to quantities of the same order (we have assumed that $U \approx 4$). If $\beta' a/\beta = 0.1$, then we find on the basis of (25) that $b = 0.015$–0.013, and from (20) it follows that $b \approx 0.1$. An estimate of b from the relation $b = (\beta')^2/\beta M\omega^2$ using the TCNQ mass and Debye temperature $\Theta_D = 100$ K leads to a value $b \approx 10^{-2}$.

The foregoing analysis of expression (14) and (15) leads also to the conclusion that at medium values of the electron-interaction parameter U a one-dimensional system with Hamiltonian (1) is unstable to periodic deformations of the type (2) with $Q = 2k_F$ and $Q = 4k_F$. In experiments on x-ray scattering by the quasi-one-dimensional crystal TTF-TCNQ,[18] characteristic symptoms of instability to both deformations were observed, and the x-ray scattering amplitudes were comparable in magnitude for $Q = 2k_F$ and $Q = 4k_F$. In light of our analysis this means that the effective interelectron interaction in this crystal is of the order of (or larger than) the width of the conduction band (0.5 eV, Ref. 29).

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Multiphoton boundary of the excitation spectrum in He II

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The behavior of the dynamic form factor in He II near the sound line $\omega = up$ is obtained. General formulas are derived for the asymptotic forms of the imaginary parts of the Green's function at high frequencies and large momenta.

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1. FORMULATION OF PROBLEM. KINEMATIC RELATIONS

The purpose of the present study was to investigate the decay of elementary excitations into phonons in superfluid helium in the case when the conservation laws permit such a decay only simultaneously into a large number of phonons.

It is known (see, e.g., Ref. 1) that at small momen-

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ta the excitation spectrum in helium takes the form shown schematically in Fig. 1. At normal pressure the spectrum curve $\varepsilon = \varepsilon(p)$ initially deviates from the "sound line"

$$\varepsilon = up \tag{1.1}$$

(u is the speed of sound) upwards, so that the spectrum is described by the equation

$$\varepsilon = up + \gamma p^3, \quad \gamma > 0. \tag{1.2}$$

The group velocity of the excitations then decreases again, and at a certain $p = p^*$ the spectrum crosses the sound line and goes under it. This form of the spectrum leads to a peculiar behavior of the damping of the excitations.

We deal throughout this paper only with the case of absolute zero temperature. At $T = 0$ the damping is due only to the decay of an excitation into a large number of excitations. This process conserves, of course, the total energy and momentum of the excitations.

At sufficiently small momenta, a decay in the spectrum of Fig. 1 into is possible two excitations satisfying the conservation law[1)]

$$\varepsilon(\mathbf{p}) = \varepsilon(\mathbf{k}) + \varepsilon(\mathbf{p}-\mathbf{k}), \tag{1.3}$$

where $\mathbf{p}$ is the momentum of decaying excitation, and $\mathbf{k}$ and $\mathbf{p}-\mathbf{k}$ are the momenta of the excitations produced in the decay. The process (1.3) leads to damping of the excitation in accordance with the law[2]

$$\Gamma = -\operatorname{Im}\varepsilon \sim p^5.$$

Decays into three and more excitations are possible simultaneously with (1.3).

With further increase of p, the damping for the spectrum of Fig. 1 begins to decrease. At a certain point $p = p_2$ the relation (1.3), if regarded as an equation for $\mathbf{k}$, ceases to have a solution, and two-particle decay becomes impossible. It is easily understood that at the point p_2 itself the two produced excitations have equal but opposite momenta, so that p_2 is determined from the relation

$$\varepsilon(p_2) = 2\varepsilon(p_2/2).$$

At $p > p_2$, decay into three and more excitations is possible. Three-particle decay stops, in turn, at the point p_3, $\varepsilon(p_3) = 3\varepsilon(p_3/3)$. Similarly, decay into n excitations is possible at $p < p_n$, where

$$\varepsilon(p_n) = n\varepsilon(p_n/n). \tag{1.4}$$

There is thus a sequence of thresholds p_n which condense towards the point $p_\infty = p^*$. In fact, going to the limit as $n \to \infty$ in (1.4), we get

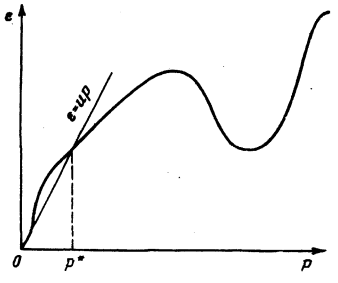

FIG. 1.

$$\varepsilon(p_n)_{n\to\infty} \approx n\frac{p_n}{n}\frac{d\varepsilon}{dp}\Big|_{p=0} = up_\infty,$$

so that p_∞ coincides in fact with the point p^* at which the curve of the spectrum crosses the sound line. At $p > p^*$ no decay of the excitation into any number of phonons is possible. Therefore as $p \to p^*$ the damping of the excitations vanishes. The determination of the law that governs this vanishing is in fact the purpose of the present paper. [Formula (1.4) was derived in Ref. 2.]

The posed problem is a particular case of a more general one. Assume that when a neutron is inelastically scattered in helium the liquid acquires an energy ε and a momentum p. This process (with energy and momentum transferred to the phonons and not, say, to rotons) is impossible if the point (ε, p) lies below the sound line. On the other hand, if the point lies above the sound line, then the number of produced phonons should be larger the closer the sound line, i.e., the smaller the difference $\delta\varepsilon = \varepsilon - up$.

The probability of scattering of a neutron with given ε and p is determined by the dynamic form factor of the liquid (see, e.g., Ref. 4, Sec. 86). According to the foregoing, this form factor should vanish as $\delta\varepsilon \to 0$ and we can raise the question of determining the law that governs this vanishing on the entire line $\varepsilon = up$. This is precisely the general problem to be solved here.[2)]

We determine first the minimum number n of the phonons among which the energy ε and the momentum p can be distributed. It is again clear beforehand that the most favorable situation is when the phonon momenta are almost equal in magnitude and in direction—in this case the total momentum at a given energy is maximal. Let n be the number of produced phonons and let ω and $\mathbf{k}$ be the energy and momentum of each of them. Then $\omega = \varepsilon/n$, $\mathbf{k} = \mathbf{p}/n$, and since ω and k are connected by Eq. (1.2) we get

$$\frac{\varepsilon}{n} = u\frac{p}{n} + \gamma\left(\frac{p}{n}\right)^3$$

or

$$n = p^{3/2}\gamma^{1/2}/(\delta\varepsilon)^{1/2}, \quad \gamma > 0, \tag{1.5}$$

so that actually $n \to \infty$ when $\delta\varepsilon \to 0$. The assumption that $n \gg 1$ serves as a basis of our entire analysis.

So far we have referred specifically to the situation in liquid helium. We shall bear this case in mind in the entire exposition that follows. It is clear, however, that the question is meaningful also when applied to the phonon spectrum of the crystal, provided the spectrum takes the form shown in Fig. 1. The solution method described is also suitable in the case of a crystal, although the actual expressions for the probability can be altered when the anisotropy is taken into account.

2. LEADING PREMISES. PRESCRIPTION FOR OBTAINING THE ANSWER

The expression for the probability of two-phonon decay (the p^5 law) can be easily obtained in first-order approximation of perturbation theory in third-order anharmonicity. To obtain the probability of n-phonon decay, it is necessary, however, to consider the nth

approximation of perturbation theory in this anharmonicity, or else take into account anharmonicities of higher order. It is in fact impossible to obtain the answer for large n. The number of different Feynman diagrams turns out to be very large, and the asymptotic and alternating-sign character of the series makes it difficult to estimate its sum.[3)]

It is clear beforehand, however, that at large n the situation is quasiclassical and the corresponding approximation can be used. In fact, the calculations, as will be shown below, turn out to be quite simple. The solution method and the character of the employed approximations are easiest to understand by using as an example the solution of a simpler problem with one degree of freedom. We have in mind the calculation of the probability of excitation of an oscillator with small anharmonicity by a high-frequency field.

We consider an oscillator with a potential energy

$$U(x)=\tfrac{1}{2}m\omega_0^2x^2+gx^3. \tag{2.1}$$

The probability of its excitation by a weak external field, corresponding to a contribution to the potential energy

$$V=-Fx(e^{-i\varepsilon t}+\text{c.c.}),$$

takes the form (in the approximation quadratic in the field)

$$w=2\pi\delta(\varepsilon-\varepsilon_n+\varepsilon_0)|M_{n0}|^2F^2, \quad M_{n0}=-\int\psi_n^*(x)x\psi(x)\,dx. \tag{2.2}$$

We shall assume that the frequency of the field ε is much higher than the natural frequency of the oscillator ω_0:

$$\varepsilon\approx(\tfrac{1}{2}+n)\omega_0, \quad n\gg1. \tag{2.3}$$

A nonzero value of the matrix element M_{n0} can then be obtained only in nth order of perturbation theory in the "interaction constant" g. In this sense, there is a far-reaching analogy between our fundamental problem and the considered auxiliary problem. Under the condition (2.3) the matrix element (2.2) is exponentially small and its calculation calls for the use of a procedure due to Landau (see Ref. 6, Sec. 51). According to Landau

$$M_{0n}\sim\psi_0(x_0)\psi_n^+(x_0),$$

where

$$\psi_0\sim\exp\Big(\int^{x}[2m(U(x)-\varepsilon_0)]^{1/2}dx\Big),$$

$$\psi_n^+\sim\exp\Big(-\int^{x}[2m(U(x)-\varepsilon_n)]^{1/2}dx\Big),$$

and x_0 is a singular point of $U(x)$ in the upper half of the complex x plane, and it is necessary to choose from among the singular points the one that makes the largest contribution to M_{0n}.

For the potential energy (2.1), which has no singular point, we must put $x_0\to\infty$. As a result we have with exponential accuracy (the phase factors have been left out)

$$M\sim\exp\Big\{\int_0^{\infty}([2m(U(x)-\varepsilon)]^{1/2}-[2mU(x)]^{1/2})\,dx\Big\}. \tag{2.4}$$

We assume the anharmonicity constant g to be small in the sense that $\varepsilon\ll\varepsilon_g$, where $\varepsilon_g\sim m^3\omega_0^6g^{-2}$ is that energy value at which the anharmonicity effects become of the order of unity (see Fig. 2). In this case the essential region of x in the integral of (2.4) is the one in which

$$\varepsilon\ll U(x)\ll\varepsilon_g. \tag{2.5}$$

Expanding by virtue of the left-hand inequality in powers of ε, we reduce (2.4) to the form

$$M\sim\exp(-\varepsilon\tau), \quad \tau=\int^{\infty}\Big[\frac{m}{2U(x)}\Big]^{1/2}dx. \tag{2.6}$$

This formula can be interpreted more illustratively by recognizing that in the classically forbidden region (2.5) the coordinate of the particle is real, and the momentum is imaginary: $p=i[(U(x)-\varepsilon)2m]^{1/2}$, a fact that can be understood as motion along the imaginary time axis. Then τ is the imaginary time necessary for the oscillator to go off to infinity. Next, by virtue of the first inequality of (2.5), we can neglect the anharmonicity in the essential region, i.e., we can simply put $U=\frac{1}{2}m\omega_0^2x^2$, and the logarithmically diverging integral can be cut off from below at the value $x_1\sim(\varepsilon/m\omega_0^2)^{1/2}$ at which the left-hand inequality is violated, and from above at the value $x_2\sim m\omega_0^2/g$ at which the right-hand inequality is violated. As a result we have with logarithmic accuracy

$$\tau\approx\frac{1}{\omega_0}\ln\frac{x_2}{x_1}=\frac{1}{\omega_0}\ln\Big|\frac{(m\omega_0^2)^{3/2}}{\varepsilon^{1/2}g}\Big|,$$

and finally we obtain for the excitation probability

$$w\sim|M_{n0}|^2\sim\exp\Big\{-\frac{2\varepsilon}{\omega_0}\ln\frac{\varepsilon_g}{\varepsilon}\Big\}=\frac{\varepsilon^ng^{2n}}{(m\omega_0^2)^{3n}}. \tag{2.7}$$

We emphasize that the relative simplicity of the calculations is essentially connected with the process of the large logarithm in τ or in the formula (2.7). At the same time, it is clear from (2.7) that the presence of this logarithm ensures a correct ($\sim g^{2n}$) dependence of the probability on the coupling constant g corresponding to the n-th order of perturbation theory. A logarithm of this kind is therefore a characteristic of a large group of problems with weak anharmonicity.

Proceeding to solve our principal problem concerning the simultaneous emission of n phonons, we show first that a formula of type (2.6) for the matrix element is valid for a system with many degrees of freedom.

Let our system be described by the coordinate x_i. Then the quasiclassical wave function of a state with energy ε is

$$\psi\sim\exp[iS_0(x,\varepsilon)],$$

where $S_0(x_i,\varepsilon)$ is the "abbreviated" action connected with the total action $S(x_i,t)$ by the relation

$$S(x_i,t)=S_0(x,\varepsilon)-\varepsilon t. \tag{2.8}$$

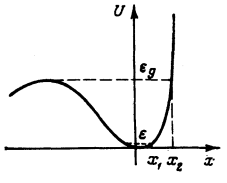

FIG. 2.

Landau's reasoning shows then that the matrix element of the transition from the ground state is

$$M \sim \exp\{i[S_0(x_i^0, \varepsilon) - S_0(x_i^0, 0)]\} \approx \exp\left[i\varepsilon \frac{\partial S_0(x_i^0)}{\partial \varepsilon}\Big|_{\varepsilon=0}\right].$$

It is easy to understand that the derivative $\partial S_0/\partial \varepsilon = t(x_i^0)$, where $t(x_i^0)$ is the time necessary for the system to reach the point x_i^0 (from a certain point in the classically allowed region). In fact, from (2.8) and from the identity for $S(x_i, t)$

$$dS = p_i dx_i - \varepsilon dt$$

it follows that

$$dS_0 = p_i dx_i + t d\varepsilon, \quad t = (\partial S_0/\partial \varepsilon)_{x_i}.$$

As to the values of x_i^0 themselves, in problems with weak anharmonicity, similar to that considered above, in which the time is logarithmically large, x_i^0 must be taken to mean the coordinate value at which the anharmonicity becomes significant. In these problems, the values of x_i^0 lie in the classically forbidden region.[4)] Therefore the time t is imaginary: $t = i\tau(x_i^0)$, so that the final estimate of the matrix element is

$$M \sim \exp(-\varepsilon\tau), \tag{2.9}$$

which agrees with (2.6).

We use formula (2.9) to solve the problem posed in Sec. 1—to calculate the probability of phonon production by a neutron near the sound line. The coordinates describing our system must be chosen to be the change of the density of the liquid at each point of space

$$\rho'(\mathbf{x}) = \rho(\mathbf{x}) - \bar{\rho}$$

or of the Fourier component ρ_k of this change. The role of the momenta will then be assumed by the Fourier components of the velocity potential of the liquid (see Ref. 4, Sec. 24). The interaction of the liquid with the neutron is described by an interaction potential in the form

$$V = \frac{2\pi a}{m\mu}\rho'(\mathbf{x})$$

(a is the length for neutron scattering by the atom, m is the mass of the atom, μ is the reduced mass of the atom and neutron). The problem reduces therefore to a calculation of the matrix element of the Fourier component of the density with wave vector $\mathbf{p}$ between the ground state and a state with energy ε and momentum $\mathbf{p}$:

$$M \sim \langle \varepsilon | \rho_{\mathbf{p}} | 0 \rangle. \tag{2.10}$$

The character of the final state ε is clear beforehand from physical considerations. Since the phonons are produced, as explained in Sec. 1, with practically equal momenta, this state represents in momentum space a narrow packet near the momentum value

$$\mathbf{k}_0 = \mathbf{p}/n, \quad n = p^{3/2}\gamma^{1/2}/\delta\varepsilon^{1/2}.$$

The packet should be such that its total energy is equal to ε, and the momentum is $\mathbf{p}$. (In the oscillator problem this normalization condition corresponds to cutting off the integral at an energy dependent value of x. We shall show below that the dependent of τ on the energy is logarithmic.)

The answer is now given directly by formula (2.9), where τ must be taken to mean the imaginary time necessary for the density perturbation in the packet to reach the unperturbed density $\bar{\rho}$. It is clear that when $\rho' \sim \bar{\rho}$ the anharmonicity effects become of the order of unity. The shape of the packet must be chosen such that the time τ is minimal.

We proceed now to perform the described program. We write the Fourier components of the density in the packet in the form

$$\rho_k = A f\left(\frac{k_\parallel - k_0}{\Delta_\parallel}, \frac{k_\perp}{\Delta_\perp}\right), \tag{2.11}$$

where $k_0 = \mathbf{p}/n$ is the average momentum of the phonon in the packet, the function $f(\alpha, \beta)$ is normalized by the condition $f(0,0) = 1$, A is a normalization constant, $\Delta_\parallel$ and $\Delta_\perp$ are respectively the widths of the packet along and across $\mathbf{p}$, while $k_\parallel$ and $k_\perp$ are the corresponding projections of the wave vector. At the accuracy with which we can solve the problem within the framework of our method, we need know only the dependences of A, $\Delta_\parallel$, and $\Delta_\perp$ on the symbolic parameters of the problem, i.e., in order of magnitude. In the calculations that follow we shall therefore omit the numerical coefficients. We establish first the connection between A and $\Delta_\parallel$ or $\Delta_\perp$. To this end, we satisfy the energy normalization condition, and it is sufficient to calculate the packet energy in the harmonic approximation

$$\varepsilon_p = \int \left(\frac{\bar{\rho}}{2} v^2 + \frac{u^2}{2\bar{\rho}}\rho'^2\right) d\mathbf{r} = \frac{u^2}{\bar{\rho}} \int |\rho_k|^2 \frac{d\mathbf{k}}{(2\pi)^3} = \varepsilon. \tag{2.12}$$

An estimate of the last integral yields $\varepsilon_p \sim (u^2/\bar{\rho}^2) A^2 \Delta_\parallel \Delta_\perp^2$, so that $A \sim (\bar{\rho}\varepsilon/u^2\Delta_\parallel\Delta_\perp^2)^{1/2}$ and

$$\rho_k \sim \frac{\bar{\rho}^{-1/2}}{u}\varepsilon^{1/2}\Delta_\parallel^{-1/2}\Delta_\perp^{-1} f\left(\frac{k_\parallel - k_0}{\Delta_\parallel}, \frac{k_\perp}{\Delta_\perp}\right).$$

The time dependence of the Fourier components is given by the factors $\rho_k \sim \exp(i\omega_k t) \sim e^{\omega_k \tau}$, so that the density of the liquid at the center of the packet depends on τ like

$$\rho(r=0) \sim \frac{\bar{\rho}^{1/2}}{u}\varepsilon^{1/2}\Delta_\parallel^{1/2}\Delta_\perp \exp(\omega_{k_0}\tau), \quad \omega_{k_0} = \frac{\varepsilon}{n}. \tag{2.13}$$

Therefore the time at which $\rho(\bar{r}=0) \sim \bar{\rho}$ is

$$\tau \approx \frac{n}{\varepsilon} \ln \frac{u\bar{\rho}^{1/2}}{\varepsilon^{1/2}\Delta_\parallel^{1/2}\Delta_\perp}. \tag{2.14}$$

It is seen from (2.14) that to decrease τ it is more convenient to make the packet not too narrow, i.e., to make $\Delta_\parallel$ and $\Delta_\perp$ not too small. On the other hand, an excessive increase of the width will obviously violate the momentum normalization. We have in analogy with (2.12)

$$\mathbf{p} = \frac{u^2}{\bar{\rho}} \int \frac{\mathbf{k}}{\omega(k)} |\varphi_k|^2 \frac{d\mathbf{k}}{(2\pi)^3}.$$

At small $k_\parallel - k_0$ and $k_\perp$ we get

$$\frac{k_\parallel}{\omega(k)} \approx \frac{1}{u}\left(1 - \frac{k_\perp^2}{k_0^2} - \frac{\gamma}{u}k_\parallel^2\right),$$

so that for the packet we have

$$\frac{\varepsilon_p - p_p u}{\varepsilon_p} \sim \frac{\gamma k_0^2}{u} + \frac{\gamma}{u}\Delta_\parallel^2 + \frac{\Delta_\perp^2}{k_0^2}. \tag{2.15}$$

If the last two terms (2.15) are absent, then, in accordance with the definition of k_0, the deviation from the sound line $\varepsilon_p - p_p u$ for the packet is automatically equal to the correct value of $\delta\varepsilon$. This means that these terms should at any rate be small compared with the first, i.e., it is necessary to have $\Delta_\parallel^2 \ll u\delta\varepsilon/\gamma\varepsilon$, $\Delta_\perp^2 \ll (\delta\varepsilon/\varepsilon)^2 u/\gamma$. But since the dependence of τ on $\Delta_\parallel$ and $\Delta_\perp$ is according to (2.13) logarithmic, we can replace, with logarithmic accuracy, the symbol $\ll$ in these inequalities by $\sim$, i.e., we can assume that

$$\Delta_\parallel \sim \left(\frac{u}{\gamma}\right)^{1/2}\left(\frac{\delta\varepsilon}{\varepsilon}\right)^{1/2}, \quad \Delta_\perp \sim \left(\frac{u}{\gamma}\right)^{1/2}\frac{\delta\varepsilon}{\varepsilon}. \tag{2.16}$$

Substituting (2.16) in (2.14), we obtain in the usual units:

$$\tau \approx \frac{n}{\varepsilon}\ln\left(\frac{\hbar^3 p^{11/2} u^4}{\varepsilon^3 \gamma^{1/2}} n^{1/3}\right), \tag{2.17}$$

and substituting this formula in (2.9) we obtain ultimately for the emission probability (i.e., for the dynamic form factor):

$$w \sim e^{-2\varepsilon\tau} \sim \left(\frac{\varepsilon^6\gamma}{\hbar^3 p u^8}\right)^n \exp(-5n\ln n), \tag{2.18}$$

with the connection between n and $\delta\varepsilon$ given by (1.5). As explained above, the presence of a large logarithm in (2.17) serves as a justification of the entire employed procedure. We emphasize that formulas (2.17) and (2.18) were obtained with logarithmic accuracy, i.e., the numerical coefficient under the logarithm sign is indeterminate.

The same formula (2.18) determines also the probability of the decay of the excitation near the point p^*. In this case $\delta\varepsilon$ in (1.5) should be taken to mean

$$\delta\varepsilon = (v^* - u)\Delta p,$$

where v^* is the velocity of the decaying excitation at the point p^*, and $\Delta p = p - p^*$.

The assumption that the damping of the excitation vanishes exponentially as $\Delta p \to 0$ was advanced in Ref. 3. Because of the presence of the logarithm in (2.17), this vanishing turns out to be even faster.

We might ask whether expression (2.18) for the probability should be multiplied by certain statistical weight corresponding to the system of phonons of given energy and momentum. Actually, however, there is no such factor. The wave packet (2.11) already has the correct normalization. On the other hand, inclusion of the degrees of freedom that are connected with its form, at the given width and length, would obviously be an exaggeration of the accuracy in the present approximation.

We note that according to (2.16) the transverse width of the packet in p-space is much smaller than its length. The reason is that in the absence of dispersion the smearing of the packet along p does not lead at all to a shift of the corresponding point on the $(\varepsilon, \mathbf{p})$ plane away from the sound line. Yet the transverse "smearing" decreases momentum at a given energy even in the absence of dispersion, and therefore is more dangerous. In r-space the packet, on the contrary, is a "pancake" which is more oblate is the direction of motion the smaller $\delta\varepsilon$. A similar form is possessed, according to Ref. 7, by a soliton propagating in a liquid.

3. IMAGINARY PART OF GREEN'S FUNCTION AT LARGE ω AND THE CONTINUAL INTEGRAL

In this section we obtain in a somewhat more formal manner the results of the preceding section. We shall see that there exists a special asymptotic form of the imaginary part of the Green's function at high frequencies, and that a large part of our reasoning is quite general in character and should be valid for any field.

We shall use the representation of the causal Green's function in terms of a continual integral[8] in order to obtain the corresponding generalization of the Landau formula. Some general derivations will be made for the Bose field $\rho(\mathbf{x}, t)$, which acts as the generalized coordinate, and for the canonically conjugate momentum $\varphi(\mathbf{x}, t)$ using the Hamiltonian formalism with a certain Hamiltonian $H(\rho, \varphi)$ which is a specified functional of φ and ρ. The system will be assumed translationally invariant.

The single-particle Green's function for a Bose field ρ at zero temperature is a continual integral over all the fields:

$$\mathcal{G}(\mathbf{p}, \omega) = -i\int e^{i\omega t} e^{-i\mathbf{p}\mathbf{x}} e^{iS}\rho(\mathbf{x}, t)\rho(0,0)D(\rho)D(\varphi)\,dt\,d\mathbf{x}\left[\int e^{iS}D(\rho)D(\varphi)\right]^{-1}, \tag{3.1}$$

where the action S takes in the Hamiltonian representation the form

$$S = \int \varphi\frac{\partial\rho}{\partial t}d\mathbf{x}dt - \int H(\rho, \varphi)\,dt. \tag{3.2}$$

We assume that the frequency ω and the momentum p are large quantities (much larger than the characteristic frequencies and momenta of the problem), and use the saddle-point method to calculate the asymptotic form of the numerator of (3.1). We are interested here only in the imaginary part of the Green's function and do not consider terms proportional to powers of $1/\omega$ and corresponding to the known Green's-function asymptotic form connected with its discontinuous character at $t = 0$, and which enter only in its real part.

We break up the integration with respect to t into two parts: Re $t > 0$ and Re $t < 0$. Both parts make equal contributions, and we consider for the sake of argument Re $t > 0$. We break up the functional integration over the field $\varphi(\mathbf{x}, t'), \rho(\mathbf{x}, t')$ into three parts:

I. $t' \in [-\infty, 0]$, II. $t' \in [0, t]$, III. $t' \in [t, \infty]$.

In accordance with the saddle-point method, the variation of the exponential under the integral sign with respect to any quantity should yield zero, and this in fact determines the saddle-point. The result of variation of S with respect to φ and ρ in each of the regions I, II, and III leads to the Hamilton equations of motion, which determine certain classical trajectories. We stipulate, in accordance with the character of the saddle-point, the fields at $t \to \pm\infty$ vanish, meaning cessation of the oscillations of e^{iS}. As a result, the conserved Hamilton function H is equal to zero on the sections of the extre-

mal trajectories I and III adjacent to $t=\pm\infty$ (we choose this reference from which to measure the field energy).

Since ω is a large quantity, it is necessary also to vary with respect to t:

$$i\omega+i\frac{\partial S_{II}}{\partial t}+i\frac{\partial S_{III}}{\partial t}=0.$$

Since $\partial S_{III}/\partial t=H_{III}=0$, we obtain as a result

$$H_{II}=\omega, \tag{3.3}$$

i.e., the energy of the field on section II of the extremal trajectory is equal to ω, a natural physical result. Sections I and III correspond to the classically forbidden region of motion, since $E=H=0$, and the classically allowed region corresponds to the equilibrium position (point in functional space). This corresponds to the asymptotic form of the wave function of the ground state (see, e.g., Ref. 9). The functional integration on section II corresponds to the wave function of the excited states, and if $\omega\gg\omega_i$, where ω_i is the energy of an individual elementary excitation, then these states are quasiclassical.

To find the saddle-point we must carry out variation with respect to the remaining arbitrary quantities $\rho(0,0)$ and $\rho(\mathbf{x},t)$. We carry out a variation of the displacement type

$$\rho(x,t)\to\rho_0(\mathbf{x}+\mathbf{a},t),\quad \rho(x,0)\to\rho_0(\mathbf{x}+\mathbf{b},0),$$

where $\rho_0(\mathbf{x},t)$ is the extremal function, and $\mathbf{a}$ and $\mathbf{b}$ are arbitrary infinitesimally small constants. Using the spatial homogeneity and the properties of the Fourier transformations of $\rho(\mathbf{x},t)$, we find that the integrand is multiplied by

$$\exp(-i\mathbf{pa}+i\mathbf{pb}+i\delta S),$$

where

$$\delta S=\left(\frac{\partial S_{II}}{\partial\mathbf{a}}+\frac{\partial S_{III}}{\partial\mathbf{a}}\right)_{\mathbf{a}=0}\mathbf{a}+\left(\frac{\partial S_{I}}{\partial\mathbf{b}}+\frac{\partial S_{II}}{\partial\mathbf{b}}\right)_{\mathbf{b}=0}\mathbf{b}.$$

According to the Noether theorem we have $\partial S_I/\partial\mathbf{b}|_{\mathbf{b}=0}=\mathbf{P}_I$, where the field momentum

$$\mathbf{P}=-\int\varphi\nabla\rho d\mathbf{x} \tag{3.4}$$

is an integral of the motion. For the trajectories that enter in the point $\varphi=\rho=0$ at $t=\pm\infty$ we have $\mathbf{P}_I=\mathbf{P}_{III}=0$. From the conditions that the variations with respect to $\mathbf{a}$ and $\mathbf{b}$ vanish, we obtain

$$\mathbf{P}_{II}=\mathbf{p}. \tag{3.5}$$

i.e., the extremal trajectory on the section $[t,0]$ should have a classical field momentum equal to the momentum of the calculated Green's function.

Thus, the classically allowed motion has an energy $E=\omega$ and a momentum $\mathbf{P}=\mathbf{p}$. The conditions for joining together the trajectories at $t'=0$, t require, as usual, equality of the coordinates $\rho(\mathbf{x},t')$ and, in addition, equality of the generalized momenta $\varphi(\mathbf{x},t')$ (this corresponds to an extremum of the action with respect to the coordinates of the junction points).

Since the energies of the joined trajectories are different, the last condition cannot be satisfied at any value of $\rho(\mathbf{x},t')|_{t'=0,t}$, if only this $\rho(\mathbf{x},t')$ is not a singular point of the equation of motion. For the functional $H(\rho,\varphi)$, which is an entire function of its arguments, the only such singular point can be infinity. The customarily employed Hamiltonians are polynomials in their functional arguments, for which this is all the more valid.

It must be noted that the extremal trajectory does not correspond to real t', since dt' is pure imaginary in regions I and III in the classically accessible region dt' is pure real, but also becomes pure imaginary beyond the turning points (see Fig. 3). Our procedure is therefore more general than the Wick rotation with pure imaginary t'. In particular, the time t at the saddle-point is complex and corresponds to a certain integral number of transitions in the classically allowed region nT, plus double the pure imaginary time $i\tau$—of reaching the joining position $\rho(x,t)$ from the boundaries of the classically allowed region. The mathematical justification for this analytic continuation of the continual integral is beyond the scope of the present article.

It is easily seen, however, that the procedure for obtaining the extremal trajectory is similar to the Landau method for the calculation of matrix element in the one-dimensional case, and that the final results agree fully.

In fact, the Green's function for a nonlinear oscillator is determined in analogy with the (3.1) except that in place of the field coordinate $\rho(\mathbf{x},t)$ and of the canonical momentum $\varphi(\mathbf{x},t)$ it is necessary to choose the coordinate $\xi(t)$ and the oscillator momentum $\pi(t)$. The Hamilton function takes the form $H(\xi,\pi)=\pi^2/2m+U(\xi)$, where $U(\xi)$ is the potential energy.

On the trajectory sections with $E=0$ it is necessary to choose the branch of the momentum such that the action vanishes at $t=\pm i\infty$,

$$\pi=\pm[2m(-U)]^{1/2}.$$

The choice of the action on section II with energy $E=\omega$ is not single valued because of the presence of a classically allowed region and because of the possibility of an arbitrary number of oscillations.

Figure 4 shows symbolically the entire extremal trajectory: at $t'=-i\infty$ it goes out of the equilibrium position and proceeds with pure imaginary momentum and with zero energy to the first junction point $\xi=\xi(0)$ ($\xi(0)\to-\infty$ since $U(\xi)$ is assumed to be a polynomial) after

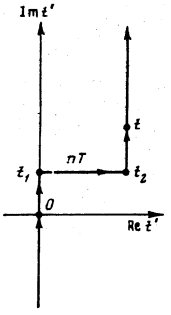

FIG. 3.

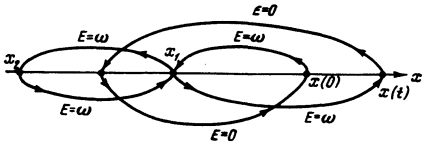

FIG. 4.

which it continues with energy ω and as before with pure imaginary momentum (large ξ are classically inadmissible) up to a certain turning point $\xi_1 = \xi(t_1)$, after which the trajectory enters the classically allowed region. In this region, the trajectory can execute an arbitrary number of oscillations between the turning points, after which at the instant of time t_2, $\xi_2 = \xi(t_2)$ $(U(\xi_2) = \omega)$ it again goes into the classically forbidden region, where at the point $\xi = \xi(t)(\xi(t) \to \infty)$ it joins the trajectory that goes at $t \to +i\infty$ into the equilibrium position with $E = 0$. The oscillations in the classically allowed region lead to the appearance of a real term of the action

$$\Delta S = \int_{t_1}^{t_2} \pi \frac{d\xi}{dt} dt.$$

If we have an arbitrary energy ω, then the increment of the phase is arbitrary and causes the result to vanish after summation over an arbitrary number of oscillations (a different number of oscillations corresponds to different saddle-points). On the other hand, if the Bohr-Sommerfeld conditions

$$\Delta S(\omega) = \oint \pi d\xi = 2\pi n \tag{3.6}$$

(n is an integer) are satisfied, then we obtain a δ function contribution to the imaginary force of the imaginary part of the Green's function.

The coefficient of the corresponding δ function, i.e., the corresponding matrix element, is determined by the classically forbidden region and will coincide with the Landau result

$$\operatorname{Im} \mathcal{G} \sim \delta(\omega - E_n + E_0) \exp\left\{ -\int_0^{\xi(0)} [2mU]^{1/2} d\xi + \int_{\xi_1}^{\xi(0)} [2m(U-\omega)]^{1/2} d\xi \right.$$
$$\left. + \int_{\xi_2}^{\xi(t)} [2m(U-\omega)]^{1/2} d\xi - \int_0^{\xi(t)} [2mU]^{1/2} d\xi \right\}. \tag{3.7}$$

In the case of a system with an infinite number of degrees of freedom [field $\rho(\mathbf{x}, t)$] the Bohr-Sommerfeld conditions cannot be satisfied for any finite wave packet (they're satisfied rigorously only for a specified Fourier component, and furthermore in an approximation quadratic in the field). They should be satisfied approximately, however, for a sufficiently narrow wave packet for a certain time determined by the width of the packet. As a result, the summation over different numbers of oscillations in the classically attainable region makes a considerable contribution only so long as $\Delta S \approx 2\pi n$, after which it begins to decrease rapidly for the saddle-points with large Re t. Thus, our procedure separates automatically those intermediate states which make an appreciable contribution to the imaginary part of the Green's function, and give directly the final result. This is precisely why it is not necessary to sum the individual δ functions and determine the state densities. For the general case we find ultimately that the imaginary part of the Green's function is

$$\operatorname{Im} \mathcal{G}(\omega, \mathbf{k}) \sim \exp\left\{ -2 \sum_{\mathbf{k}} \int_0^{\rho_{\mathbf{k}}(0)} \varphi_{\mathbf{k}} d\rho_{\mathbf{k}} \Big|_{\substack{E=0 \\ \mathbf{P}=0}} \right.$$
$$\left. + 2 \sum_{\mathbf{k}} \int_{\rho_{\mathbf{k}}^*}^{\rho_{\mathbf{k}}(0)} \varphi_{\mathbf{k}} d\rho_{\mathbf{k}} \Big|_{\substack{E=\omega \\ \mathbf{P}=\mathbf{p}}} \right\} = \exp 2\{ -\Delta S_{\mathrm{I}}(0, 0) + \Delta S_{\mathrm{II}}(\omega, \mathbf{P}) \}, \tag{3.8}$$

where we have changed over to spatial Fourier components of the field. Here (ρ^*_k, φ^*_k) is a point on the classical trajectory of the field with field energy $E = \omega$ and field momentum $\mathbf{P} = \mathbf{p}$, from which the trajectory is continued into the classically forbidden region, yielding the maximum exponential. The quantity $\Delta S(0,0) - \Delta S(\omega, \mathbf{p})$ is the difference between the increments of the imaginary part of the action along the trajectory that goes from the equilibrium position to the junction point $\rho(\mathbf{x})|_{t=0} \to \infty$, and along the trajectory that goes from ρ^* to the same junction point. Both these sections of the extremal trajectory correspond to a pure imaginary change of time. Formula (3.8) yields only the most substantial factor in the principal term of $\operatorname{Im}\mathcal{G}$.

4. CALCULATION OF THE MULTIPHOTON-DECAY PROBABILITY

In the general case the determination of the optimal trajectory, or equivalently the determination of the optimal point $(\varphi^*_k, \varphi^*_k)$ from which the continuation to complex times takes place and yields the smallest exponential, is an extremely complicated variational problem. It must be noted that the point of continuation does not agree with the stopping point, where all the velocities are equal to zero $(\varphi^*_k = 0)$, inasmuch as in this case the field momentum is also $\mathbf{P} = 0$, and consequently a trajectory with $\mathbf{P} = \mathbf{p} \neq 0$ does not reach such points at all.

There are various methods of continuation. The general method consists of a canonical transformation to new coordinates and momenta Q_k and π_k of the field. In this new representation there will be different stopping points Q^*_k, and we can continue the classical trajectory beyond these points in the usual manner, assuming the subsequent increment of the time to be pure imaginary. This procedure covers all the continuation methods.

We confine ourselves to the case of significance to us, where the nonlinear terms in the equations of motion are small in the classically allowed region. In this case the imaginary time τ to reach the joining point from the boundaries of the allowed region is logarithmically large and the main contribution to it is made the part of the trajectory where the nonlinear terms become of the order of the linear ones, and therefore the individual terms in the Hamiltonian (if separation into potential and kinetic energy is possible, then this can be one of them) will be much larger than the energy $E = \omega$. Consequently the quantity E is relatively small and, expanding the action ΔS_{11} in powers of E, we obtain directly from (3.8)

(in analogy with the procedure used in Sec. 2):

$$\operatorname{Im}\mathcal{G}(p,\omega)\sim\exp\{-2\omega\tau(p,\omega)\}, \tag{4.1}$$

where $\tau(p,\omega)$ is the imaginary time necessary to go from the classically allowed region to the joining point ($\rho\to\infty$). Since the quantity $\tau(p,\omega)$ depends on that point in the allowed region from which the imaginary time increment begins, it is necessary to find the point that yields the minimum τ.

To solve this variational problem we must find for the linearized field equations a solution that satisfies the conditions $H(\rho,\varphi)=\omega$, $\mathbf{P}(\rho,\varphi)=\mathbf{p}$ as well as the Bohr-Sommerfeld conditions

$$\int\varphi_{\mathbf{k}}d\rho_{\mathbf{k}}=2\pi n.$$

In view of the translational invariance and of the independence of the Hamilton function of the time, the solution constitutes a wave packet

$$\begin{aligned}\rho'(x,t)&=\sum\{\rho_{\mathbf{k}}^{(+)}e^{i\mathbf{kx}}\exp(-i\omega_{\mathbf{k}}t)+\rho_{\mathbf{k}}^{(-)}e^{i\mathbf{kx}}\exp(i\omega_{\mathbf{k}}t)\},\\ \varphi(x,t)&=\sum\{\varphi_{\mathbf{k}}^{(+)}e^{i\mathbf{kx}}\exp(-i\omega_{\mathbf{k}}t)+\varphi_{\mathbf{k}}^{(-)}e^{i\mathbf{kx}}\exp(i\omega_{\mathbf{k}}t)\}.\end{aligned} \tag{4.2}$$

with a certain ratio between $\rho_{\mathbf{k}}^{(\pm)}$ and $\varphi_{\mathbf{k}}^{(\pm)}$, as required by the field equations. When this solution is continued into the region of complex t, the term with the symbol (+) increases exponentially and determines that trajectory from $E=P=0$ to which our trajectory with $E=\omega$ and $\mathbf{P}=\mathbf{p}$ will be close (it is assumed that (4.2) satisfies these conditions). Thus, the smallest τ is obtained from solutions with the largest ω_k. The ω_k, however, cannot be very large since it is impossible to satisfy the Bohr-Sommerfeld condition $E_k/\omega_k=2\pi n\hbar$ (E_k is the energy of the $\mathbf{k}$-wave) for any of the individual waves in the packet, since $E=\Sigma E_{\mathbf{k}}$ and consequently $E_{\mathbf{k}}<E$. Therefore

$$\omega_k<E/2\pi\hbar n<\omega_{k_0}.$$

Thus, the sums (4.2) must be concentrated near $\mathbf{k}=\mathbf{k}_0$, which gives the maximum value of the frequency, or near $n=n_0$, where n_0 is the smallest number of excitations that add up to the field energy $E=\omega$ and to the momentum $\mathbf{P}=\mathbf{p}$ [see (1.5)].

However, if we take one solitary wave satisfying this requirement, then τ tends to infinity, inasmuch as at a finite field energy $E=\omega$ the energy density is vanishingly small (the volume of the system is large) and the nonlinear interaction also tends to zero. Thus, a certain smearing about k_0 is necessary in order that the anharmonic term not be too small. Since τ is determined by those values of $\rho'=\bar\rho$ and φ at which $H_{\text{harm}}/H_{\text{anharm}}\sim 1$, we have with logarithmic accuracy

$$\tau\sim\frac{1}{\omega_{k_0}}\ln\left|\frac{\bar\rho}{\rho'}\right|.$$

We see therefore that, with logarithmic accuracy, the width of the packet in $\mathbf{k}$-space should be logarithmically small:

$$\Delta k\sim\ln^{-1}\left|\frac{\bar\rho}{\rho'}\right|, \tag{4.3}$$

to find the principal logarithmic term in τ. The imaginary part of the Green's function is then

$$\operatorname{Im}\mathcal{G}(\omega,\mathbf{p})\sim\exp\left(-\frac{2\omega}{\omega_{k_0}}\ln\left|\frac{\bar\rho}{\rho'}\right|\right). \tag{4.4}$$

We carry out the corresponding calculations for the imaginary part of the Green's function in He II in the case of multiphonon decay.

It is possible either to use the Bogolyubov model of a weakly nonideal Bose gas, or write the hydrodynamic Hamiltonian for Bose fields that vary slowly in space and in time; this Hamiltonian presupposes integration over the rapidly varying part in the corresponding continuing integral,[8] an assumption that will be made from now on.

Thus, the effective long-wave hydrodynamic Hamiltonian takes the form

$$\begin{aligned}H&=H_0+H_i,\\ H_0&=\int dx\left[\bar\rho\frac{(\nabla\varphi)^2}{2}+\frac{u^2}{2\bar\rho}\rho'^2+\gamma\frac{u^2}{2\bar\rho}(\nabla\rho')^2\right].\\ H_i&=\int dx\left[\rho'\frac{(\nabla\varphi)^2}{2}+\frac{u^2}{6\bar\rho^2}\alpha\rho'^3+\dots\right].\end{aligned} \tag{4.5}$$

where we have explicitly separated the quadratic Hamiltonian H_0 corresponding to the sound waves with dispersion, from the Hamiltonian corresponding to the nonlinear interaction H_i, in which we have confined ourselves to the cubic terms. Actually, however, we do not need the concrete form of H_i, since the only important fact is that $H_i(\rho',\varphi)\sim H_0(\rho',\varphi)$ at $\rho'\sim\bar\rho$ and $|\nabla\varphi|\sim u$.

As already explained in Sec. 1, at energies close to the start of the spectrum it is impossible to excite phonons with momentum larger than $k_0=(\delta\varepsilon/\gamma p)^{1/2}$. We must therefore form a packet of phonons with momenta close to $\mathbf{k}_0$. We consider in this case a packet of maximum width (in $\mathbf{k}$-space), compatible with the conservation laws, so as to increase the nonlinear interaction. We can confine ourselves in this case to order-of-magnitude quantities, since the width is contained in the answer only under the logarithm sign. We introduce the longitudinal dimension l (in the direction of $\mathbf{p}$) and the transverse dimension L. Then, as usual, in a sound wave we have

$$\rho'\sim\bar\rho\varphi/ul,\qquad P\sim\rho'\varphi L^2. \tag{4.6}$$

The terms that take the dispersion into account must be of the order of the transverse kinetic energy (this corresponds to the minimum estimate for the transverse width), from which we get

$$\bar\rho\frac{\varphi^2}{L^2}\sim\frac{\gamma\rho'^2u^2}{l^2\rho_0},\quad L^2\sim\frac{l^2}{\gamma}. \tag{4.7}$$

Since the energy is $\varepsilon=uP+\delta\varepsilon$, where $\delta\varepsilon$ is connected with the dispersion, it follows that $\delta\varepsilon\sim\gamma\rho'^2L^2/l$; whence, using (4.6) and (4.7), we get

$$l^2\sim\gamma\frac{uP}{\delta\varepsilon}\quad\frac{\rho'}{\bar\rho}\sim\frac{1}{\gamma^{1/4}}\left(\frac{P}{\bar\rho u}\right)^{1/2}\left(\frac{\delta\varepsilon}{uP}\right)^{3/4}. \tag{4.8}$$

It is easily seen that the ratio of the nonlinear terms to the terms that take into account the dispersion is indeed small, since

$$\frac{u^2\rho'^3}{\bar{\rho}^2\gamma(\nabla\rho')^2}\sim\frac{1}{\gamma^{1/4}}\left(\frac{P}{\bar{\rho}u}\right)^{1/2}\left(\frac{\delta\varepsilon}{uP}\right)^{1/4}\ll 1 \tag{4.9}$$

at small $\delta\varepsilon=\varepsilon-uP$, and this in fact justifies the neglect of the nonlinear terms in the initial stage (i.e., in the classical accessible region).

The foregoing order-of-magnitude estimate of the width of the packet and of the density perturbation lead to a packed width in k-space $\Delta k_x\sim k_0$, $\Delta k_\perp\sim k_0^2$ and consequently to a frequency change $\Delta\omega_k\sim\omega_{k0}$. In this sense, the constructed packet is not optimal. It can be made optimal by increasing l (or equivalently, by decreasing Δk_x), by putting

$$l^2=\nu\frac{uP}{\delta\varepsilon}\ln\left|\frac{\bar{\rho}}{\rho'}\right|, \tag{4.10}$$

which leads to a logarithmic factor in ρ', φ, and L. This however, is of no importance in the calculation of the value of τ with logarithmic accuracy, since it leads only to doubly logarithmic terms. The average frequency ω_k averaged over the packet will differ from ω_{k_0} by an amount of the order of $(\ln\bar{\rho}/\rho')^{-1}$, which likewise does not affect the principal logarithmic term in τ.

The final answer is of the form

$$\begin{aligned}\operatorname{Im}\mathcal{G}(\omega,\mathbf{p})&\sim\exp\left[-2\left(\frac{\gamma p^3}{\delta\varepsilon}\right)^{1/2}\ln\left(\gamma^{3/4}\left(\frac{\bar{\rho}u}{p}\right)^{1/2}\left(\frac{up}{\delta\varepsilon}\right)^{5/4}\right)\right]\\&\sim\exp\left[-\frac{5}{2}\left(\frac{\gamma p^3u}{\omega-up}\right)^{1/2}\ln\frac{\gamma^{3/5}up}{\omega-up}\right],\end{aligned} \tag{4.11}$$

and demonstrates the presence of an essential singularity in the Green's function on the initial section of the spectrum.

Formula (4.11) coincides with formula (2.18) obtained from more intuitive reasoning.

In conclusion it can be stated that the use of the saddle-point method for the continual integral, which represents the Green's function, has made it possible to obtain for its imaginary part of a nontrivial asymptotic expression which is extremely difficult to obtain by summing diagrams. We hope that the described method can find use also in other problems connected with the production of a large number of particles.

[1)] We assume $\hbar=1$, i.e., we make no distinction between the energy and the momentum and accordingly between the frequency and the wave vector.

[2)] A brief exposition of the results was published earlier.[5]

[3)] It is useful to trace the connection between the quantum picture of the decay of the excitation with the classical picture of instability of a sinusoidal wave. If the conservation laws permit the decay of an excitation into two, then in the classical approach the wave will be unstable to small perturbations even in first-order perturbation theory. If the excitation can decay only into n excitations, then the classical wave is unstable only in the $(n-1)$-*st* order of perturbation theory, and the instability growth rate will depend on the amplitude of the initial perturbation. We emphasize that in the classical theory, of course, the wave does not have a definite lifetime, since the instability-evolution time depends on the initial amplitude of the perturbations, which in quantum theory is given by the zero-point oscillations.

[4)] Of course, the very possibility of expanding in powers of ϵ requires satisfaction of the condition $\epsilon\ll U$ in the essential region of values of x_i^0. It is easy to verify that this inequality is satisfied in our case.

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