can occur also at $\omega/k < c$ such that the dispersion equation $\varepsilon(\omega, k) = 0$ for the equilibrium plasma has no solutions.

A quantitative analysis of this group of questions will be presented in the following sections. At the same time, simple qualitative arguments can be advanced to explain the situation.

It follows from the definition of a relativistic plasma that its particles have velocities v close to the speed of light c, and its momenta p exceed mc (m is the rest mass of the particle). The degree of relativism of the plasma can be characterized by the dimensionless parameter $\bar{p}/mc$, where $\bar{p}$ is the characteristic momentum of the particle, so that particles with $p \gg \bar{p}$ are either totally nonexistent or their number is small enough (say, exponentially small). If the wave has a phase velocity $\omega/k < c$, then the velocity of the particles that are at Cerenkov resonance with this wave is $v_{res} = \omega/k$, and their momentum is

$$p_{res}=mc[(ck/\omega)^2-1]^{-1/2} \tag{1.1}$$

Should waves with $\omega/k < c$ be strongly damped because of the resonant interaction with the particles? Strong damping can be expected at $p_{res} \lesssim p$. Obviously, however, this damping should be small enough or be completely absent if

$$p_{res} \gg \bar{p} \tag{1.2}$$

because of the small number or absence of particles with such momenta. From (1.1) and (1.2) follows, in analogy with Tsytovich and Kaplan,[7] an estimate of the interval of the phase velocities of weakly damped waves with $\omega/k < c$:

$$0<1-\omega/kc<(mc/\bar{p})^2. \tag{1.3}$$

The analysis in Sec. 2 shows that such waves actually exist and that it is precisely this interval which contains the weakly damped waves obtained in Ref. 6. In contrast, Silin and Rukhadze,[5] as well as Suvorov and Chugunov,[8] calculated the permittivity and analyzed the possible realization of waves with $\omega/k < c$ in the opposite limiting case

$$1-\omega/kc \gg (mc/\bar{p})^2. \tag{1.4}$$

Assuming, in addition, that $1 - \omega/ck \ll 1$, Silin and Rukhadze,[4,5] having demonstrated the absence of waves with

$$(mc/\bar{p})^2 \ll 1-\omega/ck \ll 1, \tag{1.5}$$

have interpreted this result as pertaining to the entire interval of the possible values of the difference $1 - \omega/ck \ll 1$, that it is clear from the foregoing that this is incorrect.

The foregoing arguments concerned the possibility of accepting natural plasma oscillations by a beam. This explains also the interest to the problem of natural plasma oscillations. It must be borne in mind, however, that even in the absence of natural oscillations with certain $\omega/k < c$, it does not follow, generally speaking, that the beam cannot excite perturbations with such ω/k. This circumstance is well known in the theory of the instabilities of a nonrelativistic plasma. In particular, the instability of a slow monoenergetic beam in a hot nonrelativistic plasma is due to the buildup of natural oscillations of the plasma (see Ref. 16, Sec. 3.3). (These perturbations are natural oscillations of the complete plasma + beam system.) It is natural to assume that an analogous instability can take place also in a relativistic plasma. This is confirmed by the calculations that follow.

In Sec. 2 we present a quantitative analysis of longitudinal waves with $\omega/k < c$ in a relativistic plasma neglecting beam effects. In Sec. 3 we consider Cerenkov buildup of such waves by a high-energy beam, while in Sec. 4 we consider a similar buildup by a low-energy beam. By high-energy beam we mean here a beam in which the average particle momentum p_0 is large compared with the average plasma-particle momentum, $p_b \gg \bar{p}$, while a low-energy beam is such that $p_b \ll \bar{p}$. In either case, both the beam and the plasma are assumed relativistic, $(p_b, \bar{p}) \gg mc$.

In connection with the problem of the low-energy beam, we note that according to the condition (1.1) and the relation $p_{res} \approx p_b \ll \bar{p}$ such a beam interacts resonantly with perturbations whose phase velocity satisfies the inequality (1.4). However, as noted above, it is precisely to this interval of phase velocities that the expressions for the longitudinal permittivity calculated in Refs. 5 and 8 pertain. These results of the cited papers are used by us in Sec. 4.

The question of two-stream instability of a relativistic plasma is important for a correct understanding of nonlinear plasma processes in such a plasma. On the whole, this far-ranging problem is beyond the scope of the present article. We, however, will touch upon the simplest and at the same time important part of this problem, namely that of quasilinear relaxation in two-stream instability in a relativistic plasma. This is the subject of Sec. 5. The results are discussed in Sec. 6.

2. LONGITUDINAL WAVES IN A RELATIVISTIC PLASMA

We start from the dispersion equation for the longitudinal oscillations

$$\varepsilon(\omega, k)=0 \tag{2.1}$$

with the following expression for the permittivity[4,5,9]

$$\varepsilon(\omega, k)=1-\omega_p^2\langle\gamma^{-3}(\omega-kv)^{-2}\rangle. \tag{2.2}$$

It is assumed that the perturbations depend on the coordinates and on the time like $\exp(-i\omega t+ikz)$. The direction z is arbitrary in the absence of a magnetic field, and coincides with the direction of the intensity vector of this field if such a field is present in the plasma. The symbol $\langle\ \rangle$ denotes averaging over the particle momenta, $\langle(\ldots)\rangle=\int f_0(p)(\ldots)dp/n$, $n=\int f_0(p)dp$ is the plasma density, $f_0(p)$ is the equilibrium particle momentum distribution function (one/dimensional), $\gamma=(1+p^2/m^2c^2)^{1/2}$ is the relativistic factor, $v=c(1+m^2c^2/p^2)^{-1/2}$ is the particle velocity, m is its mass, $\omega_p^2=4\pi e^2n/m$ is the square of the plasma frequency, and e is the particle charge.

We confine ourselves to a one-dimensional particle distribution in momentum (the case of greatest interest for applications to the pulsar problem). We find with the aid of (2.1) and (2.2) that the longitudinal oscillations have a phase velocity equal to that of light, $\omega/k=c$ at $k=k_0$, where k_0 is defined by the relation $k_0^2=2\omega_p^2\langle\gamma\rangle/c^2$. In this case $\omega=\omega_0\equiv ck_0$. Using this result, we can obtain by successive approximations (cf. Refs. 6 and 7) a solution of (2.1) with wave numbers close to k_0. Assuming $k+k_0+\Delta k$, $\omega=\omega_0+\Delta\omega$, where Δk and $\Delta\omega$ are small quantities, and taking into account the result of the zeroth approximation $\varepsilon(\omega_0,k_0)=0$, we obtain in the first approximation

$$\Delta\omega=-\left(\frac{\partial\varepsilon/\partial k}{\partial\varepsilon/\partial\omega}\right)_0\Delta k. \tag{2.3}$$

The zero subscript in the right/hand side of this equation means that the corresponding derivatives are taken at $k=k_0, \omega=\omega_0$. In this case[2]

$$\left(\frac{\partial\varepsilon}{\partial\omega}\right)_0=\frac{2\omega_p^2}{k_0^3c^3}\left\langle\gamma^3\left(1+\frac{v}{c}\right)^3\right\rangle,\quad \left(\frac{\partial\varepsilon}{\partial k}\right)_0=-c(1-\alpha)\left(\frac{\partial\varepsilon}{\partial\omega}\right)_0,$$
$$\alpha=\left\langle\gamma\left(1+\frac{v}{c}\right)^2\right\rangle\Big/\left\langle\gamma^3\left(1+\frac{v}{c}\right)^3\right\rangle. \tag{2.4}$$

From (2.3) and (2.4) follows the dispersion law

$$\omega=c[k-\alpha(k-k_0)]. \tag{2.5}$$

Since $\alpha>0$, it is seen from (2.5) that waves with $k>k_0$ have a phase velocity lower than that of light $\omega/k<c$ (at $k<k_0$ we have $\omega/k>c$). By the same token we have demonstrated the existence of longitudinal waves with $\omega/k<c$ at an arbitrary particle momentum distribution. We note also that the dispersion law (2.5) agrees qualitatively with the one obtained in Ref. 7 for a particular form of f_0.

We consider now the limits of applicability of (2.5). We recall that we have used the assumption that Δk and $\Delta\omega$ are small, and recognize that in accordance with (2.2) and (2.5) this assumption means that

$$k_0(c-v)\gg\Delta k(c-v)-\alpha c\Delta k. \tag{2.6}$$

Since the dimensionless parameter α is of the same order of magnitude as $1-v/c$ the condition (2.6) actually means that $\Delta k\ll k_0$. It is also clear that, in order of magnitude, the dispersion Eq. (2.5) is valid right up to $\Delta k\approx k_0$.

We proceed now to calculate the damping decrement of the oscillations with $\omega/k<c$ when they interact with resonant particles. Taking into account the imaginary part of (2.2) and assuming it to be a small quantity, we obtain the perturbation decrement Γ (we assume that $\omega=\mathrm{Re}\,\omega+i\Gamma$):

$$\Gamma=-\mathrm{Im}\,\varepsilon/(\partial\varepsilon/\partial\omega)_0. \tag{2.7}$$

The denominator of the right-hand side of this equation is determined by the first relation of (2.4), while the numerator is equal to

$$\mathrm{Im}\,\varepsilon=-\frac{4\pi^2e^2}{k}\int\frac{\partial f_0}{\partial p}\delta(\omega-kv)\,dp=-\frac{4\pi^2e^2m}{k^2}\left(\gamma^3\frac{\partial f_0}{\partial p}\right)_{p=p_{res}}. \tag{2.8}$$

The value of the resonant momentum p_{res} is given by formula (1.1), and if $\omega(k)$ is of the form (2.5) we have $p_{res}=mc(2\alpha\Delta k/k_0)^{-1/2}$.

From (2.7), (2.4), and (2.8) we get

$$\Gamma=\frac{\pi}{2n}\frac{k_0c^3m^2}{\langle\gamma^3(1+v/c)^3\rangle}\left(\gamma^3\frac{\partial f_0}{\partial p}\right)_{p=p_{res}}. \tag{2.9}$$

In particular, in the case of an ultrarelativistic plasma with a distribution function in the form $f_0=(nc/2T)\exp(-c|p|/T)$, the expression for Γ means that (cf. Ref. 6)

$$\Gamma=-\frac{\pi k_0c}{96\gamma_0^2}\left(\frac{6k_0}{\Delta k}\right)^{3/2}\exp\left\{-\left(\frac{6k_0}{\Delta k}\right)^{1/2}\right\}, \tag{2.10}$$

where $\gamma_0=T/mc^2$. It is seen that at small $\Delta k/k_0$ the damping is exponentially small.

The general relation (2.7) can be represented also in the form of the energy balance of the oscillations (cf. Ref. 16, Sec. 2.2):

$$\frac{\partial W}{\partial t}=-\omega\,\mathrm{Im}\,\varepsilon\frac{|E_k|^2}{8\pi}, \tag{2.11}$$

where $W_k=\omega(\partial\epsilon/\partial\omega)|E_k|^2/8\pi$ is the energy of the oscillations, and $|E_k|^2 8\pi$ is the energy of the electrostatic field of the oscillations. In a nonrelativistic plasma we have $\varepsilon=1-\omega_p^2/\omega^2$, so that $\omega\partial\varepsilon/\partial\omega=2$, i.e., the total energy of the oscillations is equal to double the energy of the electrostatic field. On the other hand, for an ultrarelativistic plasma, $\gamma\gg1$, as follows from (2.4), we have $\omega\partial\varepsilon/\partial\omega\approx8\langle\gamma^2\rangle/\langle\gamma\rangle$, i.e., the oscillation energy is much higher than the electrostatic-field energy.

We discuss now the paper of Suvorov and Chugunov.[8] Their analysis (see also Sec. 5.10 of Zheleznyakov's book[9]) is based on the assumption that the investigated one-dimensional momentum distribution is the result of synchrotron radiation with an initial isotropic distribution in the form $F(v)=Q\delta(v-c)$, where W is a normalization factor. This approximation is insufficient, since the particle velocity is assumed to be exactly equal to that of light. To understand the result of this seemingly insignificant difference between v and c, we assume the initial isotropic velocity distribution in a single-velocity form $f=Q\delta(v-v_0)$, where $1-v_0/c\ll1$. Just as Suvorov, Chugunov, and Zheleznyakov,[8,9] we find that after the transverse energy has been emitted, a one-dimensional distribution in the longitudinal momenta $p_\parallel$ is established, in the form

$$f(p_\parallel)=\begin{cases}\dfrac{n}{2p_0}\dfrac{(1+p_0^2/m^2c^2)^{1/2}}{(1+p_\parallel^2/m^2c^2)^{1/2}}, & p\leqslant p_0,\\ 0, & p>p_0,\end{cases}\tag{2.12}$$

where $p_0=mv_0(1-v_0^2/c^2)^{-1/2}$. It is seen from (2.12) that the system has no particles with momenta $p>p_0$. Therefore, in analogy with (1.3), there exists a phase-velocity interval

$$0<1-\omega/ck<\tfrac{1}{2}(mc/p_0)^2,\tag{2.13}$$

which corresponds to a real value of the longitudinal permittivity. On the other hand, Suvorov and Chugunov[8] have assumed $p_0=\infty$, and in this case the interval (2.13) does not exist.[3]

We see thus that the reason why longitudinal waves with $\omega/k<c$ were not observed in Ref. 8 was the choice of an equilibrium distribution function unsuitable for this purpose.

As to the work of Tsytovich and Kaplan[7] they considered distribution functions in energy (E) in the form

$$f_E=\frac{(\varkappa-1)nE_*^{\varkappa-1}}{(E+E_*)^{\varkappa}}\begin{cases}1, & E<E_{max}\\ 0, & E>E_{max}\end{cases},\tag{2.14}$$

where E_* and E_{max} are the average and maximum particle energy, and $\varkappa$ is a dimensionless parameter that characterizes the rate of decrease of distribution function at high energies. In addition, they used[7] a successive-approximation method similar to that described above, and calculated the damping decrement of the longitudinal waves in two cases: at $2<\varkappa<4$ and at $\varkappa>4$. At $\varkappa<4$, however, this method cannot be used, so that the corresponding results of Tsytovich and Kaplan[7] are in error. This error was in fact the basis for the criticism of Suvorov and Chugunov.[8] The latter, however, criticized Kaplan and Tsytovich's results pertaining both to $\varkappa<4$ and $\varkappa>4$, and this is an error on the part of Suvorov and Chugunov.[8]

3. EXCITATION OF LONGITUDINAL WAVES IN A RELATIVISTIC PLASMA BY A BEAM OF HIGH-ENERGY PARTICLES

The possibility of buildup of longitudinal waves in a relativistic plasma by a beam of high-energy particles is clear even from formula (2.9): when $(\partial F/\partial p)_{p=p_{res}}>0$ the growth rate of the oscillations is positive. The corresponding instability is a kinetic variant of two-stream instabilities and occurs in the case of a beam with sufficiently large momentum spread. On the other hand, if the beam-particle momentum spread is not too large then, just as in the case of a nonrelativistic plasma, development of hydrodynamic two-stream instability is possible.

To ascertain the conditions for the particular instability, we consider by way of example a Gaussian distribution of the beam-particle momenta:

$$f_b(p)=\frac{n_b}{\pi^{1/2}p_{Tb}}\exp\left[-\frac{(p-p_b)^2}{p_{Tb}^2}\right].\tag{3.1}$$

Here p_b is the average beam-particle momentum, p_{Tb} is the "thermal" spread of the momenta in the beam. It is assumed that $p_b>p_{Tb}\gg mc$. It is also implied that the total plasma + beam system distribution function consists of two parts: $f=f_0+f_b$, where f_0 is the plasma distribution function. The plasma is assumed to be relativistic, $\bar{p}>mc$ ($\bar{p}$ is the average plasma-particle momentum), and such that $\bar{p}\ll p_b$.

Under these assumptions the dispersion equation of the longitudinal oscillations of the plasma + beam system takes the form

$$1+\varepsilon_0+\varepsilon_b=0.\tag{3.2}$$

Here ε_0 denotes the second term of the right-hand side of (2.2), and ε_b is given by

$$\begin{gathered}\varepsilon_b=\frac{2\omega_{pb}^2}{k^2c^2}\frac{\gamma_b^3}{\gamma_{Tb}^2}[1+i\pi^{1/2}xW(x)],\\ \omega_{pb}^2=4\pi e^2n_b/m,\quad \gamma_b=p_b/mc,\quad \gamma_{Tb}=p_{Tb}/mc,\\ x=\frac{(\omega-kV_b)\gamma_b^3}{|k|c\gamma_{Tb}},\quad V_b=c\left(1+\frac{1}{\gamma_b^2}\right)^{-1/2},\quad W(x)=e^{-x^2}\left(1+\frac{2i}{\pi^{1/2}}\int_0^x e^{t^2}dt\right)\end{gathered}\tag{3.3}$$

is the probability integral with complex argument, and n_b is the beam density.

Equations (3.2) and (3.3) are similar to the standard equations of the theory of two-stream instability of a nonrelativistic plasma (see Ref. 16, Sec. 3.1). Using this analogy, we conclude that the beam can be regarded as monoenergetic, and that the instability it excites is hydrodynamic if $x>1$, i.e., if

$$\gamma_{Tb}/\gamma_b<|\omega-kV_b|\gamma_b^2/kc\approx\Gamma_h\gamma_b^2/k_0c,\tag{3.4}$$

where Γ_h is the growth rate of the hydrodynamic instability. If the inequality in (3.4) is reversed, then it must be assumed that the beam has a large momentum spread and that the instability it excites is kinetic.

To express the condition (3.4) in terms of the equilibrium parameters of the plasma + beam system we must solve the dispersion Eq. (3.2) in the monoenergetic-beam approximation. In this approximation it follows from (3.3) that

$$\varepsilon_b = -\omega_{pb}^2/\gamma_b^3(\omega - kV_b)^2. \tag{3.5}$$

Assuming that $\omega = \omega_0 + \Delta\omega$, $k = k_0 + \Delta k$, where ω_0 and k_0 are defined in Sec. 2, and using (2.4), we reduce the dispersion Eq. (3.2) to the form

$$\Delta\omega - c(1-\alpha)\Delta k = (n_b/n_0)(k_0 c/2\bar{\gamma}\gamma_b)^3(\omega - kV_b)^{-2}. \tag{3.6}$$

Here $\bar{\gamma} = (\langle\gamma^3\rangle)^{1/3}$ and the angle brackets $\langle\cdots\rangle$ denote averaging over the plasma distribution functions; in other words, the quantity $\bar{\gamma}$ characterizes the average relativistic factor of the plasma particles. [At $f_0 \sim \exp(-c|p|/T)$ we have $\bar{\gamma} = 6^{1/3}\gamma_0$, $\gamma_0 = T/mc^2$, cf. Sec. 2.]

Next, in analogy with nonrelativistic-plasma theory (see Ref. 16, Sec. 1.5), we introduce the quantity $\tilde{\omega}$ defined by the relation

$$\tilde{\omega} = \Delta\omega - c(1-\alpha)\Delta k = \omega - kV_b. \tag{3.7}$$

From (3.6) we obtain for $\tilde{\omega}$ a cubic equation that yields the maximum growth rate of the hydrodynamic instability of a monoenergetic beam

$$\Gamma_h = \frac{3^{1/2}}{4}\left(\frac{n_b}{n_0}\right)^{1/3}\frac{k_0 c}{\bar{\gamma}\gamma_b}. \tag{3.8}$$

When (3.8) is taken into account, (3.4) means that

$$\frac{\gamma_{Tb}}{\gamma_b} < \left(\frac{n_b}{n_0}\right)^{1/3}\frac{\gamma_b}{\bar{\gamma}}. \tag{3.9}$$

This inequality is a generalization, to the case of a relativistic plasma ($\bar{\gamma} \gg 1$), of the known condition $\gamma_{Tb} < (n_b/n_0)^{1/3}\gamma_b$ under which a beam in a nonrelativistic plasma is monoenergetic.[18] It is seen from (3.9) that with increasing relativism of the plasma the condition for a monoenergetic beam becomes more stringent. According to (3.8), when the relativism increases the instability growth rate also decreases. Both effects can be regarded as a consequence of the increase, noted in Sec. 2, of the vibrational energy of the particles of the relativistic plasma relative to the electrostatic energy, $\omega\partial\varepsilon_0/\partial\omega \approx \bar{\gamma}^2 \gg 1$.

It is known from the theory of the interaction of a relativistic beam with a nonrelativistic plasma[18] that even a beam with a large thermal scatter can be regarded as monoenergetic if $(n_b/n_0)^{1/3}\gamma_b \gtrsim 1$. In the case of a relativistic plasma, as seen from (3.9), this inequality is replaced by the following:

$$(n_b/n_0)^{1/3}\gamma_b > \gamma_0. \tag{3.10}$$

We note also that from the second equality in (3.7) we obtain the following estimate for the value of $\Delta k/k_0$ that corresponds to the maximum of growth rate:

$$\Delta k/k_0 \approx (\bar{\gamma}/\gamma_b)^2. \tag{3.11}$$

By assumption $\bar{\gamma}/\gamma_b \ll 1$, so that the role of the resonant particles of the plasma at such $\Delta k/k_0$ is negligibly small.

We assume now that a condition inverse to (3.9) is satisfied, and consider the kinetic instability. The corresponding analysis can be carried out starting either from (3.2) and (3.3) or directly from the general expression for the kinetic growth rate (2.9). As a result we obtain for the maximum growth rate of the kinetic instability the expression

$$\Gamma_{\text{кин}} = \left(\frac{\pi}{2e}\right)^{1/2}\frac{n_b}{n_0\gamma_{Tb}^2}\left(\frac{\gamma_b}{2\bar{\gamma}}\right)^3 k_0 c. \tag{3.12}$$

It is seen that the relativism of the plasma influences the kinetic instability much more than the hydrodynamic instability, $\Gamma_{kin} \sim \gamma^{-5/2}$ (it is recognized that $k_0 \sim \bar{\gamma}^{1/2}$).

4. INSTABILITY OF LOW-ENERGY BEAM

Unlike in Sec. 3, we shall show here that longitudinal oscillations of a relativistic plasma can be excited also by a low-energy relativistic beam with $mc \ll p_b \ll \bar{p}$. We neglect the thermal scatter of the beam. Its permittivity then takes the form (3.3). Assuming ω_{pb}^2 to be a small parameter, we obtain from (3) by successive approximations

$$\omega = kV_b \pm \omega_{pb}(1 + \varepsilon_0(\omega, k)_{\omega = kV_b})^{-1/2}. \tag{4.1}$$

Taking into account our assumption $mc \ll p_b \ll \bar{p}$, we conclude that the quantity $\varepsilon_0(\omega, k)$ in (4.1) should be calculated for ω/k satisfying the condition

$$1/\bar{\gamma}^2 \ll 1 - \omega/kc \ll 1. \tag{4.2}$$

According to the statements made in Secs. 1 and 2, this is precisely the phase-velocity interval to which the calculations in the cited papers[4,5] pertain (see also below concerning Ref. 8). We use the results of these papers to demonstrate, without further calculations, the effect investigated by us.

1. Plasma with three-dimensional distribution function proposed in Ref. 5.

This function is

$$f_0(\mathbf{p}) = \frac{n_0}{8\pi}\left(\frac{c}{T}\right)^3 \exp\left(-\frac{c|\mathbf{p}|}{T}\right); \tag{4.3}$$

where $\mathbf{p}$ is the three-dimensional momentum and $p \gg mc$. With the aid of formula (13.9) of Ref. 5 we find that with f_0 so defined we have

$$\varepsilon_0(\omega, k)_{\omega = kV_b} = -\frac{1}{2k^2d^2}\left(\ln\frac{2}{1 - V_b/c} - 2 - i\pi\right); \tag{4.4}$$

$d^2 = T/4\pi e^2 n_0$. It follows from (4.1) and (44) that at small k the instability is of the hydrodynamic type, and at large k it is kinetic. The kinetic instability can be interpreted as a dissipative buildup of negative-energy waves (see Ref. 16, Sec. 2.3). The maximum growth rate is obtained at the junction of the hydrodynamic and the kinetic instabilities, namely at

$$k \approx \frac{1}{d}(\ln\gamma_b^2)^{1/2}. \tag{4.5}$$

In order of magnitude we have then

$$\Gamma \approx (\omega_{pb}/\gamma_b^{3/2})(\ln\gamma_b^2)^{1/2}. \tag{4.6}$$

Taking (4.5) and (4.6) into account, we find that the beam can be regarded as monoenergetic if

$$\gamma_{Tb}/\gamma_b < (n_b/n_0)^{1/2}\gamma_0^{1/2}\gamma_b^{1/2}(\ln\gamma_b^2)^{1/2}. \tag{4.7}$$

where $\gamma_0 = T/mc^2$. At a much larger thermal spread of the beam, the plasma + low-energy beam system is stable.

We note that in our approximation of large γ_0/γ_b the growth rate turns out to be independent of the degree of relativism of the plasma (of the parameter γ_0). This is due to the character of the dependence of the optimal wave number k on γ_0 [see (4.5)]. It is also of interest to note that the limiting thermal spread of an unstable beam increases with increasing γ_0 [see (4.7)], and does not decrease as in the case of the high-energy beam [cf. (3.9)].

2. Plasma with one-dimensional distribution function of the type (2.12)

As noted in Sec. 2, such a one-dimensional distribution function is established as a result of synchrotron radiation of the plasma with an isotropic δ-function distribution in momentum. If the momentum of the particles with such a distribution is large compared with the momentum of the beam particles, $p_0 \gg p_b$, then it can be assumed in the calculation of the permittivity $\varepsilon_0(\omega, k)_{\omega \approx k v_b}$ that $p_0 \to \infty$. It was in this approximation that $\varepsilon_0(\omega, k)$ was calculated by Suvorov and Chugunov.[8] Using Eq. (5) of that paper, we have

$$\varepsilon_0(\omega, k)_{\omega=kv_b} = -\frac{3\pi}{4}\left(\frac{\omega_p}{ck}\right)^2\left[1-2i\left(2\left(1-\frac{V_b}{c}\right)\right)^{1/2}\right]. \tag{4.8}$$

With the aid of (4.1) and (4.8) we find that the maximum instability growth rate is reached at $k=(3\pi)^{1/2}\omega_p/2c$, and in this case it is of the order of $\Gamma \approx \omega_{pb}/\gamma_b$. The condition that the thermal scatter of the beam be small takes the form

$$\gamma_{Tb}/\gamma_b < (n_b/n_0)^{1/2}\gamma_b. \tag{4.9}$$

This does not contain the relativistic factor of the plasma, a natural result, since the distribution (2.12) corresponds to a moderately relativistic plasma $\langle\gamma\rangle \approx 1$, although with a long ultrarelativistic tail.

5. QUASILINEAR RELAXATION OF HIGH-ENERGY BEAM IN AN ULTRARELATIVISTIC PLASMA

We consider now the kinetic stage of quasilinear relaxation of a high-energy beam. By the standard method (see Refs. 1, 18, 19) we obtain the initial system of quasilinear equations

$$\frac{\partial f}{\partial t} = \frac{\pi e^2}{\alpha c}\frac{\partial}{\partial p}|E_k|^2_{k=k(p)}\frac{\partial f}{\partial p}, \tag{5.1}$$

$$\frac{\partial W}{\partial t} = \frac{\pi e^2 mc}{k_0}\left(\gamma^3\frac{\partial f}{\partial p}\right)_{p=p(k)}|E_k|^2. \tag{5.2}$$

The oscillation energy W is connected with the electric field by formula (2.11), and the relation between the resonant wave number and the resonant momentum is $\Delta k(p)=k_0/2\gamma^2$.

In analogy with Ref. 1, we get from (5.1) and (5.2) an equation for the quasilinear integral

$$\frac{\partial}{\partial t}\left[f - \frac{\partial}{\partial p}\frac{k_0 W m^2 c}{\alpha p^3}\right] = 0. \tag{5.3}$$

From this we find that the quasilinear relaxation causes the beam distribution function to acquire a plateau in the interval $p_{\min} \lesssim p \lesssim p_b$, where $p_{\min}$ is the scale of the average plasma particle momentum. The distribution of oscillation energy over the wave numbers takes in this case the form

$$W(\Delta k) = \begin{cases} \dfrac{n_b mc^2 k_0}{4\alpha(\gamma_b-\gamma_{\min})(\Delta k)^2}, & \Delta k(p_b) < \Delta k < \Delta k(p_{\min}) \\ 0, & \Delta k > \Delta k(p_{\min}),\ \Delta k < \Delta k(p_b) \end{cases}. \tag{5.4}$$
$$\gamma_{\min} = p_{\min}/mc.$$

Just as in the case of a nonrelativistic plasma, in the case of relaxation the beam loses half its initial energy.[18]

Following Ref. 1, we consider also the dynamics of the quasilinear relaxation of the beam. The distribution function of the beam is then represented in the form

$$f(p,t) = \begin{cases} n_b/(p_b - p_*(t)), & p_*(t) < p < p_b \\ 0, & p < p_*(t) \end{cases}. \tag{5.5}$$

For $p_*(t)$ we get the equation

$$dX/dt = -AX^3/(1-X);$$
$$X = \frac{p_*(t)}{p_b},\quad A = \frac{\pi}{2}\frac{n_b}{n_0}\frac{\omega_p^2\alpha\gamma_b}{\Lambda k_0 c},\quad \Lambda = \ln\frac{W(p_*+0)}{W(p_*-0)} \tag{5.6}$$

is a quantity of the order of the Coulomb logarithm.[1] From (5.6) it follows that

$$p_*(t) = p_b/[1+(2At)^{1/2}]. \tag{5.7}$$

From this we get the time at which the width of the plateau reaches half its final value ($p_* = p_b/2$):

$$t_{1/2} = \frac{1}{\pi}\frac{n_0}{n_b}\frac{\Lambda k_0 c}{\omega_p^2\alpha\gamma_b}. \tag{5.8}$$

For a plasma with a momentum distribution function in the form $f \sim \exp(-c|p|/T)$ we have $\alpha = \frac{1}{12}\gamma_0^2$, $k_0 c = \omega_p \gamma_0^{1/2}$, where $\gamma_0 = T/mc^2$. In this case

$$t_{1/2} = \frac{12\Lambda}{\pi}\frac{n_0}{n_b}\frac{\gamma_0^{5/2}}{\gamma_b}\frac{1}{\omega_p}. \tag{5.9}$$

We see that the relaxation time increases with increasing plasma relativism like $\gamma_0^{5/2}$.

We can consider in similar fashion spatial rather than the temporal relaxation of the beam.[1] However, since the group velocity of the oscillations is close to the light velocity, it suffices for this purpose to make the substitution $\partial/\partial t \to c\partial/\partial z$. Therefore,

$$l_{1/2} = ct_{1/2} \tag{5.10}$$

is the distance over which the beam loses half its energy.

6. DISCUSSION OF RESULTS

We have demonstrated that in a relativistic plasma there can exist longitudinal waves with phase velocities smaller than the phase velocity of light, and have established that such waves can be exicted via Cerenkov resonance by beams in which the particle energy is large compared with the plasma-particle energy. In addition, in accordance with our analysis, the longitudinal waves can be excited by beams of low-energy particles. By the same token we have shown that the same variants of two-stream instability exist in a relativistic plasma as in a nonrelativistic plasma.

Besides the linear theory of the two-stream instabilities of a relativistic plasma, we have analyzed the quasilinear relaxation that accompanies the development of such instabilities, and obtained quantitative characteristics of this process.

It follows from our analysis that the use of the two-stream instability concepts to interpret the radioemission from pulsars does not contradict the assumption that the near-pulsar plasma is relativistic. Investigations in this direction were performed by a number of workers.[10-15] It must be noted, however, that in these studies the relativism was taken into account by using rather crude and not always consistent simplifying assumptions. This raised objections (see Ref. 3). The approach described above makes it possible to improve the earlier analysis. The development of a consistent theory of collective processes in a relativistic plasma would be useful both for the interpretation of radiation from pulsars and other astrophysical problems, and for laboratory applications.

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[1)] It is assumed, however, that the distribution function decreases rapidly enough at momenta much larger than the average; for details see below.

[2)] In the derivation of (2.4) we take into account the one-dimensional character of the particle momentum distribution. For this reason, expressions (2.4) cannot be obtained from the analogous formulas (38) and (39) of Ref. 6, which pertain to an isotropic distribution.

[3)] The need for introducing a finite p_0 is clear also from the Maxwellian-distribution example considered in Ref. 17.

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Nonlinear effects in the excitation of a magnetic field in a strongly ionized plasma

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We consider the nonlinear dynamics of the magnetic instability of a one-dimensionally inhomogeneous plasma. It is shown that the principal nonlinear effect that limits the growth of the magnetic perturbations is the dissipation of the temperature inhomogeneity of the plasma. An estimate of the maximum value of the exciting magnetic field is obtained.

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1. It is known that to obtain a high temperature dense plasma with the aid of powerful photon beams or charged particles, the optimal procedure is to produce adiabatically spherically symmetrical heating. Therefore serious misgivings were expressed when spontaneous excitation of a magnetic field, reaching 10^4–10^6 G, was observed in experiments with laser plasma.[1-5] The appearance of strong magnetic fields can lead to asymmetry of the heat flux and affect most adversely the regime of compression and heating of the plasma.

Several assumptions were made concerning the cause of excitation of the magnetic field. It is shown in Refs. 6–8 that the magnetic field can be produced in the re-

