Spatial dispersion of spin susceptibility of conduction electrons in a superconductor

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(Submitted 12 June 1978)
Zh. Eksp. Teor. Fiz. **76**, 578–587 (February 1979)

The method of temperature Green's functions is used to investigate the spatial dispersion of the spin susceptibility of the conduction electrons in pure and "dirty" (i.e., containing magnetic and nonmagnetic impurities) superconductors. The Hamiltonian of the indirect exchange interaction of the localized moments is also obtained. It is shown that in a pure superconductor the susceptibility and the indirect-exchange Hamiltonian have an additional nonoscillating and longer-range term that is exponentially cut off at a distance on the order of the coherence length ξ_0. For a dirty superconductor, the Fourier transform $\chi(\mathbf{q})$ of the susceptibility is obtained for an arbitrary wave vector $\mathbf{q}$. In the $p_0 r \gg 1$ approximation, a transition is effected to the coordinate space. It is shown that the effective radius of the indirect exchange interaction potential, which is equal to the mean free path l_p in a normal metal, increases abruptly on going into the superconducting state, to a new coherence length $\xi \approx (\xi_0 l_p)^{1/2} \gg l_p$.

PACS numbers: 74.30.Ci, 74.70.Nr

1. INTRODUCTION

The study of the magnetic properties of normal and superconducting metals with paramagnetic impurities is connected with the investigation of the spatial dispersion of the spin susceptibility of the conduction electrons. This is determined primarily by the fact that the indirect exchange interaction of the localized moments is expressed directly in terms of the nonlocal[1)] spin susceptibility (see, e.g., Ref. 1)

$$\mathcal{H}_{ex}(\mathbf{r}_1-\mathbf{r}_2)=-\frac{J^2}{g^2\mu_B{}^2}(\mathbf{S}_1\mathbf{S}_2)\int\frac{d\mathbf{q}}{(2\pi)^3}\chi(\mathbf{q})e^{i\mathbf{q}(\mathbf{r}_1-\mathbf{r}_2)}, \tag{1.1}$$

where J is the $s-d(f)$ exchange integral, g and μ_B are respectively the conduction electron g factor and the Bohr magneton, and $\chi(\mathbf{q})$ is the Fourier transform of the conduction electron susceptibility.

It is known that the main contribution to the integral (1.1) in a normal metal is made by the logarithmic singularity of $\chi(\mathbf{q})$ in the region $q \sim 2p_0$ (p_0 is the Fermi momentum). It is natural to expect that the derivation of the Hamiltonian $\mathcal{H}_{ex}(\mathbf{r}_1-\mathbf{r}_2)$ in a superconductor calls for knowledge of $\chi(\mathbf{q})$ for an arbitrary wave vector. Unfortunately, the paramagnetic susceptibility of the conduction electrons in superconductors has been investigated so far only at small values of q. The homogeneous spin susceptibility with account taken of the spin-orbit and potential scattering by the impurities was obtained by Abrikosov and Gor'kov,[2] while $\chi(\mathbf{q})$ at $q \ll p_0$ was obtained by Kaufman and Entin-Wohlman.[3] This, however, is not enough to find the spin susceptibility $\chi(r)$ in the coordinate representation even at $r \gg p_0^{-1}$.

One can expect that when the metal becomes superconducting the main change of the nonlocal susceptibility, and hence also of the indirect exchange interaction (1.1), occurs in the large distances $|\mathbf{r}_1-\mathbf{r}_2|$. The reason is that the Cooper pairing can not alter substantially the local spin polarization of the conduction electrons in the immediate vicinity of a paramagnetic impurity, but the total spin polarization of the electrons by the given spin, which is proportional to $\chi(0)$, should vanish at $T=0$ (in the absence of magnetic scattering by other impurities). This means that the local polarization is compensated over considerably larger distances—of the order of the coherence length.

The long-range increment to the potential of the indirect exchange interaction of the localized moments in a pure superconductors, due to the superconducting correlations of the conduction electrons, is given in the paper of Alekseevskii *et al.*[4]

In Sec. 3 of this paper we obtain an expression for the spatial dispersion of the spin susceptibility of the conduction electron in a pure superconductor. In Sec. 4 we calculate the Fourier transform of the susceptibility of the conduction electrons in the superconductor for an arbitrary wave vector in the presence of potential and exchange scattering by the impurities. We next obtain in Sec. 5 an expression for the nonlocal susceptibility and for the Hamiltonian of the Ruderman–Kittel–Kasuya–Yosida (RKKY) exchange interaction[5] in a "dirty" (containing magnetic and nonmagnetic impurities) superconductor. In Sec. 6, finally, we discuss the result.

2. FORMULATION OF THE PROBLEM

We consider a model of a superconducting metal with an isotropic quadratic dispersion law for the conduction electrons, $\varepsilon(\mathbf{p})=\mathbf{p}^2/2m^*$ (where m^* is the effective mass of the conduction electrons), in which magnetic and nonmagnetic impurities are randomly distributed. The conduction electrons experience exchange scattering by the magnetic impurities (the so-called $s-d(f)$ exchange) and potential scattering by the nonmagnetic impurities. This system can be described by the Hamiltonian

$$\begin{aligned}
&\mathcal{H}=\mathcal{H}_0+\mathcal{H}_1+\mathcal{H}_p+\mathcal{H}_{s-d};\\
&\mathcal{H}_0=\sum_\alpha\int d\mathbf{r}\Psi_\alpha^+(\mathbf{r})\left[-\frac{\nabla^2}{2m^*}\right]\Psi_\alpha(\mathbf{r}),\\
&\mathcal{H}_1=-|\lambda|\int d\mathbf{r}\Psi_\uparrow^+(\mathbf{r})\Psi_\downarrow^+(\mathbf{r})\Psi_\downarrow(\mathbf{r})\Psi_\uparrow(\mathbf{r}),\\
&\mathcal{H}_p=V\sum_{i\alpha\beta}\Psi_\alpha^+(\mathbf{r}_i)\delta_{\alpha\beta}\Psi_\beta(\mathbf{r}_i),\quad \mathcal{H}_{s-d}=\frac{J}{2}\sum_{j\alpha\beta}\Psi_\alpha^+(\mathbf{r}_j)(\mathbf{S}_j\sigma_{\alpha\beta})\Psi_\beta(\mathbf{r}_j),
\end{aligned} \tag{2.1}$$

 0038-5646/79/020291-05$02.40

where $\mathcal{H}_0$ is the Hamiltonian of the kinetic energy of the conduction electrons, $\mathcal{H}_1$ is the BCS Hamiltonian, $\mathcal{H}_p$ characterizes the potential scattering by the impurities, and $\mathcal{H}_{s-d}$ characterizes the exchange scattering; to simplify the subsequent calculations, all the scattering potential are assumed δ-like; $\Psi_\alpha^+(\mathbf{r})$ and $\Psi_\alpha(\mathbf{r})$ are the electron field operators, the arrow subscripts designate the projection of the electron spin on the chosen z axis; S_j is the spin operator of the j-th impurity, J is the $s-d$ exchange integral, and V is the potential-scattering constant. The indices i and j run respectively over the positions of the nonmagnetic and magnetic impurities.

The nonlocal spin susceptibility of the conduction electrons in a superconductor can be expressed in the form

$$\chi(\mathbf{r}_1,\mathbf{r}_2)=-\frac{g^2\mu_B^2}{2}\lim_{\substack{\mathbf{r}'\to\mathbf{r}_1\\ \tau'\to\tau_1+0}}\sum_{\gamma\delta}\int_0^{1/T}d\tau_2 \times\langle T\Psi_\uparrow(x_1)\Psi_\gamma^+(x_2)\sigma_{\gamma\delta}^z\Psi_\delta(x_2)\Psi_\uparrow^+(x')\rangle, \tag{2.2}$$

where $x\equiv(\mathbf{r},\tau)$, and the angle brackets denote averaging over the Gibbs ensemble. After averaging over the impurity configurations, the susceptibility will depend only on the argument difference $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$.

All the calculations that follow will be carried out by the method of temperature Green's functions.[6] We introduce the following mean values:

$$G_{\alpha\beta}(x,x')=-\langle T\Psi_\alpha(x)\Psi_\beta^+(x')\rangle,$$
$$F_{\alpha\beta}(x,x')=\langle T\Psi_\alpha(x)\Psi_\beta(x')\rangle,\quad F_{\alpha\beta}^+(x,x')=\langle T\Psi_\alpha^+(x)\Psi_\beta^+(x')\rangle. \tag{2.3}$$

Here $G_{\alpha\beta}(x,x')$ is the single-particle electron Green's function, and $F_{\alpha\beta}(x,x')$ and $F_{\alpha\beta}^+(x,x')$ are the "anomalous" Green's functions of the conduction electrons and are characteristic of the superconducing state.

The mean value under the integral sign in (2.2) is a two-particle-Green's function whose calculation yields in fact the nonlocal susceptibility.

3. NONLOCAL SUSCEPTIBILITY OF THE CONDUCTION ELECTRONS IN A PURE SUPERCONDUCTOR

Owing to the presence of translational symmetry, the susceptibility (2.2) in a pure superconductor depends only on the argument difference $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$. The mean value of the four field operators can be broken up in accord with Wick's theorem into a product of paired mean values that constitute the Green's functions of the unperturbed Hamiltonian $\mathcal{H}_0+\mathcal{H}_1$:

$$\chi(\mathbf{r})=-\frac{g^2\mu_B^2}{2}\int\frac{d\mathbf{p}\,d\mathbf{q}}{(2\pi)^6}e^{i\mathbf{q}\mathbf{r}}T\sum_\omega\{G_\omega^0(\mathbf{p})G_\omega^0(\mathbf{p}+\mathbf{q}) +F_\omega^0(\mathbf{p})F_\omega^{0+}(\mathbf{p}+\mathbf{q})\}, \tag{3.1}$$

$$G_\omega^0(\mathbf{p})=-\frac{i\omega+\xi_\mathbf{p}}{\omega^2+\xi_\mathbf{p}^2+\Delta^2},\quad F_\omega^0(\mathbf{p})=F_\omega^{0+}(\mathbf{p})=\frac{\Delta}{\omega^2+\xi_\mathbf{p}^2+\Delta^2}, \tag{3.2}$$

where $G_\omega^0(\mathbf{p})$, $F_\omega^0(\mathbf{p})$, $F_\omega^{0+}(\mathbf{p})$ are the Fourier transforms of the corresponding Green's functions (2.3), $\omega=\pi T(2n+1)$, $\xi_\mathbf{p}=\mathbf{p}^2/2m^*-\varepsilon_0$, ε_0 is the Fermi energy, and 2Δ is the energy gap in the elementary-excitation spectrum.

The expression for the nonlocal susceptibility (3.1) is integrated exactly, and the result is

$$\chi(r)=-\frac{g^2\mu_B^2}{2}\left(\frac{N(0)\pi}{p_0r}\right)^2T\sum_\omega\left\{\exp\left[2ip_0r\left(1+i\frac{(\omega^2+\Delta^2)^{1/2}}{\varepsilon_0}\right)^{1/2}\right]\right.$$
$$+\exp\left[-2ip_0r\left(1-i\frac{(\omega^2+\Delta^2)^{1/2}}{\varepsilon_0}\right)^{1/2}\right]-\frac{\Delta^2}{\omega^2+\Delta^2}\left[\exp\left[ip_0r\left(1\right.\right.\right.$$
$$\left.\left.\left.+i\frac{(\omega^2+\Delta^2)^{1/2}}{\varepsilon_0}\right)^{1/2}\right]-\exp\left[-ip_0r\left(1-i\frac{(\omega^2+\Delta^2)^{1/2}}{\varepsilon_0}\right)^{1/2}\right]\right]^2\Bigg\}. \tag{3.3}$$

The Hamiltonian of the indirect exchange interaction of the localized moments in the pure superconductor can be written in simpler form at $p_0r\gg1$:

$$\mathcal{H}_{ex}(r)=\frac{J^2}{2}(\mathbf{S}_1\mathbf{S}_2)\left(\frac{N(0)\pi}{p_0r}\right)^2T\sum_\omega\left\{\frac{\omega^2}{\omega^2+\Delta^2}\cos2p_0r +\frac{\Delta^2}{\omega^2+\Delta^2}\right\}\exp\left(-\frac{(\omega^2+\Delta^2)^{1/2}}{\varepsilon_0}p_0r\right). \tag{3.4}$$

Here $N(0)$ is the state density on the Fermi surface, and r is the distance between the interacting paramagnetic impurities. The first term oscillates with a period $2p_0r$ and decreases with distance like r^{-3}. Obviously, at $\Delta=0$ this term coincides exactly with the expression for the RKKY potential in a normal metal at $p_0r\gg1$. The second term, which vanishes at $\Delta=0$, is due to the superconducting correlations of the conduction electrons. trons. It decreases with distance like r^{-2}, so that the main contribution to the sum over the frequency is made by small ω. The cutoff factor of either term is of the order $\exp(-r/\xi_0)$, where ξ_0 is the coherence length in the pure superconductor. The contribution of the superconducting increment to the total polarization is opposite in sign to the contribution of the first term, and it can be easily shown that the two cancel out completely[2)] at $T=0$. The reason is that the integral of the oscillating term converges rapidly over distances $r\ll\xi_0$, whereas the second term, as seen from (3.4), acts over a longer range and does not oscillate.

4. FOURIER TRANSFORM OF THE SUSCEPTIBILITY OF THE CONDUCTION ELECTRONS IN A DIRTY SUPERCONDUCTOR

In the presence of potential and exchange scattering by the impurities, expression (2.2) should be averaged over the positions and spin orientations of the impurities. We introduce the notation (see Ref. 2)

$$\Pi_{\alpha\beta}^{(1)}(x_1-x_2,x_2-x')=\sum_{\gamma\delta}\langle T\Psi_\alpha(x_1)\Psi_\gamma^+(x_2)\sigma_{\gamma\delta} \times\Psi_\delta(x_2)\Psi_\beta^+(x')\rangle_{av} \tag{4.1}$$

(the subscript "av" stands for averaging over the impurities). The susceptibility is expressed in this notation by

$$\chi(\mathbf{r})=\int\frac{d\mathbf{q}}{(2\pi)^3}\chi(\mathbf{q})e^{i\mathbf{q}\mathbf{r}} =-\frac{g^2\mu_B^2}{2}T\sum_\omega\int\frac{d\mathbf{q}}{(2\pi)^3}e^{i\mathbf{q}\mathbf{r}}\int\frac{d\mathbf{p}}{(2\pi)^3}\Pi_{\uparrow\uparrow}^{(1)}(\mathbf{p},\tilde\omega;\mathbf{p}+\mathbf{q},\tilde\omega), \tag{4.2}$$

where $\Pi_{\alpha\beta}^{(1)}(\mathbf{p},\tilde\omega;\mathbf{p}+\mathbf{q},\tilde\omega)$ is the Fourier transform of (4.1) and $\tilde\omega$ is defined below.

The method of calculating mean values of the type (4.1) was developed in the papers of Abrikoscov and Gor'kov.[2,8] Summing ladder-type diagrams ($p_0l\gg1$), we obtain an integral equation for the Fourier transform of (4.1):

$$\Pi^{(1)}(\mathbf{p},\tilde{\omega};\mathbf{p}+\mathbf{q},\tilde{\omega}) = G_{\tilde{\omega}}(\mathbf{p})G_{\tilde{\omega}}(\mathbf{p}+\mathbf{q}) + F_{\tilde{\omega}}(\mathbf{p})F^{+}_{\tilde{\omega}}(\mathbf{p}+\mathbf{q})$$
$$+ z^{(-)}G_{\tilde{\omega}}(\mathbf{p})G_{\tilde{\omega}}(\mathbf{p}+\mathbf{q})\int\frac{d\mathbf{p}'}{(2\pi)^3}\Pi^{(1)}(\mathbf{p}',\tilde{\omega};\mathbf{p}'+\mathbf{q},\tilde{\omega})$$
$$- z^{(+)}F_{\tilde{\omega}}(\mathbf{p})G_{\tilde{\omega}}(\mathbf{p}+\mathbf{q})\int\frac{d\mathbf{p}'}{(2\pi)^3}\Pi^{2}(\mathbf{p}',\tilde{\omega};\mathbf{p}'+\mathbf{q},\tilde{\omega})$$
$$- z^{(-)}F_{\tilde{\omega}}(\mathbf{p})F^{+}_{\tilde{\omega}}(\mathbf{p}+\mathbf{q})\int\frac{d\mathbf{p}'}{(2\pi)^3}\Pi^{(3)}(\mathbf{p}',\tilde{\omega};\mathbf{p}'+\mathbf{q},\tilde{\omega})$$
$$+ z^{(+)}G_{\tilde{\omega}}(\mathbf{p})F^{+}_{\tilde{\omega}}(\mathbf{p}+\mathbf{q})\int\frac{d\mathbf{p}'}{(2\pi)^3}\Pi^{(4)}(\mathbf{p}',\tilde{\omega};\mathbf{p}'+\mathbf{q},\tilde{\omega}). \qquad (4.3)$$

The Green's functions in this equation are complete and are averaged over the impurities. They can be obtained from the corresponding Green's functions (3.2) of the pure superconductor by making the substitution $\{\omega,\Delta\}\to\{\tilde{\omega},\tilde{\Delta}\}$, where[8]

$$\tilde{\omega}=\omega+\frac{u}{2\tau_+(u^2+1)^{1/2}},\quad \tilde{\Delta}=\Delta+\frac{1}{2\tau_-(u^2+1)^{1/2}},$$

$$u=\frac{\tilde{\omega}}{\tilde{\Delta}},\quad \frac{\omega}{\Delta}=u\left(1-\frac{1}{\Delta\tau_s}\frac{1}{(u^2+1)^{1/2}}\right),$$

$$\frac{1}{\tau_\pm}=\frac{1}{\tau_p}\pm\frac{1}{\tau_s},\quad \frac{1}{\tau_p}=\frac{mp_0nV^2}{\pi},\quad \frac{1}{\tau_s}=\frac{mp_0n'}{\pi}\frac{J^2}{4}S(S+1),$$

where τ_p and τ_s are the times of potential and exchange scattering by the impurities. In (4.3),

$$z^{(\pm)}=nV^2\pm\frac{n'J^2}{4}\frac{S(S+1)}{3},$$

and n and n' are the numbers of the potential and spin scatterers per unit volume.

It is seen from (4.3) that to find $\Pi^{(1)}(x_1-x_2,x_2-x')$ it is necessary to know three other quantities:

$$\Pi^{(2)}_{\alpha\beta}(x_1-x_2,x_2-x')=\sum_{\gamma\delta}\langle T(g\Psi^{+}(x_1))_\alpha\Psi_\gamma{}^{+}(x_2)\sigma_{\gamma\delta}\Psi_\delta(x_2)\Psi_\beta{}^{+}(x')\rangle_{cp},$$

$$\Pi^{(3)}_{\alpha\beta}(x_1-x_2,x_2-x')=\sum_{\gamma\delta}\langle T(g\Psi^{+}(x_1))_\alpha\Psi_\gamma{}^{+}(x_2)$$
$$\times\sigma_{\gamma\delta}\Psi_\delta(x_2)(\Psi(x')g)_\beta\rangle_{cp}, \qquad (4.4)$$

$$\Pi^{(4)}_{\alpha\beta}(x_1-x_2,x_2-x')=\sum_{\gamma\delta}\langle T\Psi_\alpha(x_1)\Psi_\gamma{}^{+}(x_2)\sigma_{\gamma\delta}\Psi_\delta(x_2)(\Psi(x')g)_\beta\rangle_{cp},$$

where

$$g_{\alpha\beta}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}_{\alpha\beta}.$$

The equations for the Fourier components of these quantities are similar to (4.3). The result is a system of four integral equations for the Fourier transforms of the quantities (4.1) and (4.4). To obtain these equations we used the relation

$$\Pi^{(i)}_{\alpha\beta}(\mathbf{p},\tilde{\omega};\mathbf{p}+\mathbf{q},\tilde{\omega})=\sigma_{\alpha\beta}\Pi^{(i)}(\mathbf{p},\tilde{\omega};\mathbf{p}+\mathbf{q},\tilde{\omega}),$$

which can be easily proved by iterating the equations of the system. Integral equations such as (4.3) can be reduced to algebraic ones by integrating both halves with respect to the momentum:

$$\Lambda^{(1)}(\mathbf{q},\tilde{\omega})=K^{(1)}(\mathbf{q},\tilde{\omega})+K^{(3)}(\mathbf{q},\tilde{\omega})+z^{(-)}\Lambda^{(1)}(\mathbf{q},\tilde{\omega})K^{(1)}(\mathbf{q},\tilde{\omega})$$
$$-z^{(+)}\Lambda^{(2)}(\mathbf{q},\tilde{\omega})K^{(2)}(\mathbf{q},\tilde{\omega})-z^{(-)}\Lambda^{(3)}(\mathbf{q},\tilde{\omega})K^{(3)}(\mathbf{q},\tilde{\omega})$$
$$+z^{(+)}\Lambda^{(4)}(\mathbf{q},\tilde{\omega})K^{(2)}(\mathbf{q},\tilde{\omega}),$$

where

$$\Lambda^{(i)}(\mathbf{q},\tilde{\omega})=\int\frac{d\mathbf{p}}{(2\pi)^3}\Pi^{(i)}(\mathbf{p},\tilde{\omega};\mathbf{p}+\mathbf{q},\tilde{\omega}),\quad i=1,2,3,4,$$

$$K^{(1)}(\mathbf{q},\tilde{\omega})=\int\frac{d\mathbf{p}}{(2\pi)^3}G_{\tilde{\omega}}(\mathbf{p})G_{\tilde{\omega}}(\mathbf{p}+\mathbf{q}),$$

$$K^{(2)}(\mathbf{q},\tilde{\omega})=\int\frac{d\mathbf{p}}{(2\pi)^3}G_{\tilde{\omega}}(\mathbf{p})F_{\tilde{\omega}}(\mathbf{p}+\mathbf{q}), \qquad (4.5)$$

$$K^{(3)}(\mathbf{q},\tilde{\omega})=\int\frac{d\mathbf{p}}{(2\pi)^3}F_{\tilde{\omega}}(\mathbf{p})F^{+}_{\tilde{\omega}}(\mathbf{p}+\mathbf{q}),$$

$$K^{(4)}(\mathbf{q},\tilde{\omega})=\int\frac{d\mathbf{p}}{(2\pi)^3}G_{\tilde{\omega}}(\mathbf{p})G_{-\tilde{\omega}}(\mathbf{p}+\mathbf{q}).$$

Thus, the system of integral equations has been reduced to a system of algebraic equations. It is easily seen from them that $\Lambda^{(4)}(\mathbf{q},\tilde{\omega})=-\Lambda^{(2)}(\mathbf{q},\tilde{\omega})$. Taking this into account, we obtain a system of three equations:

$$[1-z^{(-)}K^{(1)}(\tilde{\omega})]\Lambda^{(1)}+2z^{(+)}K^{(2)}(\tilde{\omega})\Lambda^{(2)}+z^{(-)}K^{(3)}(\tilde{\omega})\Lambda^{(3)}=K^{(1)}(\tilde{\omega})+K^{(3)}(\tilde{\omega}),$$
$$-z^{(-)}K^{(2)}(\tilde{\omega})\Lambda^{(1)}+[1-z^{(+)}(K^{(1)}(\tilde{\omega})-K^{(3)}(\tilde{\omega}))]\Lambda^{(2)}-z^{(-)}K^{(2)}(-\tilde{\omega})\Lambda^{(3)}$$
$$=K^{(2)}(\tilde{\omega})-K^{(2)}(-\tilde{\omega}),$$
$$z^{(-)}K^{(3)}(\tilde{\omega})\Lambda^{(1)}+2z^{(+)}K^{(2)}(-\tilde{\omega})\Lambda^{(2)}+[1-z^{(-)}K^{(1)}(-\tilde{\omega})]\Lambda^{(3)}$$
$$=-K^{(1)}(-\tilde{\omega})-K^{(3)}(\tilde{\omega}).$$

To abbreviate the notation, some of the arguments have been left out.

We represent the solution of the system of equations for $\Lambda^{(1)}(\mathbf{q},\tilde{\omega})$ in the form

$$\Lambda^{(1)}(q,\tilde{\omega})=X(q,\tilde{\omega})/Y(q,\tilde{\omega}),$$

$$X(q,\tilde{\omega})=\frac{i(m^*)^2}{4\pi}\left\{\left(\frac{\tilde{\omega}^2}{\tilde{\omega}^2+\tilde{\Delta}^2}+\frac{\tilde{\omega}}{(\tilde{\omega}^2+\tilde{\Delta}^2)^{1/2}}\right)L_1 \qquad (4.6)$$
$$-\left(\frac{\tilde{\omega}^2}{\tilde{\omega}^2+\tilde{\Delta}^2}-\frac{\tilde{\omega}}{(\tilde{\omega}^2+\tilde{\Delta}^2)^{1/2}}\right)L_2+\frac{2\tilde{\Delta}^2}{\tilde{\omega}^2+\tilde{\Delta}^2}L_3$$
$$-\frac{i}{2}\left[\frac{1}{l_+'}\left(1+\frac{\tilde{\omega}}{(\tilde{\omega}^2+\tilde{\Delta}^2)^{1/2}}\right)+\frac{1}{l_-'}\frac{\tilde{\Delta}^2}{\tilde{\omega}^2+\tilde{\Delta}^2}\right]L_1L_3+\frac{i}{2}\left[\frac{1}{l_+'}\left(1-\frac{\tilde{\omega}}{(\tilde{\omega}^2+\tilde{\Delta}^2)^{1/2}}\right)\right.$$
$$\left.+\frac{1}{l_-'}\frac{\tilde{\Delta}^2}{\tilde{\omega}^2+\tilde{\Delta}^2}\right]L_2L_3-\frac{i}{l_-'}\frac{\tilde{\omega}^2}{\tilde{\omega}^2+\tilde{\Delta}^2}L_1L_2+\frac{1}{2l_+'l_-'}L_1L_2L_3\Big\}, \qquad (4.7)$$

$$Y(q,\tilde{\omega})=1-\frac{i}{4}\left\{\left[\frac{1}{l_+'}\left(1+\frac{\tilde{\omega}^2}{\tilde{\omega}^2+\tilde{\Delta}^2}\right)+\frac{1}{l_-'}\frac{\tilde{\Delta}^2}{\tilde{\Delta}^2+\tilde{\omega}^2}\right](L_1-L_2)\right.$$
$$+2\left[\frac{1}{l_+'}\frac{\tilde{\omega}^2}{\tilde{\omega}^2+\tilde{\Delta}^2}+\frac{1}{l_-'}\frac{\tilde{\Delta}^2}{\tilde{\omega}^2+\tilde{\Delta}^2}\right]L_3\Big\}-\frac{1}{8l_-'}\Big\{\Big[\frac{1}{l_+'}\Big(1+\frac{\tilde{\omega}^2}{\tilde{\omega}^2+\tilde{\Delta}^2}\Big)$$
$$+\frac{1}{l_-'}\frac{\tilde{\Delta}^2}{\tilde{\omega}^2+\tilde{\Delta}^2}\Big]L_3(L_1-L_2)$$
$$-2\left[\frac{1}{l_+'}\frac{\tilde{\Delta}^2}{\tilde{\omega}^2+\tilde{\Delta}^2}+\frac{1}{l_-'}\frac{\tilde{\omega}^2}{\tilde{\omega}^2+\tilde{\Delta}^2}\right]L_1L_2+\frac{i}{l_+'l_-'}L_1L_2L_3\Big\},$$

where

$$L_1=\frac{1}{q}\ln\left(\frac{2p^{+}+q}{2p^{+}-q}\right),\quad L_2=\frac{1}{q}\ln\left(\frac{2p^{-}+q}{2p^{-}-q}\right),\quad L_3=\frac{1}{q}\ln\left(\frac{p^{+}-p^{-}+q}{p^{+}-p^{-}-q}\right),$$
$$\frac{1}{l_\pm'}=\frac{1}{v_0\tau_\pm'}=\frac{1}{v_0}\left(\frac{1}{\tau_p}\pm\frac{1}{3\tau_s}\right)=\frac{1}{l_p}\pm\frac{1}{3l_s},$$

v_0 is the electron velocity on the Fermi surface, l_p and l_s are the electron mean free paths due respectively to the potential and exchange scattering, and $p^{\pm}=p_0(1\pm i(\tilde{\omega}^2+\Delta^2)^{1/2}/\varepsilon_0)^{1/2}$.

Using relations (4.2), (4.5), and (4.6), we obtain for the susceptibility the expression

$$\chi(q)=-\frac{g^2\mu_B{}^2}{2}T\sum_{\omega}\Lambda^{(1)}(q,\tilde{\omega})=-\frac{g^2\mu_B{}^2}{2}T\sum_{\omega}\frac{X(q,\tilde{\omega})}{Y(q,\tilde{\omega})}. \qquad (4.8)$$

It is obvious that by letting l_p and l_s to go to infinity in (4.7) and (4.8) we obtain the Fourier transform of the susceptibility of the conduction electrons in a pure superconductor for arbitrary q. In this case $\gamma(\mathbf{q},\tilde{\omega})$ becomes simply equal to unity, and what are left of $X(\mathbf{q},\tilde{\omega})$ are the first three terms, two of which have logarithmic singularities with the real part of the wave vector $q\sim 2p_0$, while the third term has a pure imaginary logarithmic singularity. It is these singularities in momentum

space which cause the oscillations of the first term and the non-oscillating character of the second term in the RKKY Hamiltonian (3.4) of the pure superconductor.

At $\tilde{\Delta}=0$, Eq. (4.6) goes over naturally into the corresponding equation for the normal metal.[9]

For small q ($q \ll p_0$) the expression (4.8) for the susceptibility coincides with that obtained in Ref. 3.

5. NONLOCAL SUSCEPTIBILITY OF CONDUCTION ELECTRONS IN A DIRTY SUPERCONDUCTOR

To find the spatial dispersion of the susceptibility it is necessary, as follows from (4.2), (4.5), and 4.8), to take the Fourier transform of $\gamma(q)$. From expressions (4.7) and (4.8) for the Fourier transform of the susceptibility we see that the latter has the following singularities on the complex plane: two logarithmic branch points with real part of the order of $2p_0$, just as in the normal metal, one pure imaginary logarithmic singularity, and a pole that occurs at small q ($q \ll p_0$). The equation for finding the pole is of the form $Y(\mathbf{q}, \tilde{\omega})=0$. At small q it takes the form

$$1-\frac{i}{2q}\left[\frac{1}{l_+'}\frac{u^2}{u^2+1}+\frac{1}{l_-'}\frac{1}{u^2+1}\right]\ln\left(\frac{p^+-p^-+q}{p^+-p^--q}\right)=0.$$

It can be shown that this equation has a pure imaginary solution for which the approximate expression is[3)]

$$q_0=\sqrt{3}(p^+-p^-)\left\{\left(1+\frac{u^2-1}{u^2+1}\frac{l_p}{3l_s}\right)^{-1}(p^--p^+)i-1\right\}^{1/2}. \quad (5.1)$$

We choose the integration contour shown in the figure. The integrals along the large semicircle and the small circles vanish when their radii go respectively to infinity and zero. The sought integral consists thus of the residue at the pole of (5.1) and the integrals along the edges of the cuts from the branch points of the logarithms L_1, L_2, and L_3. We present only the final result, which is valid at $p_0 r \gg 1$:

$$\chi(r)=-\frac{g^2\mu_B^2}{2}\left(\frac{N(0)\pi}{p_0 r}\right)^2 T\sum_\omega\left\{\left[\frac{u^2}{u^2+1}\frac{\cos(2p_0r+2\Phi)}{1+[(1-\eta)(\ln 4p_0r)/4p_0l_p]^2}+\frac{1}{u^2+1}\right]\exp\left[-\frac{r}{l_p}\left(1+3\varkappa+\frac{(\omega^2+\Delta^2)^{1/2}}{\varepsilon_0}p_0l_p\right)\right]+3\frac{r}{l_p}\frac{1}{u^2+1}\times\left(1+\frac{(\omega^2+\Delta^2)^{1/2}}{\varepsilon_0}\frac{p_0l_p}{1+\varkappa}\right)^2\left(1+3\varkappa+\frac{(\omega^2+\Delta^2)^{1/2}}{\varepsilon_0}p_0l_p\right)\times\exp\left[-3^{1/2}p_0r\left(\frac{(\omega^2+\Delta^2)^{1/2}}{\varepsilon_0}\frac{1}{p_0l_p(1+\varkappa)}\right)^{1/2}\left(1+3\varkappa+\frac{(\omega^2+\Delta^2)^{1/2}}{\varepsilon_0}p_0l_p\right)\right]\right\}. \quad (5.2)$$

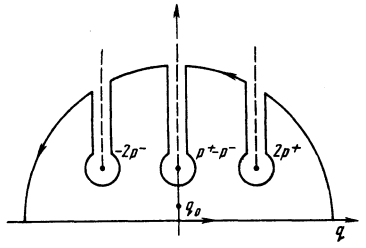

FIG. 1. Integration contour for the calculation of (5.2); $2p^+$, $-2p^-$, and p^+-p^- are the branch points of the logarithms L_1, L_2, and L_3, and q_0 is the pole of (5.1). The dashed lines show the cuts from the corresponding branch points.

Here

$$\varkappa=\frac{u^2-1}{u^2+1}\frac{l_p}{3l_s},\quad \eta=\frac{u^2}{u^2+1}\frac{l_p}{3l_s},\quad \operatorname{tg}\Phi=(1-\eta)\frac{\ln 4p_0r}{4p_0l_p}.$$

The Hamiltonian of the RKKY interaction in superconductors with impurities is obtained by substituting (5.2) in (1.1).

6. DISCUSSION OF RESULTS

The susceptibility $\chi(r)$ contains three terms, the first of which oscillates like $\cos 2p_0 r$ and goes over exactly into the susceptibility of the normal metal at $\tilde{\Delta}=0$,[9] while the second term has the same nature as the corresponding expression in (3.4). Its non-oscillating character is explained by the fact that the Cooper pairing upsets those phase relations between the conduction-electron wave functions, which lead to Friedel oscillations of the spin density in a normal metal. The third term, which is likewise non-oscillating, decreases with distance like r^{-1} and is exponentially cut off at distances on the order of the new coherence length ξ:

$$p_0^{-1}\left(\frac{(\omega^2+\Delta^2)^{1/2}}{\varepsilon_0}\frac{1}{p_0l_p}\right)^{-1/2}\sim(\xi_0 l_p)^{1/2}=\xi.$$

The non-oscillating behavior of this term is due to the joint action of the superconducting correlation and of scattering by the impurities on the phase shifts of the electron wave functions. As already noted above, the homogeneous susceptibility of the conduction electrons in the absence of magnetic scattering should vanish at $T=0$. It is easy to verify that our expression (5.2) for $\chi(r)$ satisfies this requirement. For this it is necessary to integrate it over the entire space and sum over the freqnency, putting $l_s=\infty$ (i.e., $u\equiv\omega/\Delta$, $\varkappa=\eta=0$). To prevent a formal divergence in the summation over the frequency, we must add and subtract the susceptibility of the normal metal.[9] It turns out then that the first term in (5.2) gives the Pauli susceptibility while the second term, which has compensated in the pure superconductor the contribution of the oscillating part, is cut off at distances $\sim l_p \ll \xi_0$ ($l_p \sim 10^{-6}-10^{-7}$ cm, $\xi_0 \sim 10^{-4}-10^{-5}$ cm) and makes practically no contribution to $\chi(0)$ at small l_p. The third term has a much larger range and is cut off, as already mentioned, at distances of the order of the new coherence length $\xi \gg l_p$. It is precisely this term which is responsible for the complete compensation of the normal part of the homogeneous susceptibility in a cold superconductor. Thus, the effective radius of the indirect-exchange potential, which equals the mean free path in a normal metal l_p, increases sharply to $\xi \gg l_p$ on going into the superconducting state (in strong magnetic scattering, the spin mean free path l_s begins to compete with the new coherence length ξ).

In the investigation of magnetic ordering in dilute superconducting alloys it is important to know in fact the spatial dispersion of the spin susceptibility, rather than its Fourier transform at $q=0$ or at small q. Actually $\chi(\mathbf{q}=0)$, for example, is in fact an integral of the nonlocal susceptibility over all of space, and contains contribution from distances shorter than interatomic, where the amplitude of the susceptibility oscillations is a maximum. On the other hand, in the ordering problem we

must know the molecular field at the impurity, namely the sum of $\chi(r)$ over the lattice, starting at least from the nearest neighbor. The oscillating character of the susceptibility at short distances makes all the quantities containing sums over the lattice highly sensitive to the discreteness and parameters of the lattice. This fact must be taken into account in the analysis of magnetic ordering.

The expression obtained by us for the nonlocal susceptibility can be used also in the study of the question of the line width of paramagnetic resonance on localized magnetic moments along with the study of the Korringa mechanism of relaxation in a superconductor.[4,10]

[1)]Here and below nonlocality is understood in the sense of spatial dispersion.
[2)]The RKKY potential in a pure superconductor was also obtained in Ref. 7, but it differs from (4.3) and does not satisfy the condition that the homogeneous susceptibility vanish at $T = 0$.
[3)]We note that it differs from the numerical solution by several percent when l_p, l_s and ω vary in a wide range.

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Translated by J. G. Adashko

Fluctuation diamagnetism of "dirty" superconductors

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(Submitted 3 July 1978)
Zh. Eksp. Teor. Fiz. **76**, 588–596 (February 1979)

The diamagnetic properties of "dirty" superconducting metals above the transition temperature is systematically investigated. The dependence of the fluctuation-induced increment to the magnetization on the temperature and on the magnetic field of type-II superconductors is calculated under conditions close to surface superconductivity. It is shown that the magnetization varies strongly near $T_{c3}(H)$ and near the third critical field $H_{c3}(T)$. The conditions under which the contribution made to the magnetization by surface-type superconducting nucleation centers is larger than the contribution of the volume centers are determined.

PACS numbers: 74.40. + k, 74.60. — w, 74.70.Nr

1. INTRODUCTION

The change produced in the diamagnetic susceptibility of a superconductor by fluctuations of the superconducting phase was observed in a large range of temperatures above T_c. Gollub *et al.*[1] observed diamagnetism due to electronic pair fluctuations in strong magnetic fields $[H \sim H_{c2}(0)]$ and at high temperatures $(T \sim 2T_{c0})$ in bulky samples. The fluctuation-induced mechanism was investigated by Schmidt[2] and by Prange[3] on the basis of the Ginzburg–Landau Theory. This theory describes correctly only the behavior of long-wave fluctuations, so that the theoretical results of Schmidt and Prange are not in good agreements with the experiments of Gollub *et al.*[1] at high temperatures and in strong magnetic fields, when the short-wave fluctuations must be correctly accounted for. Corrections for the short-wave fluctuations were introduced in the calculations of the magnetization of superconductors above T_c on the basis of the Gor'kov theory in a number of papers.[4-6] In these papers, the dependence of the magnetization on the temperature and on the applied magnetic field was determined by the procedure developed in Schmid's paper,[2] namely, by calculating the free energy of the system. The results agree well in the main with the experimental data.

In other experiments, Gollub *et al.*[7] measured the fluctuation-induced magnetization of bulky type-II superconductors under conditions close to surface superconductivity. If a stationary magnetic field is applied parallel to the sample surface, fluctuation-induced surface nu-

0038-5646/79/020295-05$02.40