tance of the round solenoid per unit solenoid length, $L = 2\pi RN$, n is the number of turns per unit length, and ω is the frequency of the alternating current $I(t)$.

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Recursion equation for the percolation problem

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A recursion equation allowing for an arbitrary change in scale is obtained for the percolation problem. The percolation threshold for two-dimensional space coincides with the exact value $P_c = 0.5$, and the corresponding threshold for three-dimensional space is $P_c = 0.16$, which agrees with the available data. A calculation is given of the critical index which governs the power-law behavior of the conductivity near the percolation threshold. The recursion approach confirms the hypothesis of scaling invariance.

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1. The percolation theory methods are being used widely to solve various problems in the physics of disordered systems.[1,2] Numerical calculations in the lattice problems and in the continuum analog of the percolation problem have yielded the most important characteristics of disordered systems and the hypothesis of scaling invariance has been put forward[3-6] by analogy with the theory of phase transitions. As a result of the scaling invariance and universality, percolation along channels of dimensions of the order of the lattice constant has little effect on the large-scale properties of such systems and this justifies the application of the renormalization group methods to the calculation of the percolation threshold and critical indices. The first attempts have been made[7,8] to use the recursion equation in solving the two-dimensional problem of bonds[7] and the problem of sites.[8] The doubling procedure is used in Refs. 7 and 8 and this suffers from low precision.[9]

We shall propose an improved recursion equation which can be applied to systems of any dimensions in space and which allows for an arbitrary change in scale which increases considerably the precision of the results. Our method is readily seen to be similar to the recursion approach of A. A. Migdal, who first applied this to the theory of phase transitions and gauge fields.[9]

2. We shall derive the recursion equation by selecting in a d-dimensional system a region in the form of a d-dimensional cube with the characteristic size L, which is much less than the correlation radius[2] but much greater than the lattice constant. We shall assume that this region is described approximately by a single parameter $P(L)$, which is the probability that the region is conducting. Then, the probability that the region is not conducting is $1-P(L)$. We shall combine N^d cubes into one with a characteristic size NL and calculate the probability $P(NL)$ that the cube of dimensions NL is conducting. If we connect current-carrying conductors to two opposite $(d-1)$-dimensional faces of the cube NL, we find that the probability that the current passes through the cube can be calculated as follows. We considered a "column" of N d-dimensional cubes joining these faces. The probability that the column is conducting is $[P(L)]^N$. The current can pass between the faces if at least one of the N^{d-1} columns is conducting. The probability of this happening is $1-(1-P^N)^{N^{d-1}}$. This is the probability that the cube NL is conducting:

$$P(NL) = 1-[1-P^N(L)]^{N^{d-1}}. \tag{1}$$

Equation (1) allows only for the shortest leakage paths between the opposite faces of the cube NL. The most accurate results may be expected in the limit $N \to 1$ (Ref. 9) with Eq. (1) continued analytically to nonintegral values of N. If we assume that $NL = L + dL$, where dL is an infinitesimally small change in scale, and expand both sides of Eq. (1) as series up to the first order in dL/L, we obtain the following equation which describes the dependence of P on L for an arbitrary change in scale:

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$$dP/d\ln L = P\ln P - (d-1)(1-P)\ln(1-P). \quad (2)$$

Equation (2) is scaling-invariant and it has the same meaning as the well-known equation of Gell-Mann and Low in the field theory.[10] Zeros of the function

$$\Psi(P) = P\ln P - (d-1)(1-P)\ln(1-P),$$

usually called the Gell-Mann–Low function define fixed points of the nonlinear equation (2) and in the present problem they give the percolation threshold P_c.

In one-dimensional space we have $d=1$ and $\Psi(P) = P\ln P$. There are two trivial fixed points: $P_c = 0$ is a stable point and $P_c = 1$ is unstable. The instability of $P_c = 1$ is physically self-evident: if the bare value of P for some scale L_0 is less than unity, the probability $P(L)$ tends to zero as $P(L_0)^{L/L_0}$.

In a two-dimensional space the equation

$$\Psi(P) = P\ln P - (1-P)\ln(1-P) = 0$$

has not only trival stable points $P_c = 0$ and $P_c = 1$ but also a nontrivial unstable fixed point $P_c = 0.5$. This solution is exact because there is complete symmetry between the conducting and nonconducting states of the system in a two-dimensional space.

In a three-dimensional space the equation $\Psi(P) = 0$ has trivial solutions as well as an unstable fixed point $P_c = 0.16$. This result is in good agreement with the experimental data[11] on the metal–insulator transition in the compound Na_xWO_3, for which $P_c = 0.16 \pm 0.03$ is given, as well as with numerical calculations[3] yielding $P_c = 0.17 \pm 0.01$.

It is quite clear that an increase in the dimensions of space lowers the percolation threshold P_c; for example, for $d=4$, we have $P_c = 0.05$. The coefficient $(d-1)$ on the right-hand side of Eq. (2) represents an enhancement of the conductivity on increase in the dimensions of a system because of an increase in the number of directions along which the current may flow.

3. Near the critical value of P the correlation radius behaves singularly:

$$\zeta(P) \propto (P_c - P)^{-\nu}. \quad (3)$$

According to Ref. 2, the index ν governs the behavior of the conductivity near the threshold:

$$\sigma(P) \propto (P_c - P)^{\nu(d-1)}. \quad (4)$$

We can use Eq. (2) to calculate the index ν, which is related to the derivative of the Gell-Mann–Low function at the point $P = P_c$. In fact, the linearized equation (2) becomes

$$d(P-P_c)/d\ln L = \lambda(P-P_c), \quad (5)$$

where $\lambda = d + \ln P_c + (d-1)\ln(1-P_c)$. We can easily obtain a general expression for the index ν in the case of an arbitrary change in the scale of the system:

$$\nu(N) = \ln N[d\ln N + (N-1)\ln P_c + (N^{d-1}-1)\ln(1-P_c)]^{-1}. \quad (6)$$

In the limit $N \to 1$, we obtain $\nu = \lambda^{-1}$, i.e.,

$$\nu^{-1} = d + \ln P_c + (d-1)\ln(1-P_c). \quad (7)$$

We find from Eq. (7) that if $d=2$ then $\nu = 2(1-\ln 2)^{-1}$, and if $d=3$, then $\nu = 1.22$. The values obtained from Eq. (2) for the indices are in satisfactory agreement with the results of Ref. 11, where $\nu = 0.9 \pm 0.2$ is obtained near the metal–insulator transition, and with the numerical calculations[2,5] for systems of different dimensions. The agreement between the results of the recursion approach and the experimental data and numerical calculations provides a direct confirmation of the hypothesis of scaling invariance of the percolation process near the threshold.[5,6]

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